From Seismology to Compressed Sensing and Back, a Brief History of Optimization-Based Signal Processing

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Deconvolution in seismology

Compressed sensing

Back to deconvolution: the super-resolution problem

Super-resolution via semidefinite programming

Demixing of sines and spikes

Deconvolution in seismology

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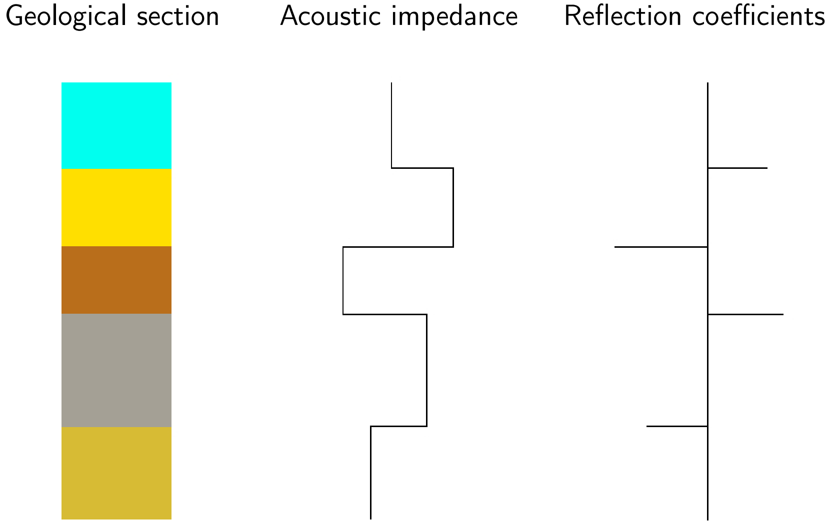
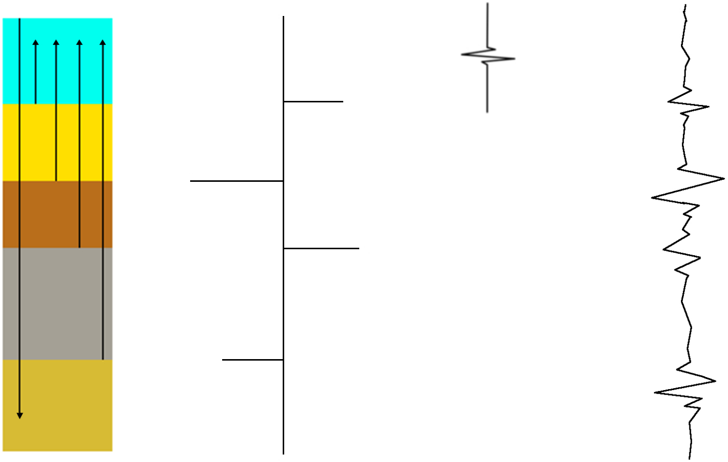
Seismology

Reflection seismology



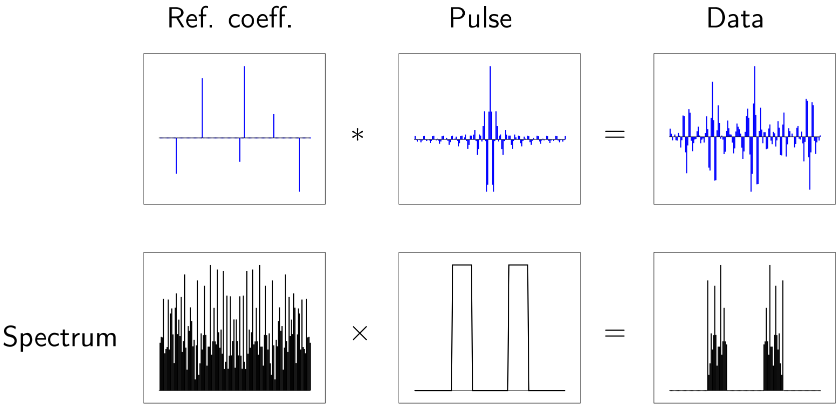
Reflection seismology

Sensing Ref. coeff. Pulse Data



Data ≈ convolution of pulse and reflection coefficients

Sensing model for reflection seismology



Convolution in time = Pointwise multiplication in frequency

Ill-posed problem! How do we choose between signals consistent with data?

Geophysicists: Minimize ℓ_1 norm

Deconvolution with the ℓ_1 norm

Howard L. Taylor,* Stephen C. Banks,‡ and John F. McCoy§

GEOPHYSICS, VOL. 44, NO. 1 (JANUARY 1979)

LINEAR INVERSION OF BAND-LIMITED REFLECTION SEISMOGRAMS*

FADIL SANTOSA† AND WILLIAM W. SYMES‡

SIAM J. SCI. STAT. COMPUT.
Vol. 7, No. 4, October 1986

Reconstruction of a sparse spike train from a portion of its spectrum and application to high-resolution deconvolution

Shlomo Levy* and Peter K. Fullagar‡

GEOPHYSICS, VOL. 46, NO. 9 (SEPTEMBER 1981)

ROBUST MODELING WITH ERRATIC DATA†

JON F. CLAERBOUT* AND FRANCIS MUIR‡

GEOPHYSICS, VOL. 38, NO. 5 (OCTOBER 1973)

Minimum ℓ_1-norm estimate

$$\begin{aligned} &\textit{minimize} && ||\text{estimate}||_1 \\ &\textit{subject to} && \text{estimate} * \text{pulse} = \text{data} \end{aligned}$$

Minimum ℓ_1-norm estimate

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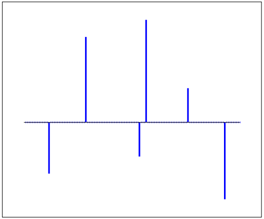
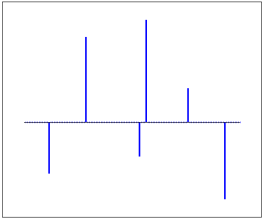
Reflection coefficients

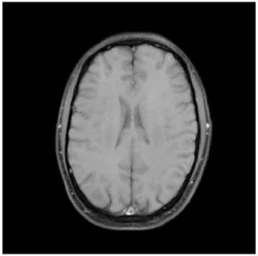


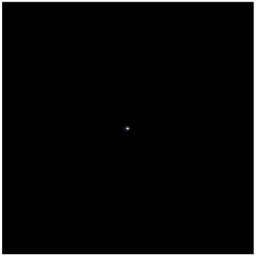
Estimate

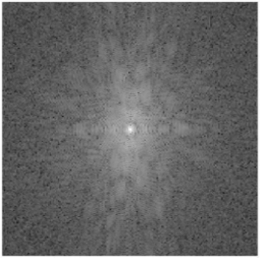


It works, but under what conditions?

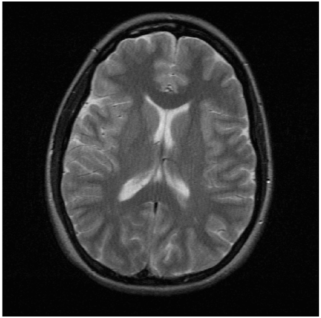
Magnetic resonance imaging



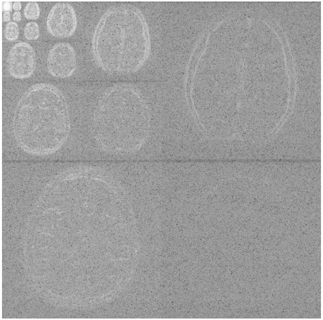




Images are sparse/compressible



Wavelet coefficients



Magnetic resonance imaging

Data: Samples from spectrum

Problem: Sampling is time consuming (annoying, patient might move)

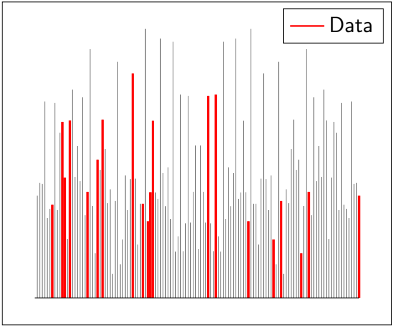
Images are compressible ($\approx$ sparse)

Can we recover compressible signals from less data?

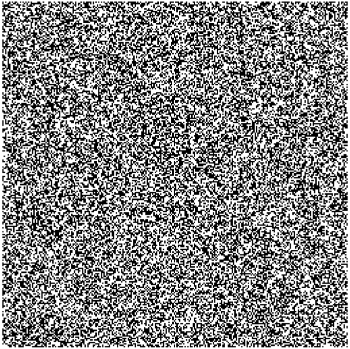
Compressed sensing

1. Undersample the spectrum randomly

1D



2D



Compressed sensing

2. Solve the optimization problem

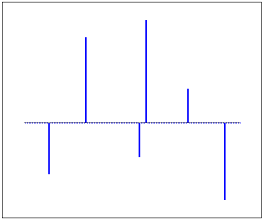
$$\begin{aligned} &\textit{minimize} \quad && ||\text{estimate}||_1 \\ &\textit{subject to} \quad && \text{frequency samples of estimate} = \text{data} \end{aligned}$$

Compressed sensing

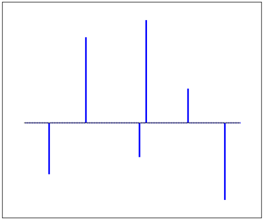
2. Solve the optimization problem

$$\begin{aligned} &\textit{minimize} && \|\text{estimate}\|_1 \\ &\textit{subject to} && \text{frequency samples of estimate} = \text{data} \end{aligned}$$

Signal

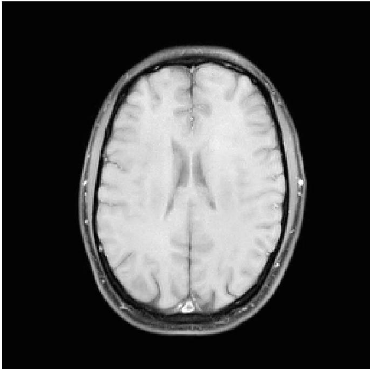


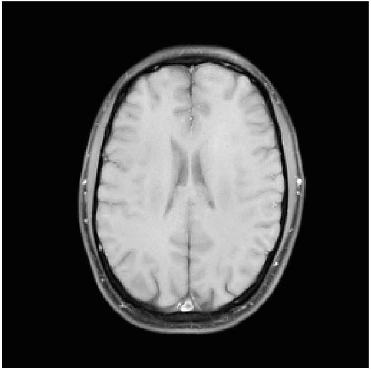
Estimate



Compressed sensing in MRI

x2 Undersampling



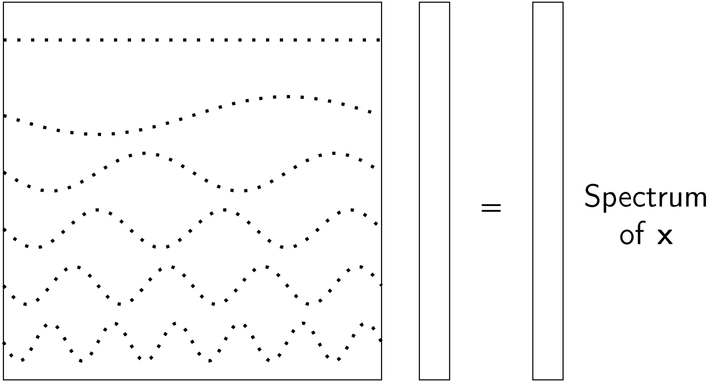


Theoretical questions

1. Is the problem well posed?

2. Does ℓ_1-norm minimization work?

Is the problem well posed?



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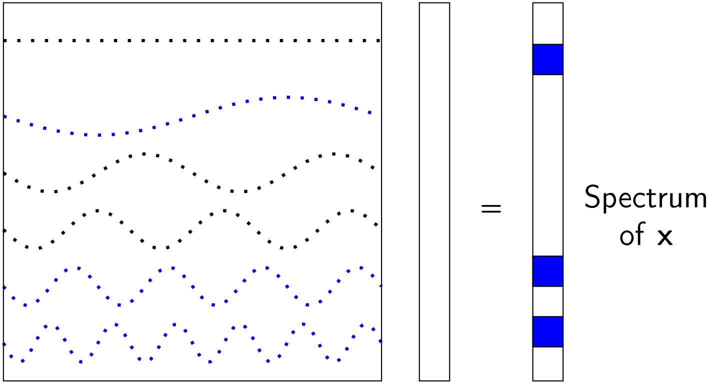
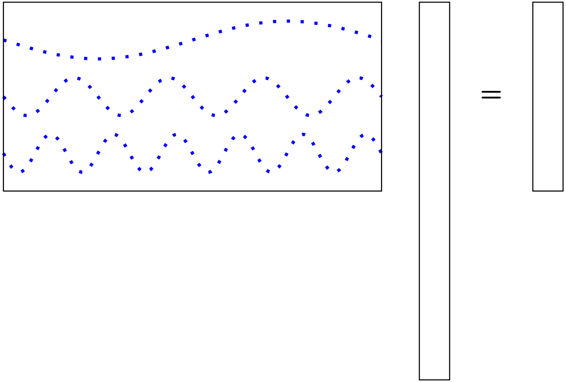


Measurement operator = random frequency samples

Is the problem well posed?



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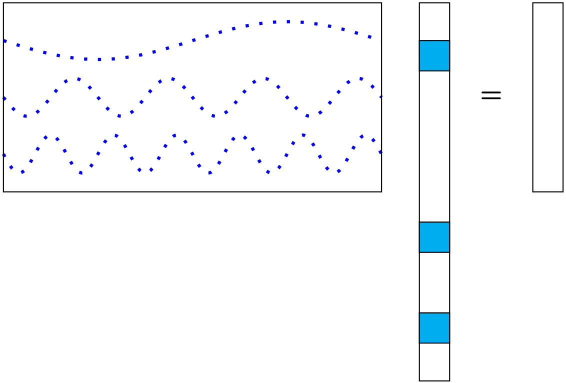
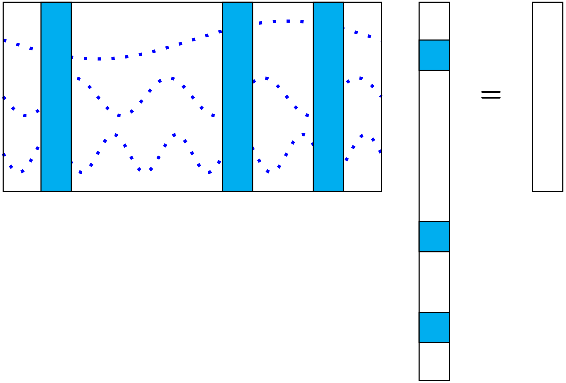


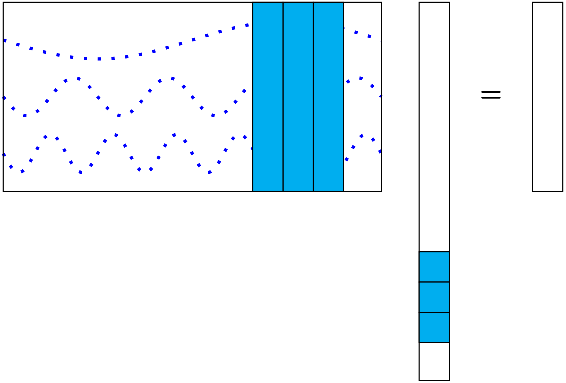
What is the effect of the measurement operator on sparse vectors?

Is the problem well posed?



Are sparse submatrices always well conditioned?

Is the problem well posed?



Are sparse submatrices always well conditioned?

Restricted isometry property (RIP)

An $m \times n$ matrix A satisfy the restricted isometry property if there is $0 < \delta < 1$ such that for any s-sparse vector x

$$(1-\delta)\,||x||_2 \leq ||Ax||_2 \leq (1+\delta)\,||x||_2$$

Random Fourier matrices satisfy the RIP with high probability if $m \geq \mathcal{O}(s)$ up to log factors (Candès, Tao 2006)

$2s$-RIP implies that for any s-sparse signals x_1, x_2

$$||y_2 - y_1||_2 \geq (1-\delta)\,||x_2 - x_1||_2$$

Theoretical questions

1. Is the problem well posed?

2. Does ℓ_1-norm minimization work?

Characterizing the minimum ℓ_1-norm estimate

- Aim: Show that the original signal x is the solution of

$$\begin{aligned} \textit{minimize} \quad & \left|\left|x'\right|\right|_1 \\ \textit{subject to} \quad & A\,x' = y \end{aligned}$$

- This is guaranteed by the existence of a dual certificate

Dual certificate

$v \in \mathbb{R}^m$ is a dual certificate associated to x if

$$q := A^T v$$

satisfies

$$\begin{aligned} q_i &= \operatorname{sign}(x_i) && \text{if } x_i \neq 0 \\ |q_i| &< 1 && \text{if } x_i = 0 \end{aligned}$$

Dual certificate

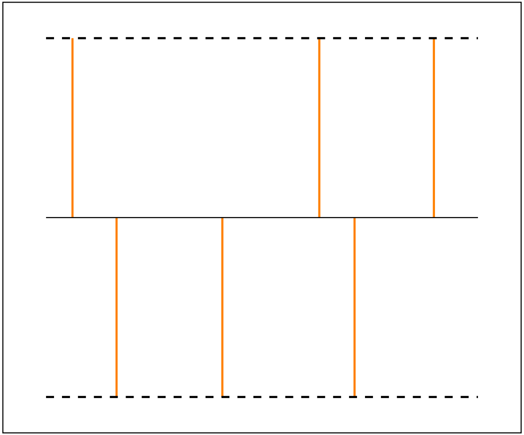
q is a subgradient of the ℓ_1 norm at x

For any $x + h$ such that $Ah = 0$

$$\begin{aligned} ||x + h||_1 &\geq ||x||_1 + q^T h \\ &= ||x||_1 + v^T Ah \\ &= ||x||_1 \end{aligned}$$

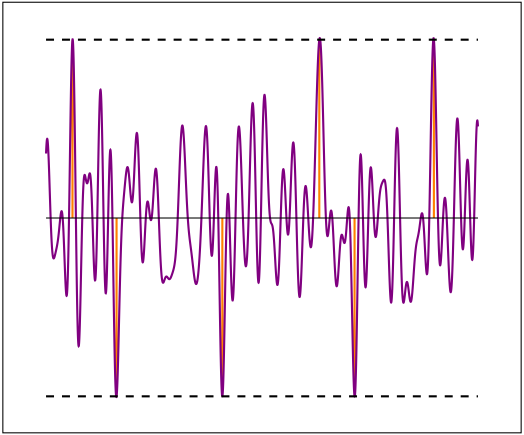
If A_T (where T is the support of x) is injective, x is the unique solution

Dual certificate for compressed sensing



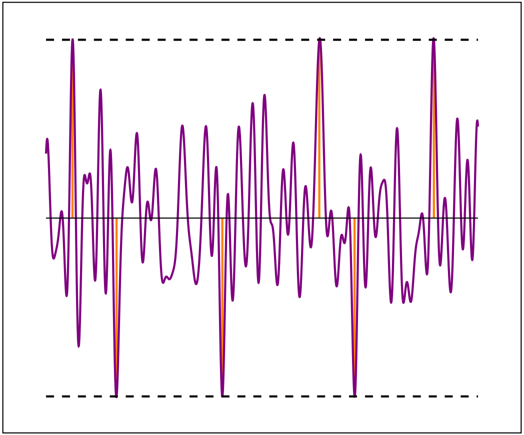
Aim: Show that a dual certificate exists for *any* sparse support and sign pattern

Certificate for compressed sensing



Idea: Minimum-energy interpolator has closed-form solution

Certificate for compressed sensing

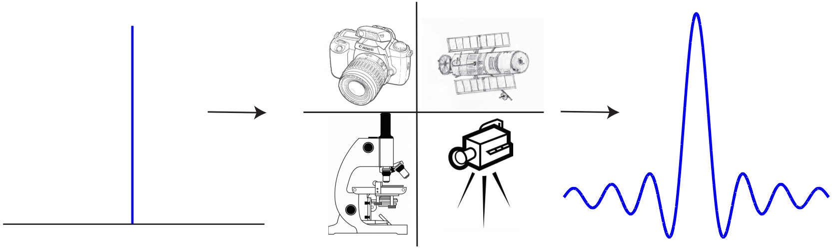


Valid certificate if $m \geq \mathcal{O}(s)$ up to log factors

(Candès, Romberg, Tao 2006)

Limits of resolution in imaging

The resolving power of lenses, however perfect, is limited (**Lord Rayleigh**)

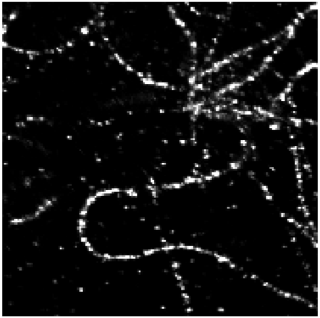


$\delta(t-\tau)$ optical system $h(t-\tau)$

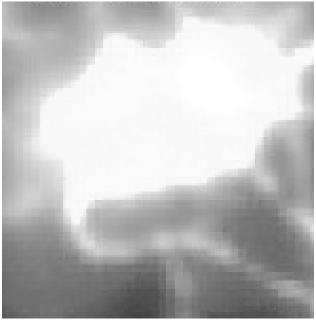
Diffraction imposes a fundamental limit on the resolution of optical systems

Fluorescence microscopy

Data



Point sources



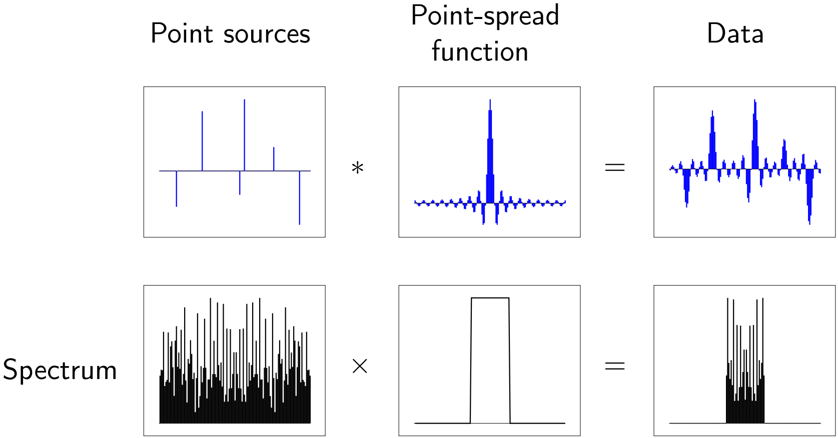
Low-pass blur

(Figures courtesy of V. Morgenshtern)

Super-resolution

- **Optics**: Data-acquisition techniques to overcome the diffraction limit
- **Image processing**: Methods to upsample images onto a finer grid while preserving edges and hallucinating textures
- **This talk**: Estimation/deconvolution from low-pass measurements

Sensing model for super-resolution



Deconvolution problem as in reflection seismology

Minimum ℓ_1-norm estimate

$$\begin{aligned} &\textit{minimize} && ||\text{estimate}||_1 \\ &\textit{subject to} && \text{estimate} * \text{psf} = \text{data} \end{aligned}$$

Minimum ℓ_1-norm estimate

$$\begin{array}{ll} \textit{minimize} & ||\text{estimate}||_1 \\ \textit{subject to} & \text{estimate} * \text{psf} = \text{data} \end{array}$$

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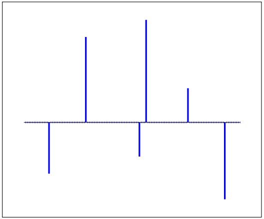
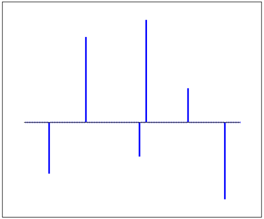


Estimate



Mathematical model

- Signal: superposition of Dirac measures with support T

$$x = \sum_j a_j \delta_{t_j} \qquad a_j \in \mathbb{C},\ t_j \in T \subset [0,1]$$

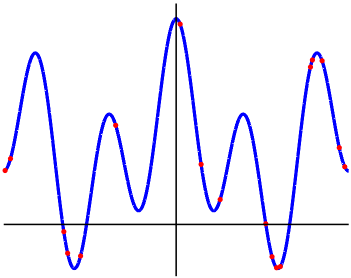
- Data: low-pass Fourier coefficients with cut-off frequency f_c

$$y = \mathcal{F}_c\, x$$

$$y(k) = \int_0^1 e^{-i2\pi kt} x\,(\mathrm{d}t) = \sum_j a_j e^{-i2\pi k t_j}, \quad k \in \mathbb{Z},\ |k| \leq f_c$$

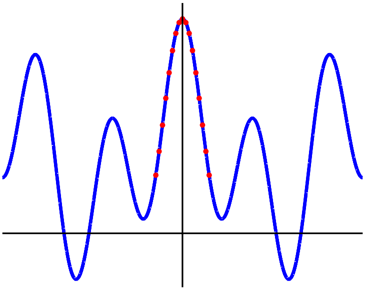
Compressed sensing vs super-resolution

Compressed sensing



spectrum interpolation

Super-resolution



spectrum extrapolation

Total-variation norm

- Continuous counterpart of the ℓ_1 norm
- If $x = \sum_j a_j \delta_{t_j}$ then $||x||_{\text{TV}} = \sum_j |a_j|$
- Not the total variation of a piecewise-constant function
- Formal definition: For a complex measure ν

$$||\nu||_{\text{TV}} = \sup \sum_{j=1}^{\infty} |\nu(B_j)|,$$

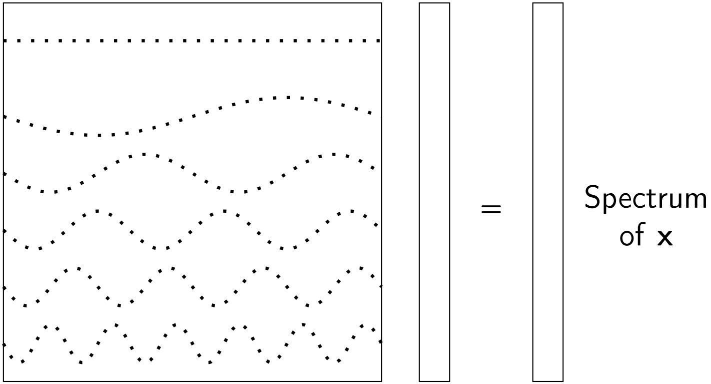
(supremum over all finite partitions B_j of $[0,1]$)

Theoretical questions

1. Is the problem well posed?

2. Does TV-norm minimization work?

Is the problem well posed?



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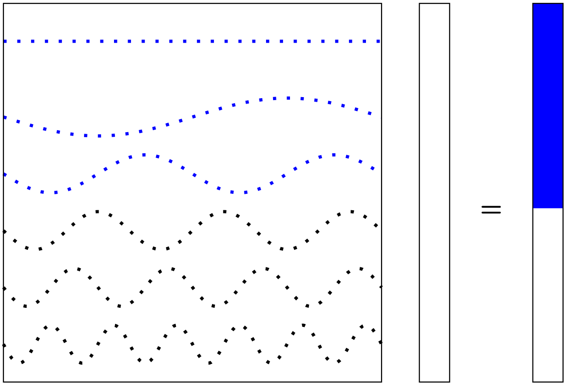
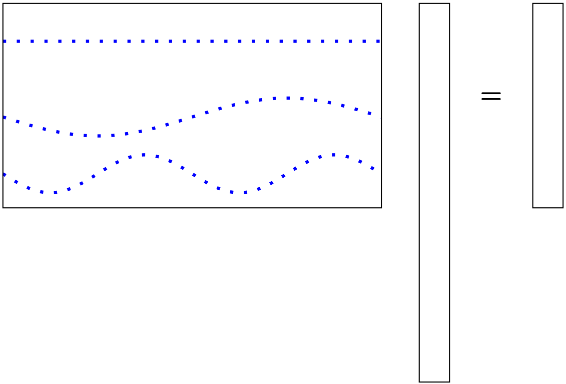


Measurement operator = low-pass samples with cut-off frequency f_c

Is the problem well posed?



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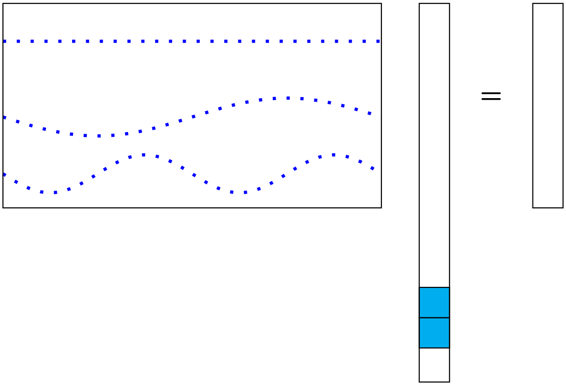
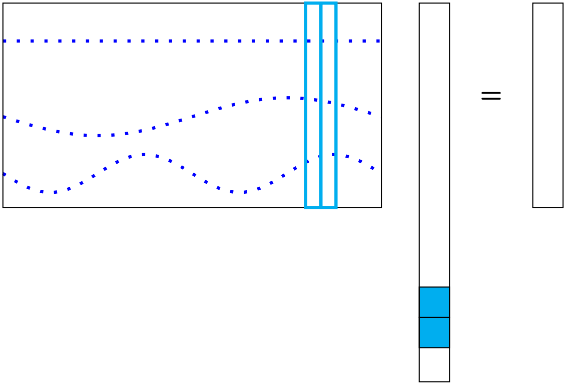


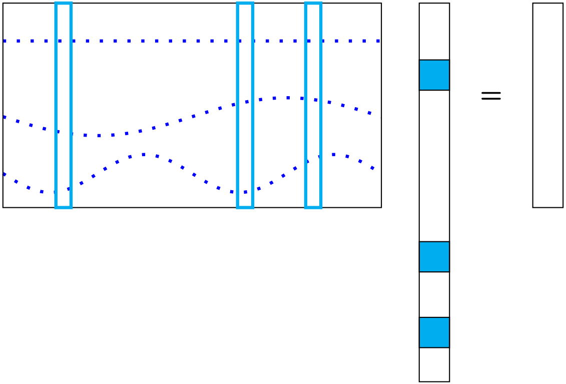
Effect of measurement operator on sparse vectors?

Is the problem well posed?



Submatrix can be very ill conditioned!

Is the problem well posed?

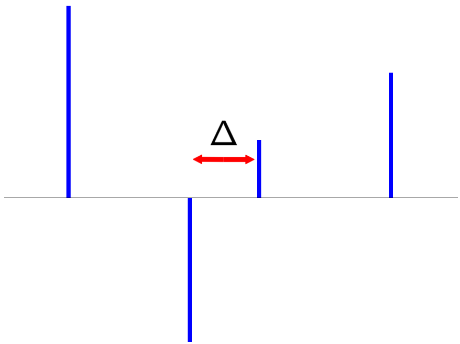


If support is spread out there is hope

Minimum separation

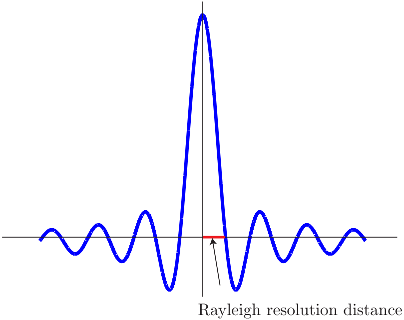
The minimum separation Δ of the support of x is

$$\Delta = \inf_{(t,t') \in \text{support}(x) : t \neq t'} |t - t'|$$



Conditioning of submatrix with respect to Δ

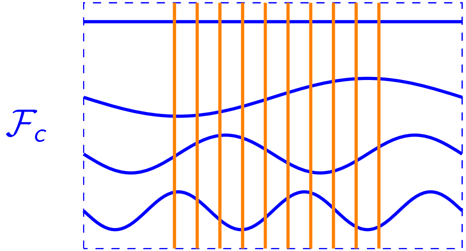
- If $\Delta < 1/f_c$ the problem is ill posed
- If $\Delta > 1/f_c$ the problem becomes well posed
- Proved asymptotically by Slepian and non-asymptotically by Moitra



$1/f_c$ is the diameter of the main lobe of the point-spread function (twice the Rayleigh distance)

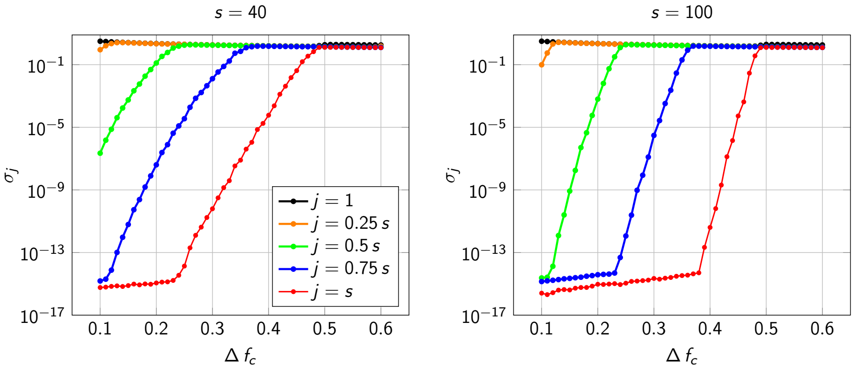
Lower bound on Δ

- Above what minimum distance Δ is the problem well posed?
- Numerical lower bound on Δ:
  1. Compute singular values of restricted operator for different values of Δ_{diff}
  2. Find Δ_{diff} under which the restricted operator is ill conditioned
  3. Then $\Delta \geq 2\Delta_{\text{diff}}$



Singular values of the restricted operator

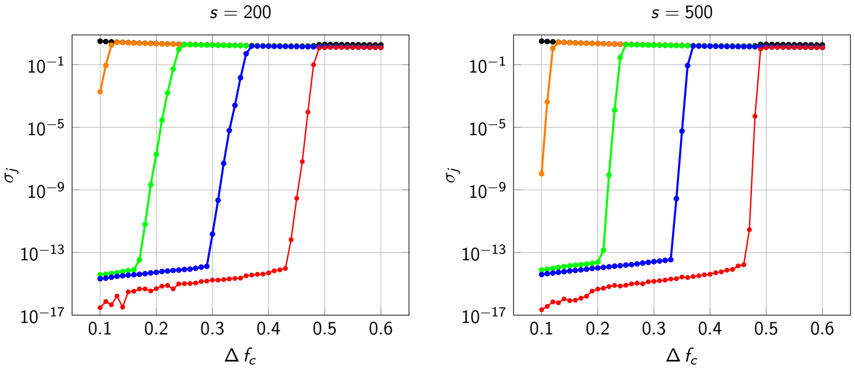
Number of spikes $= s$, $f_c = 10^3$



Phase transition at $\Delta_{\text{diff}} = 1/2f_c \rightarrow \Delta = 1/f_c$

Singular values of the restricted operator

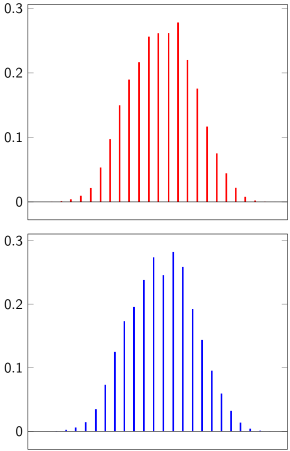
Number of spikes $= s$, $f_c = 10^3$



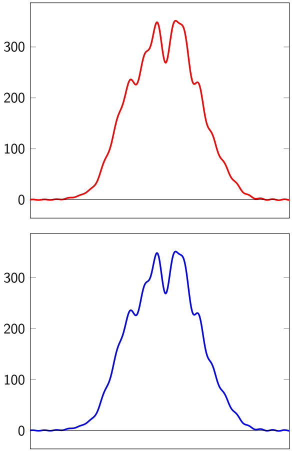
Phase transition at $\Delta_{\text{diff}} = 1/2f_c \rightarrow \Delta = 1/f_c$

Example: 25 spikes, $f_c = 10^3$, $\Delta = 0.8/f_c$

Signals



Data (in signal space)



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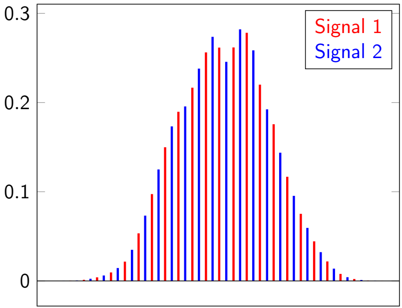
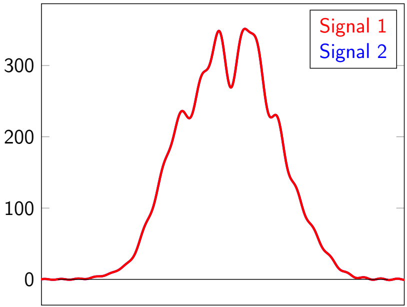


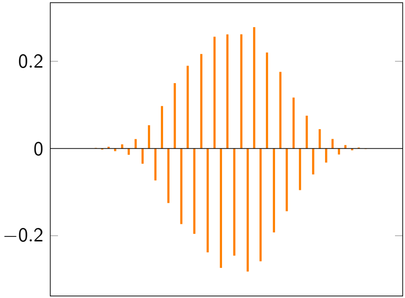
Signals



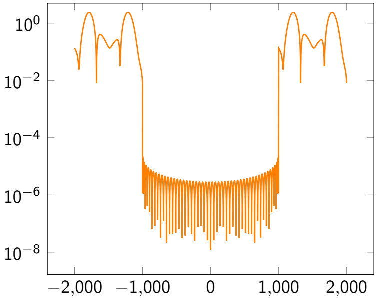
Data (in signal space)

Example: 25 spikes, $f_c = 10^3$, $\Delta = 0.8/f_c$

The difference is almost in the null space of the measurement operator



Difference



Spectrum

Theoretical questions

1. Is the problem well posed?

2. Does TV-norm minimization work?

Estimation via convex programming

For data of the form $y = \mathcal{F}_c\, x$, we solve

$$\min_{\tilde{x}} ||\tilde{x}||_{\text{TV}} \quad \text{subject to} \quad \mathcal{F}_c\, \tilde{x} = y,$$

over all finite complex measures $\tilde{x}$ supported on $[0, 1]$

Dual certificate of TV norm

A dual certificate of the TV norm at

$$x = \sum_j a_j \delta_{t_j} \qquad a_j \in \mathbb{C},\ t_j \in T$$

guarantees that x is the unique solution if

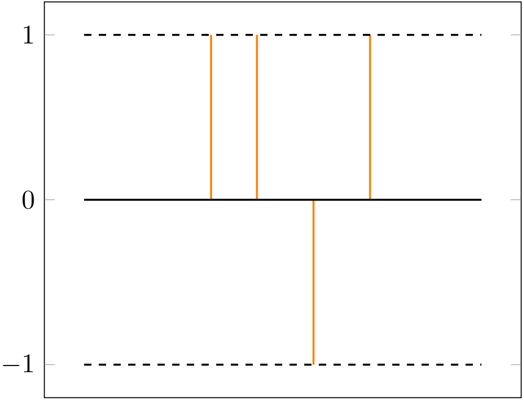
$$q := \mathcal{F}_c^* \, v = \sum_{k \leq |f_c|} v_k e^{i 2\pi k t}$$

$$q\left(t_j\right) = \operatorname{sign}\left(a_j\right) \qquad \text{if } t_j \in T$$

$$\left|q\left(t\right)\right| < 1 \qquad \text{if } t \notin T$$

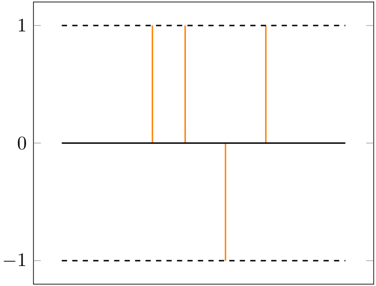
Range of $\mathcal{F}_c^*$ is spanned by low pass sinusoids instead of random sinusoids

Certificate for super-resolution



Aim: Interpolate sign pattern

Certificate for super-resolution



1st idea: Interpolation with a low-frequency fast-decaying kernel K

$$q(t) = \sum_{t_j \in T} \alpha_j \, K\left(t - t_j\right)$$

Certificate for super-resolution

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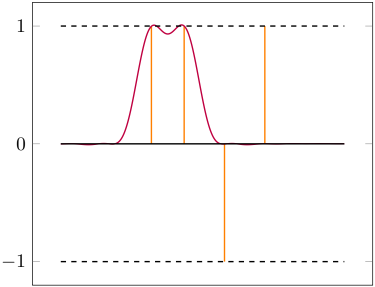

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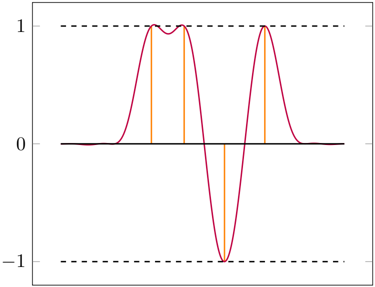

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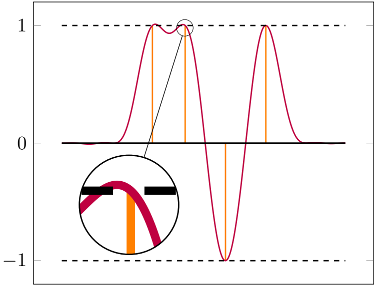
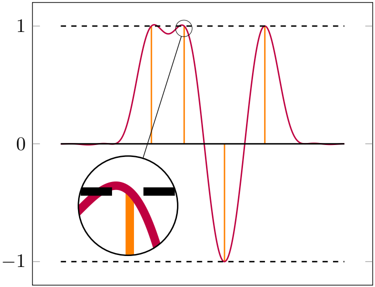


Problem: Magnitude of certificate locally exceeds 1

Certificate for super-resolution

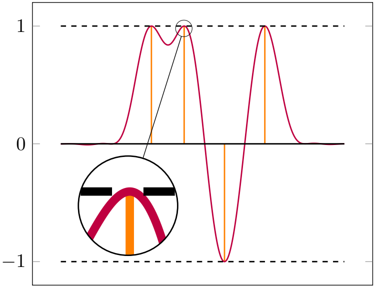


Problem: Magnitude of certificate locally exceeds 1

Solution: Add correction term and force the derivative of the certificate to equal zero on the support

$$q(t) = \sum_{t_j \in T} \alpha_j \, K(t - t_j) + \beta_j \, K'(t - t_j)$$

Certificate for super-resolution



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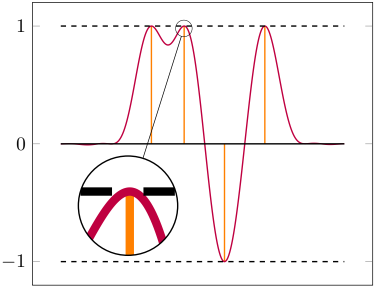
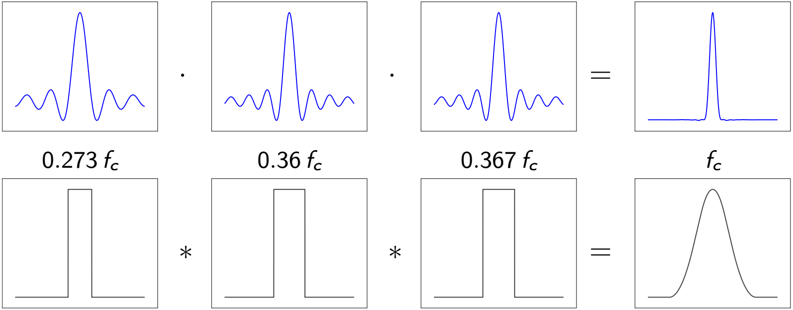
Certificate for super-resolution



Similar construction for bandpass point-spread functions relevant to reflection seismology

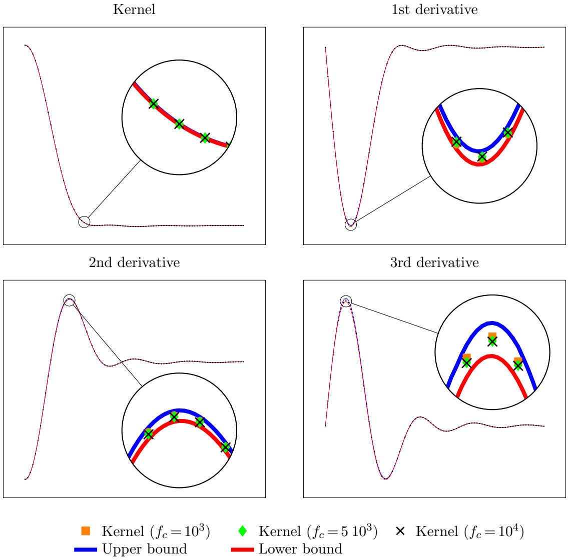
Sketch of proof: Interpolation kernel

Key step: Designing a good interpolation kernel



Trade-off between *spikiness* at the origin and asymptotic decay

Sketch of proof: Non-asymptotic bounds on kernel



Guarantees for super-resolution

Theorem [Candès, F. 2012]

If the minimum separation of the signal support obeys

$$\Delta \geq 2\,/f_c$$

then recovery via convex programming is exact

Theorem [Candès, F. 2012]

In 2D convex programming super-resolves point sources with a minimum separation of

$$\Delta \geq 2.38\,/f_c$$

where f_c is the cut-off frequency of the low-pass kernel

Guarantees for super-resolution

Theorem [F. 2016]

If the minimum separation of the signal support obeys

$$\Delta \geq 1.26 \, / f_c,$$

then recovery via convex programming is exact

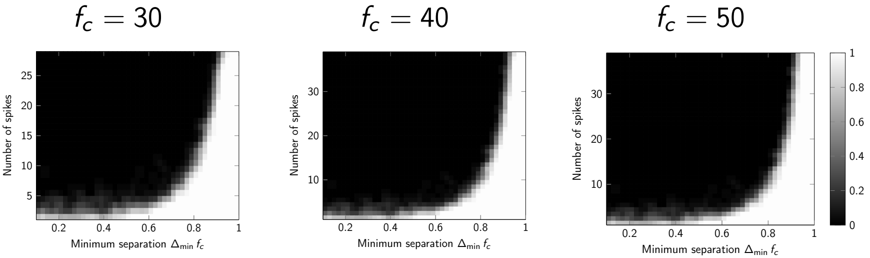
Theorem [Candès, F. 2012]

In 2D convex programming super-resolves point sources with a minimum separation of

$$\Delta \geq 2.38 \, / f_c$$

where f_c is the cut-off frequency of the low-pass kernel

Numerical evaluation of minimum separation



Conjecture: TV-norm minimization succeeds if $\Delta \geq \frac{1}{f_c}$

Dual certificate as theoretical tool

Subsequent work builds on our construction to analyze

- Stability of super-resolution [Candès, F. 2013], [F. 2013], [Azais, De Castro, Gamboa 2013], [Duval, Peyré 2013]
- Denoising of line spectra [Tang, Bhaskar, Recht 2013]
- Compressed sensing off the grid [Tang, Bhaskar, Shah, Recht 2013]
- Recovery of splines from their projection onto spaces of algebraic polynomials [Bendory, Dekel, Feuer 2013], [De Castro, Mijoule 2014]
- Recovery of point sources from spherical harmonics [Bendory, Dekel, Feuer 2013]

Practical implementation

- **Primal problem**:

$$\min_{\tilde{x}} ||\tilde{x}||_{\mathrm{TV}} \quad \text{subject to} \quad \mathcal{F}_c\, \tilde{x} = y$$

Infinite-dimensional variable $\tilde{x}$ (measure in $[0, 1]$)

First option: Discretizing + ℓ_1-norm minimization

Practical implementation

- **Primal problem**:

$$\min_{\tilde{x}} ||\tilde{x}||_{\mathrm{TV}} \quad \text{subject to} \quad \mathcal{F}_c\, \tilde{x} = y$$

Infinite-dimensional variable $\tilde{x}$ (measure in $[0, 1]$)

First option: Discretizing + ℓ_1-norm minimization

- **Dual problem**:

$$\max_{\tilde{u} \in \mathbb{C}^n} \mathrm{Re}\left[y^* \tilde{u}\right] \quad \text{subject to} \quad ||\mathcal{F}_c^*\, \tilde{u}||_\infty \leq 1, \quad n := 2f_c + 1$$

Finite-dimensional variable $\tilde{u}$, but infinite-dimensional constraint

$$\mathcal{F}_c^*\, \tilde{u} = \sum_{k \leq |f_c|} \tilde{u}_k e^{i2\pi kt}$$

Second option: Solving the dual problem

Lemma: Semidefinite representation

The Fejér-Riesz Theorem and the semidefinite representation of polynomial sums of squares imply that

$$||\mathcal{F}_c^* \, \tilde{u}||_\infty \leq 1$$

is equivalent to

There exists a Hermitian matrix $Q \in \mathbb{C}^{n \times n}$ such that

$$\begin{bmatrix} Q & \tilde{u} \\ \tilde{u}^* & 1 \end{bmatrix} \succeq 0, \qquad \sum_{i=1}^{n-j} Q_{i,i+j} = \begin{cases} 1, & j = 0, \\ 0, & j = 1, 2, \ldots, n-1. \end{cases}$$

Consequence: The dual problem is a tractable semidefinite program

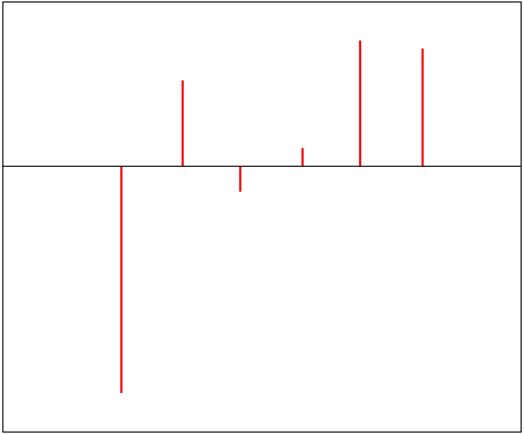
Support-locating polynomial

How do we obtain an estimator from the dual solution?

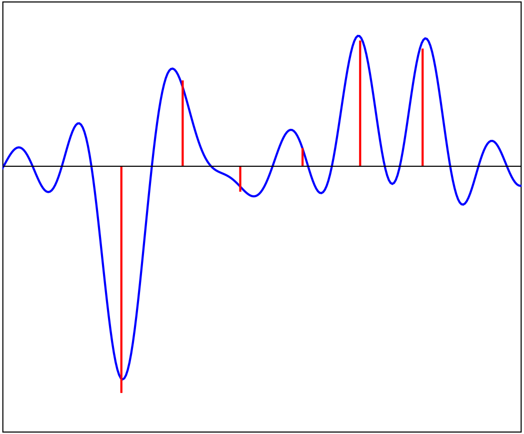
Dual solution vector: Fourier coefficients of low-pass polynomial that interpolates the sign of the primal solution (follows from strong duality)

Idea: Use the polynomial to locate the support of the signal

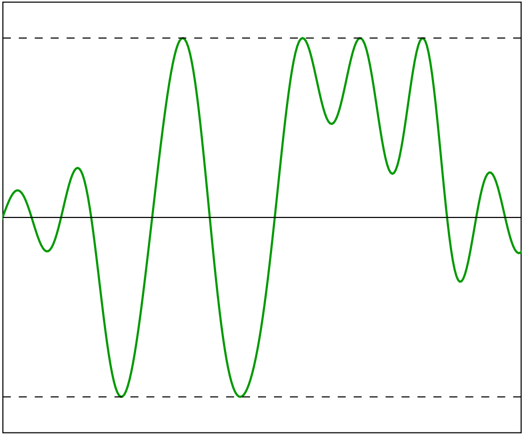
Super-resolution via semidefinite programming



Super-resolution via semidefinite programming

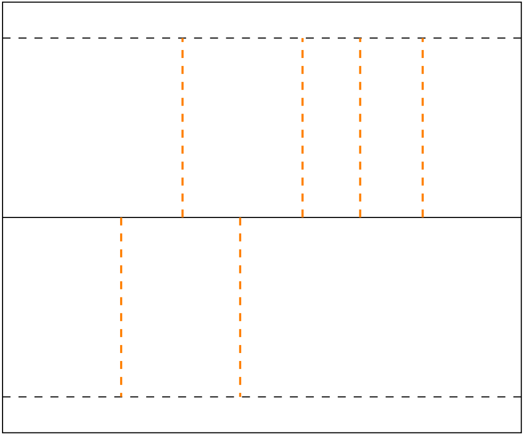


Super-resolution via semidefinite programming



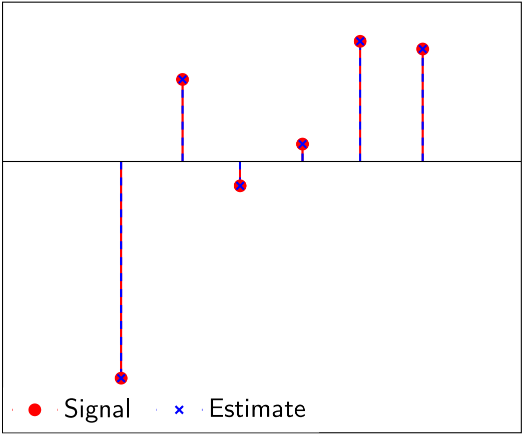
1. Solve semidefinite program to obtain dual solution

Super-resolution via semidefinite programming



2. Locate points at which corresponding polynomial has unit magnitude

Super-resolution via semidefinite programming



3. Estimate amplitudes via least squares

Support-location accuracy

f_c	25	50	75	100
Average error	$6.66\,10^{-9}$	$1.70\,10^{-9}$	$5.58\,10^{-10}$	$2.96\,10^{-10}$
Maximum error	$1.83\,10^{-7}$	$8.14\,10^{-8}$	$2.55\,10^{-8}$	$2.31\,10^{-8}$

For each f_c, 100 random signals with $|T| = f_c/4$ and $\Delta(T) \geq 2/f_c$

Spectral super-resolution

- Signal: Multisinusoidal signal

$$g(t) := \sum_{f_j \in T} c_j e^{-i2\pi f_j t}$$

$$\widehat{g} = \sum_{f_j \in T} c_j \delta_{f_j}$$

- Data: n samples measured at Nyquist rate

$$g(k) := \sum_{f_j \in T} c_j e^{-i2\pi k f_j}, \qquad 1 \leq k \leq n$$

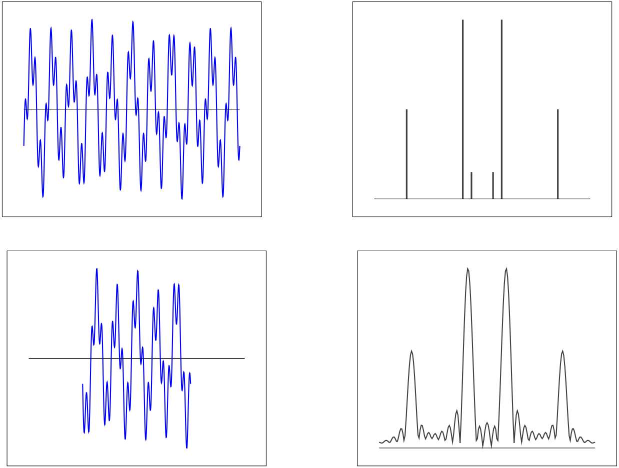
Equivalent to our super-resolution model!

Spectral Super-resolution

Spectrum

Signal

Data



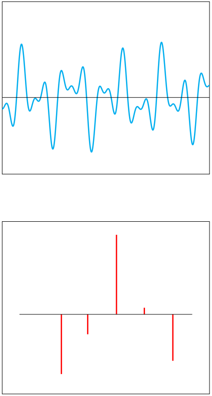
Demixing of sines and spikes

Aim: Super-resolving the spectrum of a multi-sinusoidal signal (sines) in the presence of impulsive events (spikes)

$$y = \mathcal{F}_c \, x + s$$

Demixing of sines and spikes

Sines

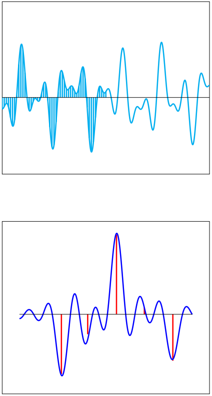


Spectrum

x

Demixing of sines and spikes

Sines

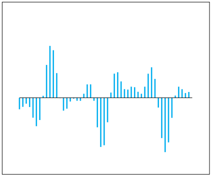


Spectrum

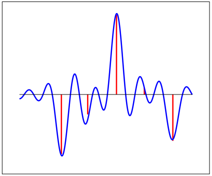
$\mathcal{F}_c\, x$

Demixing of sines and spikes

Sines

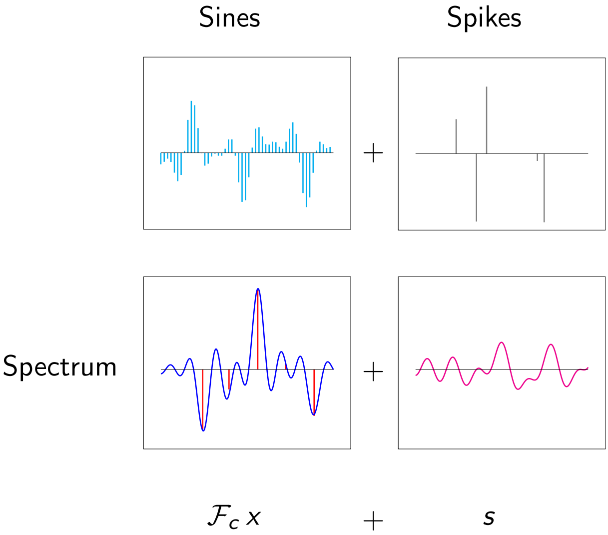


Spectrum

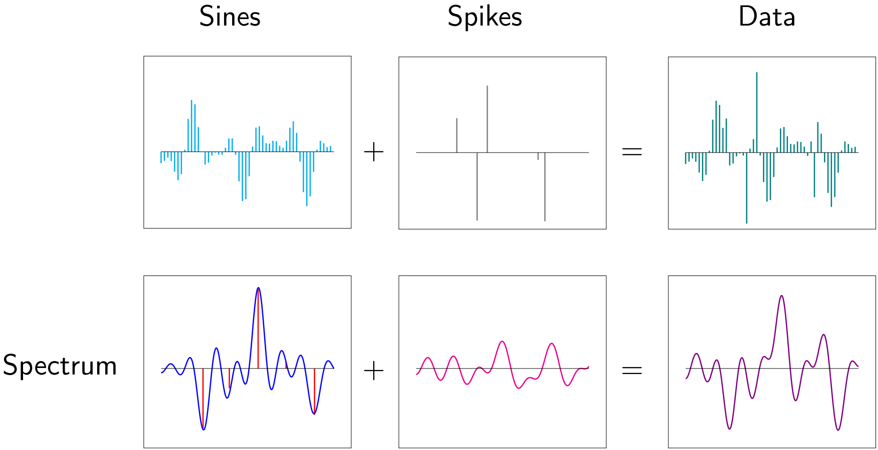


$\mathcal{F}_c \, x$

Demixing of sines and spikes



Demixing of sines and spikes



$$\mathcal{F}_c\, x \quad + \quad s \quad = \quad y$$

Demixing of sines and spikes

Estimator: Solution to

$$\min_{\tilde{x},\,\tilde{s}} ||\tilde{x}||_{\text{TV}} + \gamma\,||\tilde{s}||_1 \quad \text{subject to} \quad \mathcal{F}_c\,\tilde{x} + \tilde{s} = y$$

Dual problem:

$$\max_{\tilde{u}\in\mathbb{C}^n} \text{Re}\left[y^*\,\tilde{u}\right] \quad \text{subject to} \quad ||\mathcal{F}_c^*\,\tilde{u}||_\infty \leq 1, \quad ||\tilde{u}||_\infty \leq \gamma$$

Demixing of sines and spikes

Estimator: Solution to

$$\min_{\tilde{x},\,\tilde{s}} ||\tilde{x}||_{\text{TV}} + \gamma\,||\tilde{s}||_1 \quad \text{subject to} \quad \mathcal{F}_c\,\tilde{x} + \tilde{s} = y$$

Dual problem:

$$\max_{\tilde{u}\in\mathbb{C}^n} \text{Re}\left[y^*\,\tilde{u}\right] \quad \text{subject to} \quad ||\mathcal{F}_c^*\,\tilde{u}||_\infty \leq 1, \quad ||\tilde{u}||_\infty \leq \gamma$$

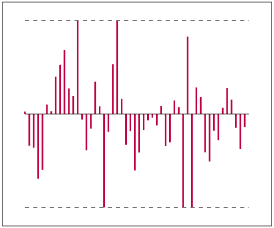
Dual solution: $\hat{u}$

- $\hat{u}$ interpolates the sign of the primal solution $\hat{s}$
- $\mathcal{F}_c^*\,\hat{u}$ interpolates the sign of the primal solution $\hat{x}$

Demixing of sines and spikes

$\hat{u}$ $\quad$ $\mathcal{F}_c^* \, \hat{u}$

Dual solution

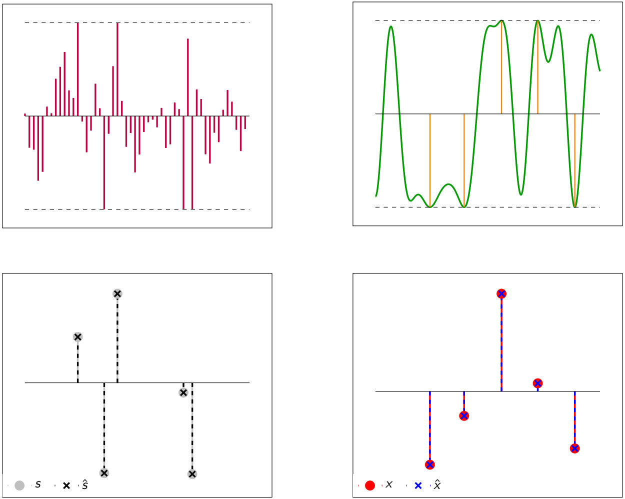


Demixing of sines and spikes

$\hat{u}$ | $\mathcal{F}_c^* \, \hat{u}$

Dual solution

Estimate

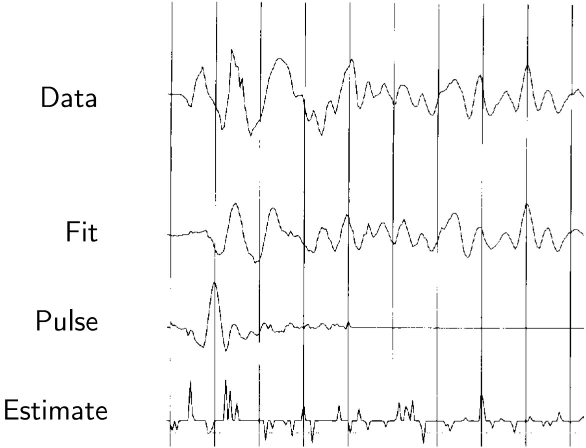


Spikes | Sines (spectrum)

Conclusion

- Geophysicists pioneered the use of ℓ_1-norm regularization for underdetermined inverse problems
- Mathematicians and statisticians developed theoretical tools to understand compressed sensing
- Adapting these insights allows to analyze the potential and limitations of convex programming for super-resolution

Deconvolution with the ℓ_1 norm (Taylor, Banks, McCoy '79)



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