Robust Super-resolution via Convex Programming

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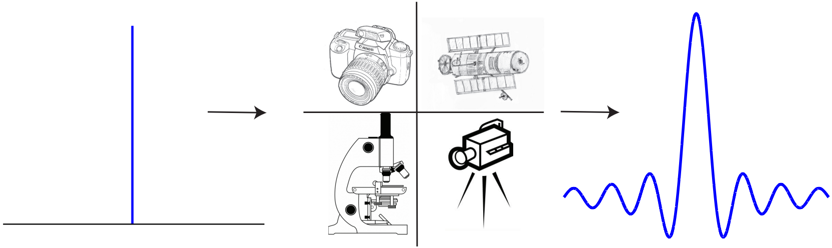
Acknowledgements

- This work was supported by a Fundación La Caixa Fellowship and a Fundación Caja Madrid Fellowship
- **Collaborator** : Emmanuel Candès (Department of Mathematics and of Statistics, Stanford)

Motivation : Limits of resolution in imaging

The resolving power of lenses, however perfect, is limited (**Lord Rayleigh**)

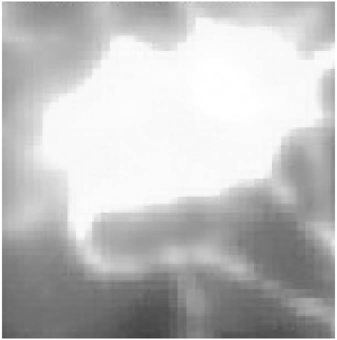


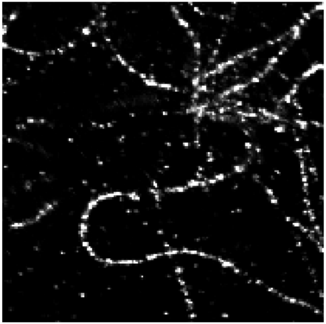
$\delta(t-\tau)$ optical system $h(t-\tau)$

Diffraction imposes a fundamental limit on the resolution of optical systems

Motivation

Similar problems arise in electronic imaging, signal processing, radar, spectroscopy, medical imaging, astronomy, geophysics, etc.





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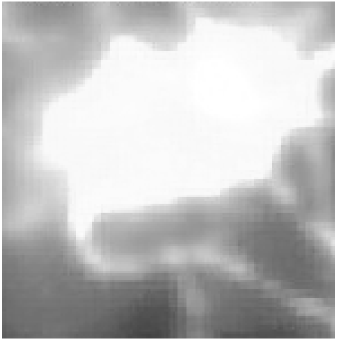
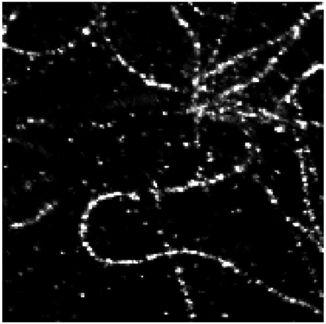
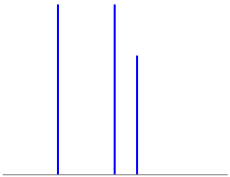
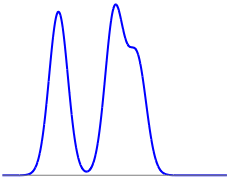




Signals of interest are often point sources : celestial bodies (astronomy), line spectra (signal processing), molecules (fluorescence microscopy), ...

Super-resolution





Aim : estimating fine-scale structure from low-resolution data

Super-resolution

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Equivalently, extrapolating the high end of the spectrum

Mathematical model

- Signal : superposition of Dirac measures with support T

$$x = \sum_j a_j \delta_{t_j} \qquad a_j \in \mathbb{C},\ t_j \in T \subset [0, 1]$$

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- Measurements : low-pass filtering with cut-off frequency f_c

$$y = \mathcal{F}_c\, x \quad \text{(vector of low-pass Fourier coefficients)}$$

$$y(k) = \int_0^1 e^{-i2\pi kt} x\,(\mathrm{d}t) = \sum_j a_j e^{-i2\pi k t_j}, \quad k \in \mathbb{Z},\ |k| \le f_c$$

Equivalent problem : line-spectra estimation

Swapping time and frequency

- Signal : superposition of sinusoids

$$x(t) = \sum_j a_j e^{i2\pi\omega_j t} \qquad a_j \in \mathbb{C},\ \omega_j \in T \subset [0,1]$$

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- Measurements : equispaced samples

$$x(1), x(2), x(3), \ldots x(n)$$

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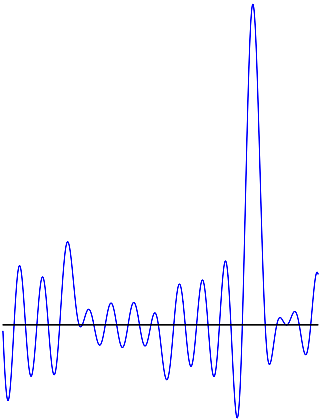
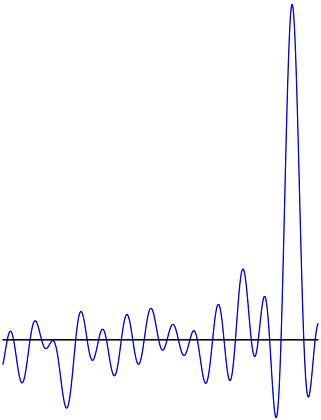
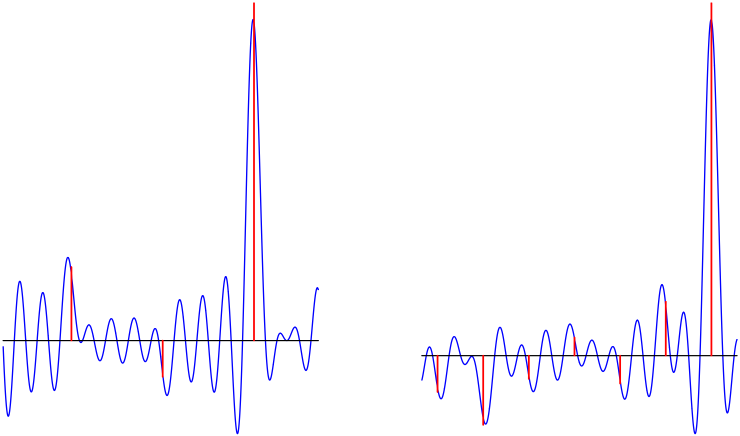
- Classical problem in signal processing

Can you find the spikes?



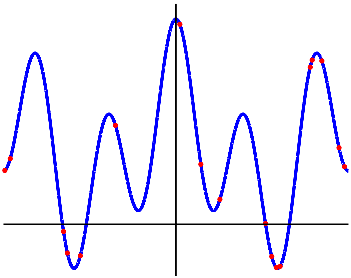


Can you find the spikes?



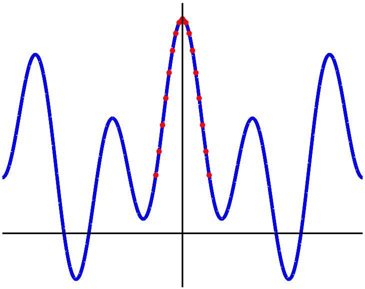
Compressed sensing vs super-resolution

Compressed sensing



spectrum interpolation

Super-resolution



spectrum extrapolation

Outline of the talk

Theory

Proof (sketch)

Robustness to noise

Algorithms

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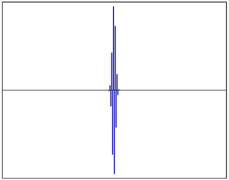
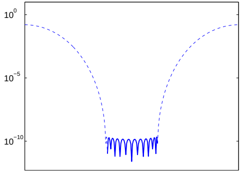
Sparsity is not enough

Sparse signals can be almost completely destroyed by low-pass filtering (shown rigorously by Slepian in the 1970s)

Signal



Spectrum



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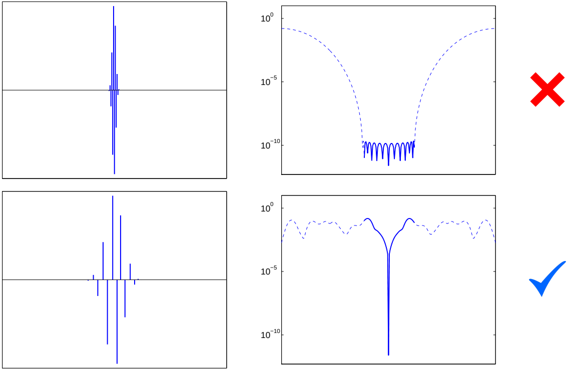
Signal Spectrum

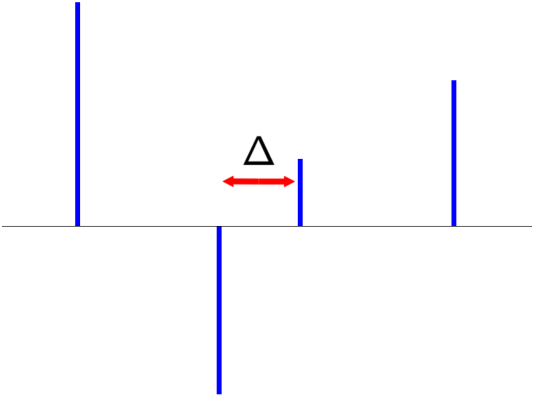


Stable reconstruction is only possible for signals with non-clustered supports

Minimum separation

To exclude highly-clustered signals from our model, we control the minimum separation Δ of the support T

$$\Delta = \inf_{(t,t') \in T \,:\, t \neq t'} |t - t'|$$



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- Not the total variation of a piecewise-constant function
- **Formal definition :** For a complex measure ν

$$||\nu||_{\text{TV}} = \sup \sum_{j=1}^{\infty} |\nu(B_j)|,$$

(supremum over all finite partitions B_j of $[0, 1]$)

Recovery via convex programming

In the absence of noise, i.e. if $y = \mathcal{F}_c\, x$, we solve

$$\min_{\tilde{x}} ||\tilde{x}||_{\text{TV}} \quad \text{subject to} \quad \mathcal{F}_c\, \tilde{x} = y,$$

over all finite complex measures $\tilde{x}$ supported on $[0, 1]$

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Theorem [Candès, F. 2012]

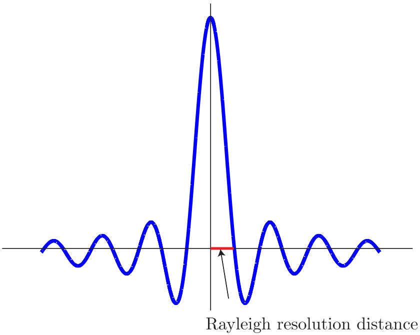
If the minimum separation of the signal support T obeys

$$\Delta \geq 2\,/f_c := 2\,\lambda_c,$$

then recovery is exact

Minimum-distance condition

- $\lambda_c/2$ is the Rayleigh resolution limit (half-width of measurement filter)



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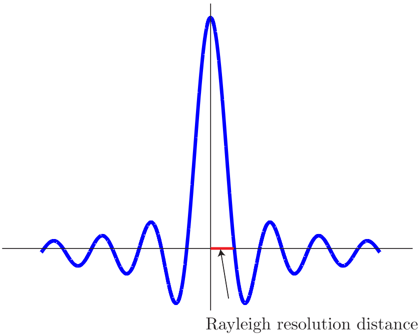

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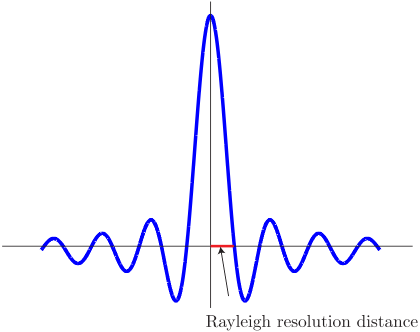

- Numerical simulations show that TV-norm minimization fails if $\Delta < \lambda_c$
- If $\Delta < \lambda_c/2$ some signals are *almost* in the nullspace of the measurement operator (no method can achieve stable estimation)

Higher dimensions

- Signal : superposition of point sources (delta measures) in 2D
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Theorem [Candès, F. 2012]

TV-norm minimization yields exact recovery if

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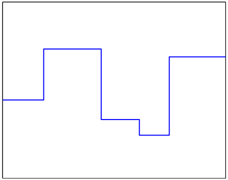
In dimension d, $\Delta \geq C_d \, \lambda_c$, where C_d only depends on d

Extensions

- Signal : piecewise-constant function
- Measurements : low-pass Fourier coefficients



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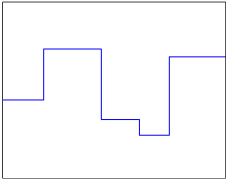


Corollary

$$\text{Solving } \min \|\tilde{x}^{(1)}\|_{\text{TV}} \quad \text{subject to} \quad \mathcal{F}_c \tilde{x} = y$$

$$\text{yields exact recovery if } \Delta \geq 2\,\lambda_c$$

Similar result for cont. differentiable piecewise-smooth functions

Theory

Proof (sketch)

Robustness to noise

Algorithms

Certificate of optimality

Proposition

For any support $T \subset [0,1]$ satisfying $\Delta \geq 2\,\lambda_c$ and every vector $v \in \mathbb{C}^{|T|}$ such that

$$|v_j| = 1 \quad \text{for all } 1 \leq j \leq |T|$$

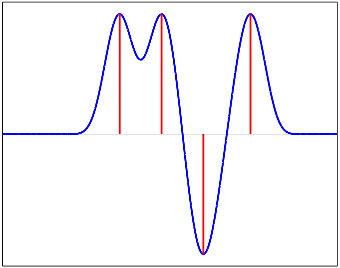
there exists a low-frequency trigonometric polynomial

$$q(t) = \sum_{k=-f_c}^{f_c} c_k e^{i2\pi kt}$$

obeying

$$\begin{cases} q(t_j) = v_j, & t_j \in T, \\ |q(t)| < 1, & t \in [0,1] \setminus T. \end{cases}$$

Certificate of optimality



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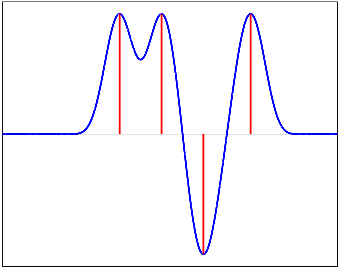

Lemma

The proposition implies that $x = \sum_{t_j \in T} a_j \delta_{t_j}$ is the unique solution to

$$\min_{\tilde{x}} ||\tilde{x}||_{\mathrm{TV}} \quad \text{subject to} \quad \mathcal{F}_c\, \tilde{x} = \mathcal{F}_c\, x$$

Certificate of optimality : Proof

Take any feasible signal $x' = x + h$ and decompose $h = h_T + h_{T^c}$, where

$$h_T = \sum_{t_j \in T} b_j \delta_{t_j} = \sum_{t_j \in T} e^{i\phi_j} |b_j| \delta_{t_j} \quad \text{(proof is easily generalized)}$$

(1) $v_j = e^{-i\phi_j}$ yields q such that $\langle q_T, h_T \rangle = \sum_{t_j \in T} |b_j| = ||h_T||_{\text{TV}}$

(2) x' is feasible so $\langle q, h \rangle = \langle q_T, h_T \rangle + \langle q_{T^c}, h_{T^c} \rangle = 0$

$$\begin{aligned} ||h_{T^c}||_{\text{TV}} &> |\langle q_{T^c}, h_{T^c} \rangle| && \text{by Hölder's inequality and } ||q_{T^c}||_{\infty} < 1 \\ &= |\langle q_T, h_T \rangle| && \text{by (2)} \\ &= ||h_T||_{\text{TV}} && \text{by (1)} \end{aligned}$$

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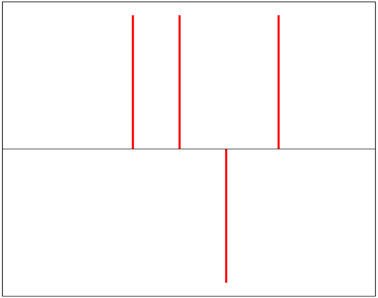
By this null-space condition and the fact that the TV norm is separable,

$$\begin{aligned} \left|\left|x'\right|\right|_{\text{TV}} &= ||x + h_T||_{\text{TV}} + ||h_{T^c}||_{\text{TV}} \geq ||x||_{\text{TV}} + ||h_{T^c}||_{\text{TV}} - ||h_T||_{\text{TV}} \\ &> ||x||_{\text{TV}} \end{aligned}$$

Construction of the certificate



1st idea : interpolation with a low-frequency fast-decaying kernel K

$$q(t) = \sum_{t_j \in T} \alpha_j K(t - t_j),$$

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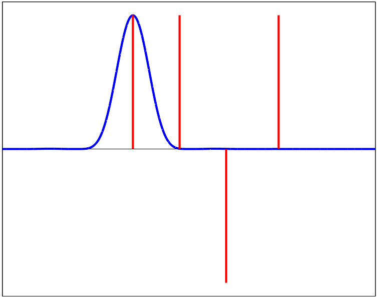

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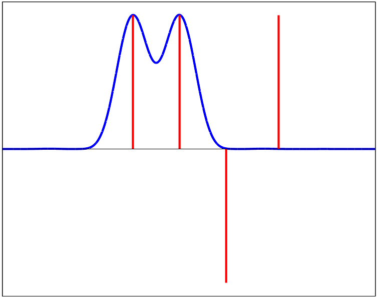

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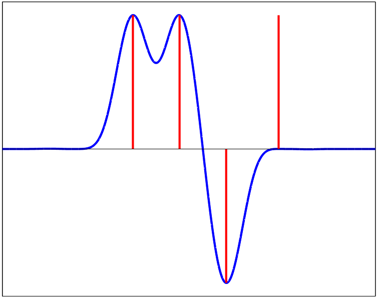

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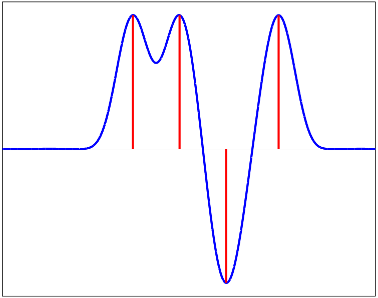

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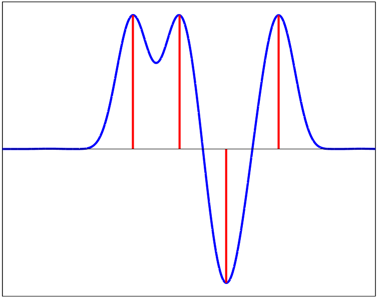

Problem : magnitude of polynomial locally exceeds 1

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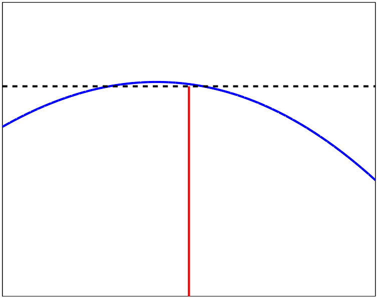

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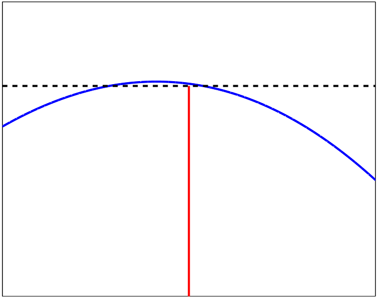

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Solution : add correction term and force $q'(t_k) = 0$ for all $t_k \in T$

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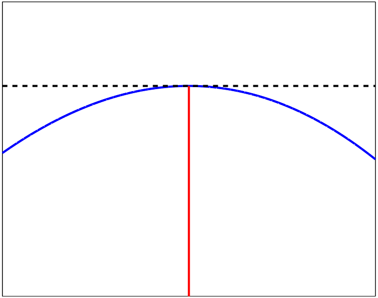

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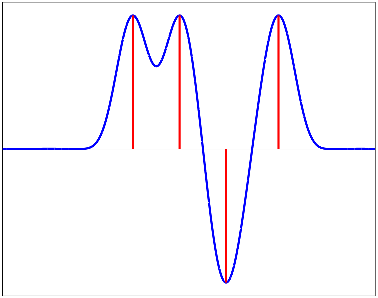

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$$||\phi_{\lambda_c} * (x_{\text{est}} - x)||_{L_1} \leq \delta$$

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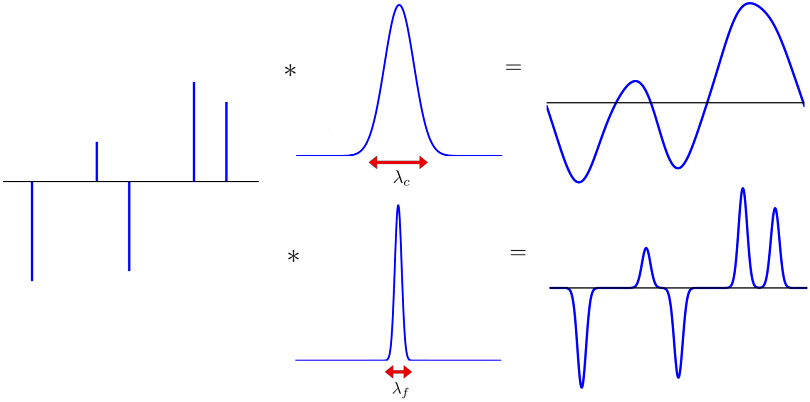
How does the estimate degrade at a higher resolution ?

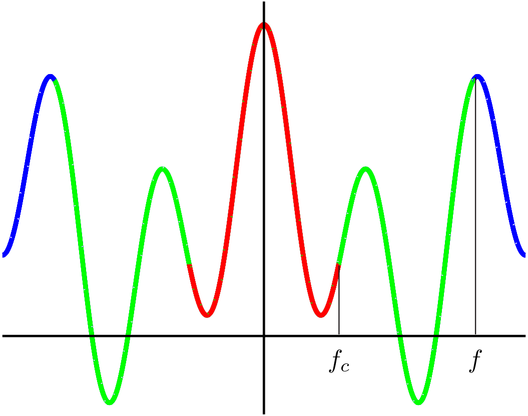
Super-resolution factor : spatial viewpoint



Super-resolution factor

$$\text{SRF} = \frac{\lambda_c}{\lambda_f}$$

Super-resolution factor : spectral viewpoint



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At a higher resolution

Theorem [Candès, F. 2012]

If $\Delta \geq 2\,/f_c$ then the solution $\hat{x}$ to

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$$\text{satisfies} \quad \left|\left|\phi_{\lambda_f} * (\hat{x} - x)\right|\right|_{L_1} \lesssim \text{SRF}^2\, \delta$$

Approximation at a higher resolution

At the resolution of the measurements

$$||\phi_{\lambda_c} * (x_{\text{est}} - x)||_{L_1} \leq \delta$$

At a higher resolution

Theorem [Candès, F. 2012]

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Another metric for stability is the accuracy of support detection :
Support detection in super-resolution. C. Fernandez-Granda. SampTA 2013

Theory

Proof (sketch)

Robustness to noise

Algorithms

Practical implementation

- **Primal problem** :

$$\min_{\tilde{x}} ||\tilde{x}||_{\mathrm{TV}} \quad \text{subject to} \quad \mathcal{F}_c\, \tilde{x} = y,$$

Infinite-dimensional variable x (measure in $[0, 1]$)

Practical implementation

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$$\max_{u \in \mathbb{C}^n} \mathrm{Re}\left[y^* u\right] \quad \text{subject to} \quad ||\mathcal{F}_c^*\, u||_\infty \leq 1,$$

$$n := 2f_c + 1$$

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- Similar for relaxed versions that account for noise

Lemma : Semidefinite representation

$$||\mathcal{F}_c^* \, u||_\infty \leq 1$$

is equivalent to

There exists a Hermitian matrix $Q \in \mathbb{C}^{n \times n}$ such that

$$\begin{bmatrix} Q & u \\ u^* & 1 \end{bmatrix} \succeq 0, \qquad \sum_{i=1}^{n-j} Q_{i,i+j} = \begin{cases} 1, & j = 0, \\ 0, & j = 1, 2, \ldots, n-1. \end{cases}$$

We can solve the dual problem, but how do we retrieve a primal solution ?

Implementation

Dual solution vector : Fourier coefficients of low-pass polynomial that interpolates the sign of the primal solution

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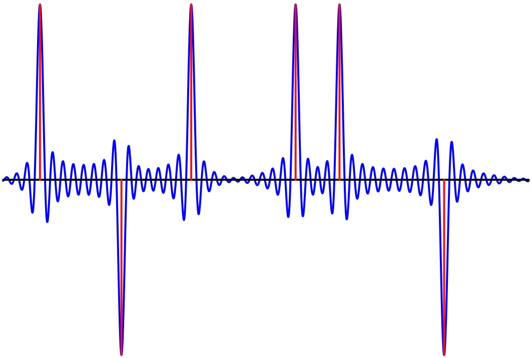

Implementation

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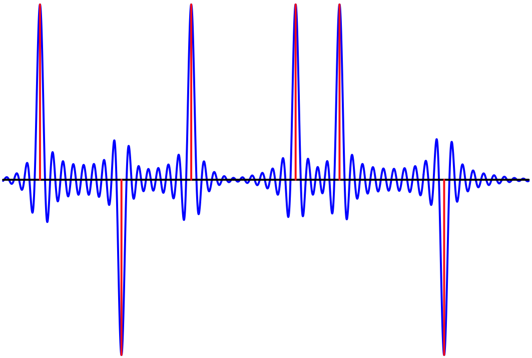

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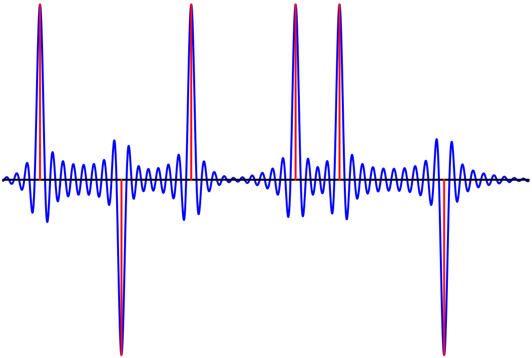

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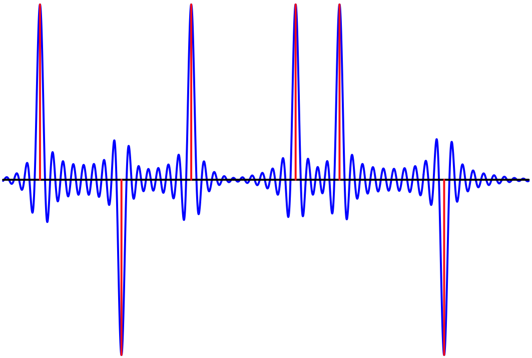
1. solve the sdp

Implementation

Dual solution vector : Fourier coefficients of low-pass polynomial that interpolates the sign of the primal solution

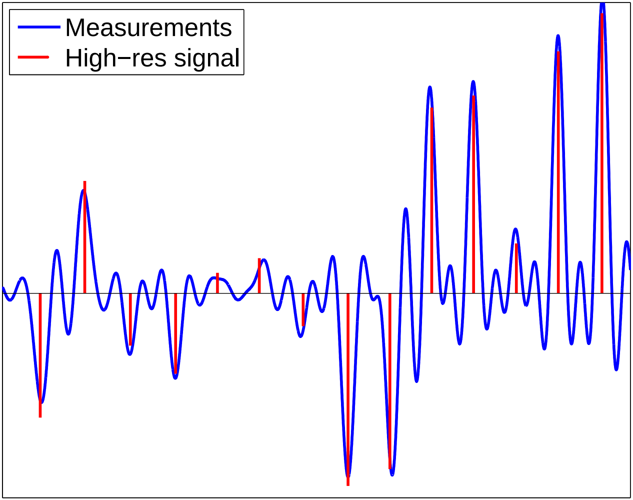


To estimate the support we

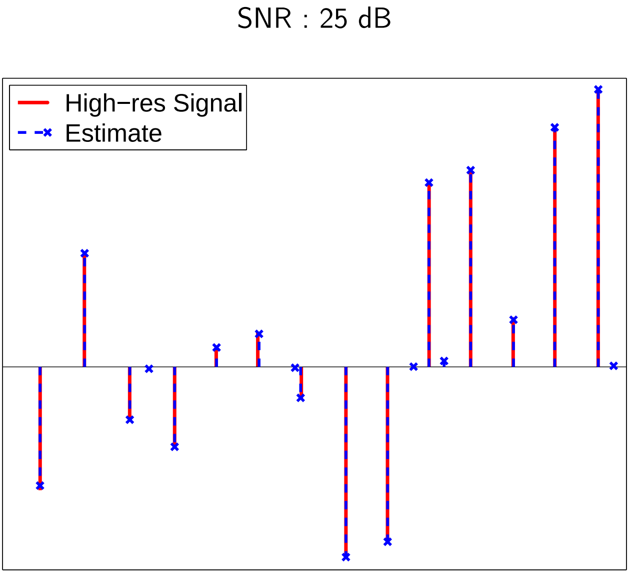
1. solve the sdp
2. determine where the magnitude of the polynomial equals 1

Example

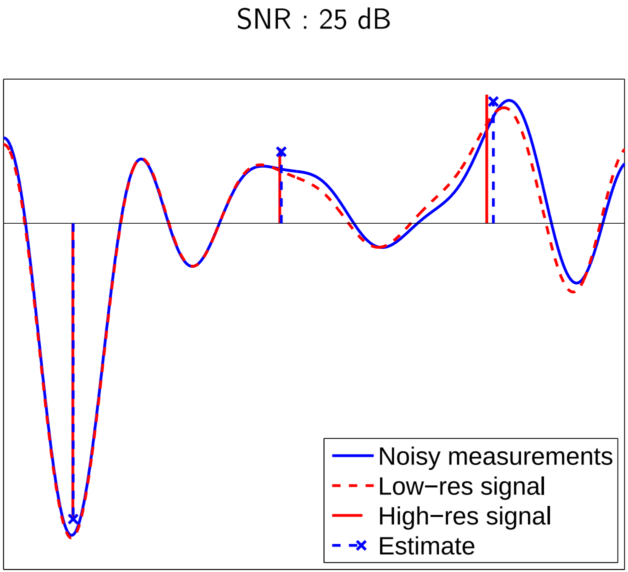
SNR : 25 dB



Example



Example


SNR : 25 dB
Noisy measurements
Low−res signal
High−res signal
Estimate

Conclusion and future work

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- To obtain theoretical guarantees for super-resolution in realistic settings, we need conditions that avoid clustered supports
- Under a minimum-separation condition, convex programming achieves exact recovery
- The method is provably robust to noise
- The optimization problem can be recast as a tractable semidefinite program
- Research directions :
  - Super-resolution of images with sharp edges
  - Developing fast solvers to solve sdp formulation

For more details

- **Towards a mathematical theory of super-resolution**. E. J. Candès and C. Fernandez-Granda. *Comm. on Pure and Applied Math.*
- **Super-resolution from noisy data**. E. J. Candès and C. Fernandez-Granda. *Journal of Fourier Analysis and Applications*
- **Support detection in super-resolution**. C. Fernandez-Granda. *Proceedings of SampTA 2013*