Super-resolution via Convex Programming

Carlos Fernandez-Granda
(Joint work with Emmanuel Candès)

Structure and Randomness in System Identification and Learning, IPAM

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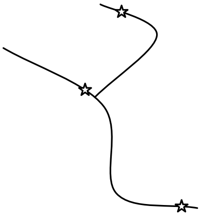
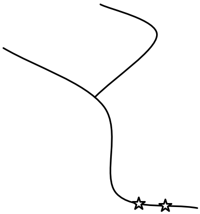
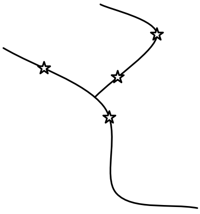
Single-molecule imaging



Frame 1

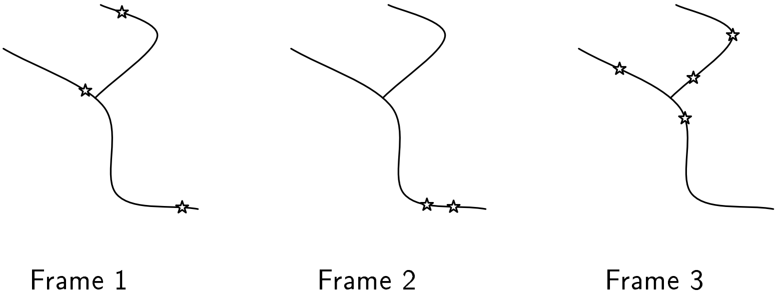


Frame 2



Frame 3

Single-molecule imaging



- Microscope receives light from fluorescent molecules

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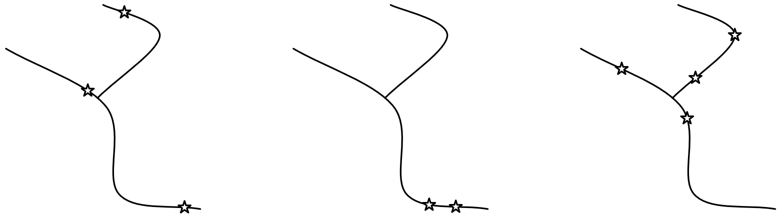

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- **Few** molecules are active in each frame $\Rightarrow$ **sparsity**

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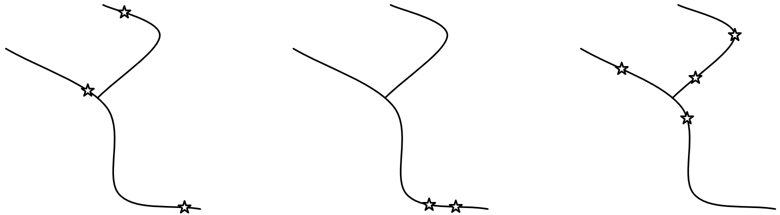

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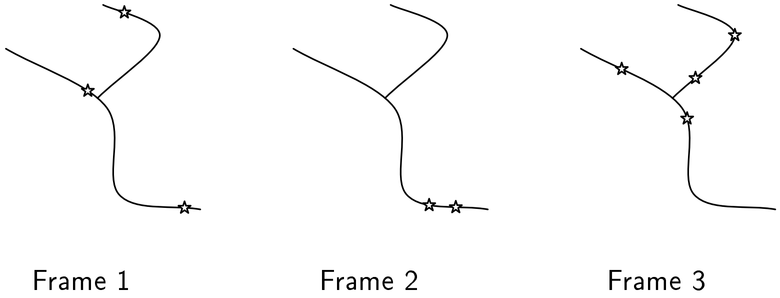

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- **Few** molecules are active in each frame $\Rightarrow$ **sparsity**
- Multiple ($\sim$ 10000) frames are recorded and processed individually
- Results from all frames are **combined** to reveal the underlying signal

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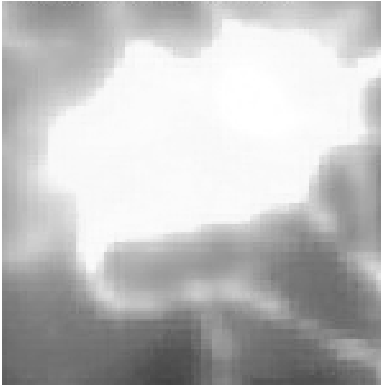

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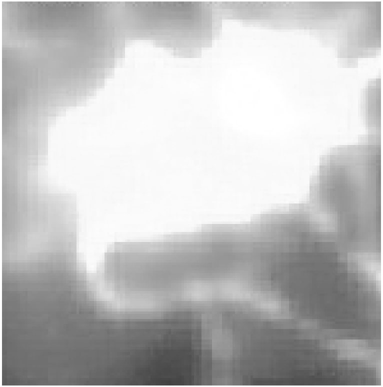

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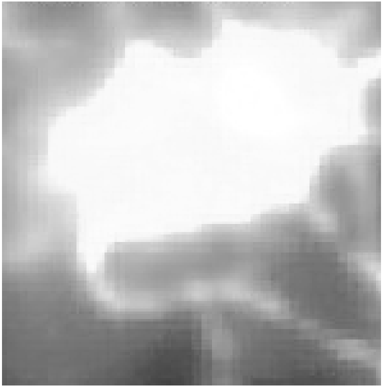

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- **Good news** : there is **structure** in the signal

Limits of resolution in imaging

In any optical imaging system **diffraction** imposes a fundamental limit on resolution

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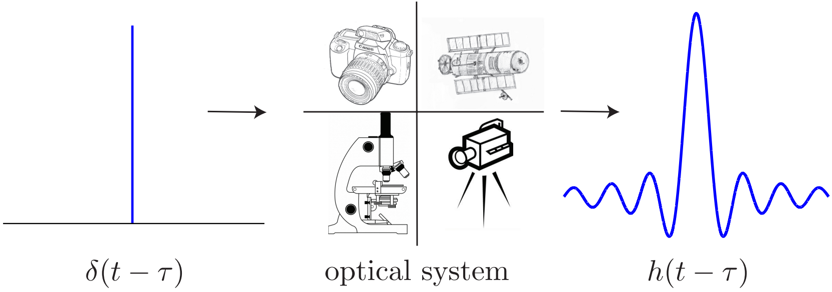
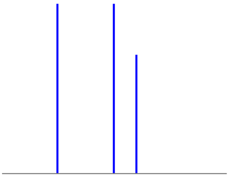
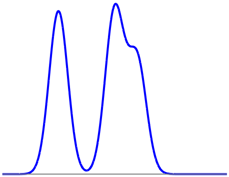


What is super-resolution ?





Retrieving fine-scale structure from low-resolution data

What is super-resolution ?

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Equivalently, extrapolating the high end of the spectrum

Super-resolution of point sources

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- Reconstruction of sub-wavelength structure in conventional imaging
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- Celestial bodies in astronomy
- Line spectra in speech analysis
- Fluorescent molecules in single-molecule microscopy

Mathematical model

- Signal : superposition of delta measures

$$x = \sum_j a_j \delta_{t_j} \qquad a_j \in \mathbb{C},\ t_j \in T \subset [0,1]$$

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Under what conditions is it possible to recover x from y ?

Equivalent problem : line spectra estimation

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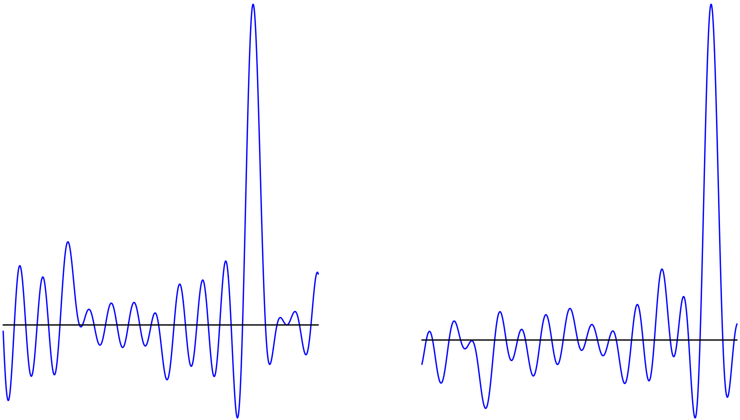
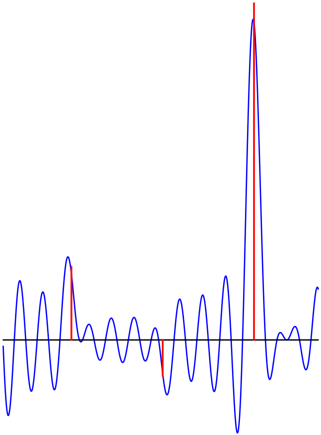
- Measurements : equispaced samples

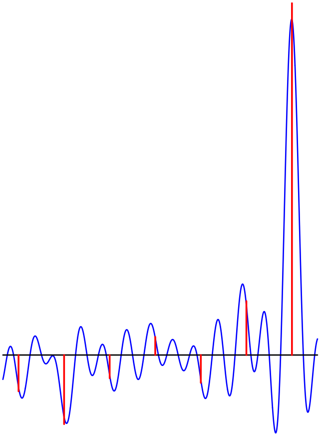
$$x(1), x(2), x(3), \ldots x(n)$$

Can you find the spikes ?



Can you find the spikes?





Compressed sensing

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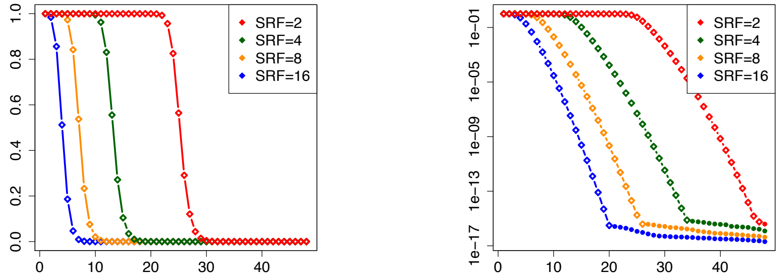
- **Crucial insight** : measurement operator is well conditioned when acting upon sparse signals (restricted isometry property)
- Is this the case in the super-resolution setting ?
- Simple experiment :
  - discretize support to $N = 4096$ points
  - restrict signal to an interval of 48 contiguous points
  - measure n DFT coefficients $\Rightarrow$ super-resolution factor (SRF) $= \frac{N}{n}$
  - how well conditioned is the inverse problem **if we know the support** ?

Simple experiment



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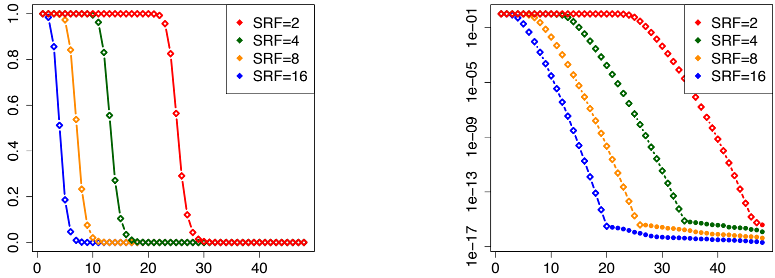

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At an SNR of 145 dB, recovery is impossible **by any method**

Prolate spheroidal sequences

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Asymptotically WT eigenvalues cluster near one while the rest are almost zero [Slepian 1978]

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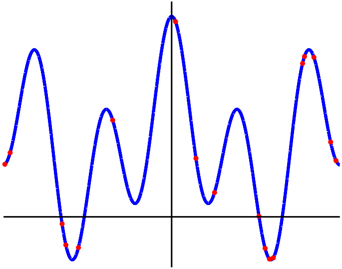
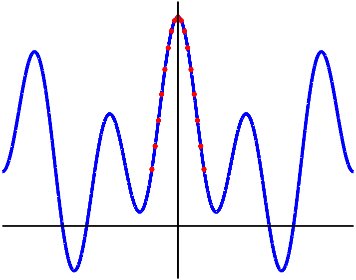
- For any interval T of length k, there is an irretrievable subspace of dimension $(1 - 1/\text{SRF})\, k$ supported on T
- When SRF > 2 many clustered sparse signals are **killed** by the measurement process

Compressed sensing vs super-resolution

What is the difference ?





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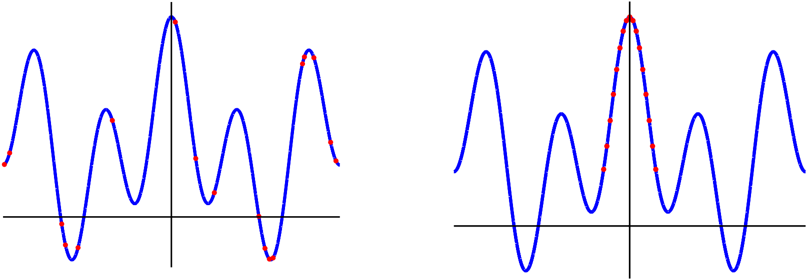

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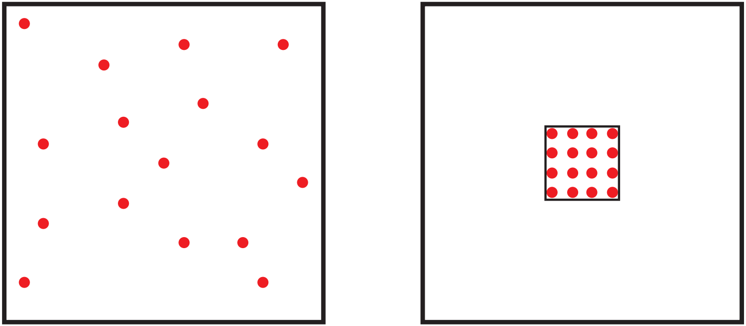

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- We can only hope to recover signals that are not too clustered
- In this work, this structural assumption is captured by introducing the **minimum separation** of the support T of a signal

$$\Delta(T) = \inf_{(t,t') \in T \,:\, t \neq t'} |t - t'|$$

Noiseless case

Recovery by solving

$$\min_{\tilde{x}} ||\tilde{x}||_{\mathrm{TV}} \quad \text{subject to} \quad \mathcal{F}_n \, \tilde{x} = y,$$

over all finite complex measures $\tilde{x}$ supported on $[0, 1]$

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- **not** the total variation of a piecewise constant function
- continuous counterpart of the ℓ_1 norm
- if $\sum_j a_j \delta_{t_j}$ then $||x||_{\mathrm{TV}} = \sum_j |a_j|$

Noiseless recovery

$$y(k) = \int_0^1 e^{-i2\pi kt} x(\mathrm{d}t) = \sum_j a_j e^{-i2\pi kt_j}, \quad k \in \mathbb{Z}, |k| \leq f_c$$

Theorem [Candès, F. 2012]

If the minimum separation of the signal obeys

$$\Delta(T) \geq 2/f_c := 2\lambda_c,$$

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- Infinite precision
- Recovers $(2\lambda_c)^{-1} = f_c/2 = n/4$ spikes from n low-frequency measurements
- If x is real, then

$$\Delta(T) \geq 2/f_c := 1.87\,\lambda_c,$$

Lower bound

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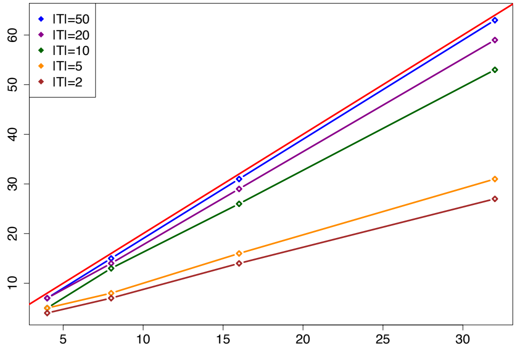

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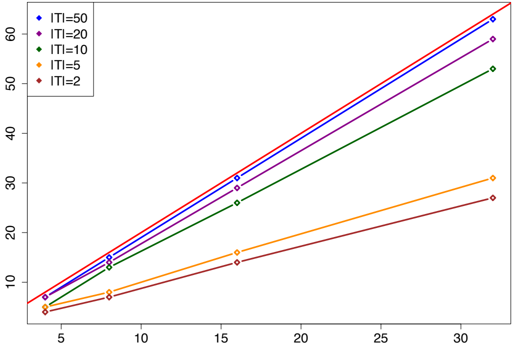

- Red line corresponds to $\Delta(T) = \lambda_c$

Higher dimensions

- Signal :

$$x = \sum_j a_j \delta_{t_j} \qquad a_j \in \mathbb{R},\ t_j \in T \subset [0,1]^2$$

- Measurements : low-pass 2D Fourier coefficients

$$y(k) = \int_{[0,1]^2} e^{-i2\pi \langle k,t \rangle} x(\mathrm{d}t) = \sum_j a_j e^{-i2\pi \langle k,t_j \rangle}, \quad k \in \mathbb{Z}^2,\ |k| \leq f_c$$

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In dimension d, $\Delta(T) \ge C_d\,\lambda_c$, where C_d only depends on d

Extensions

- Signal : $\ell - 1$ times continuously differentiable piecewise smooth function

$$x = \sum_{t_j \in T} \mathbf{1}_{(t_{j-1}, t_j)} p_j(t), \ t_j \in T \subset [0, 1]$$

where $p_j(t)$ is a polynomial of degree ℓ
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where $p_j(t)$ is a polynomial of degree ℓ
- Measurements : low-pass Fourier coefficients
$$y(k) = \int_0^1 e^{-i2\pi k t} x\,(\mathrm{d}t)\,, \quad k \in \mathbb{Z}, \, |k| \le f_c$$
- Recovery :
$$\min \|\tilde{x}^{(\ell+1)}\|_{\mathrm{TV}} \quad \text{subject to} \quad \mathcal{F}_n \tilde{x} = y$$

Corollary

TV norm minimization yields exact recovery if $\Delta(T) \ge 2\,\lambda_c$

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If we discretize the support :

- Sparse recovery problem in an overcomplete Fourier dictionary

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Additional structural assumptions allow for a more precise theoretical analysis

Sketch of the proof

Sufficient condition for exact recovery of a signal supported on T :

For any $v \in \mathbb{C}^{|T|}$ with $|v_j| = 1$, there exists a low-frequency trigonometric polynomial

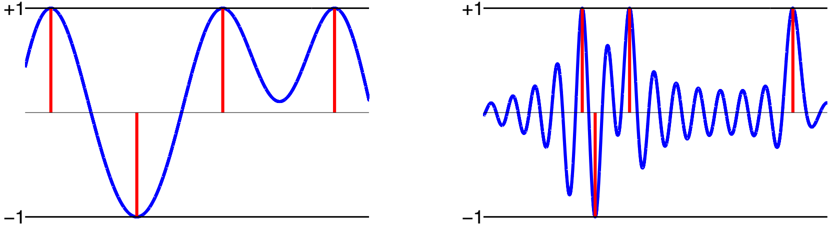
$$q(t) = \sum_{k=-f_c}^{f_c} c_k e^{i2\pi kt} \quad (1)$$

obeying

$$\begin{cases} q(t_j) = v_j, & t_j \in T, \\ |q(t)| < 1, & t \in [0,1] \setminus T. \end{cases} \quad (2)$$

Sketch of the proof

Interpolating the sign pattern with a low frequency polynomial becomes challenging if the minimum separation is small



Sketch of the proof

Interpolation with low-frequency kernel K

$$q(t) = \sum_{t_j \in T} \alpha_j K(t - t_j),$$

where α is a vector of coefficients, and q is constrained to satisfy

$$q(t_k) = \sum_{t_j \in T} \alpha_j K\left(t_k - t_j\right) = v_k, \quad \forall t_k \in T,$$

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However, this does **not** allow to construct a valid **continuous** dual polynomial

Sketch of the proof

Adding a correction term

$$q(t) = \sum_{t_j \in T} \alpha_j K(t - t_j) + \beta_j K'(t - t_j)$$

and an extra constraint

$$q(t_k) = v_k, \qquad q'(t_k) = 0, \quad \forall t_k \in T$$

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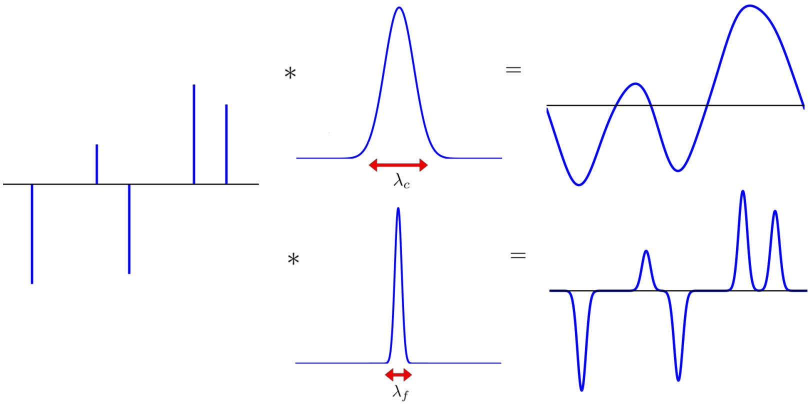
Choosing K to be the square of a Fejér kernel allows to show that

- there exist α and β satisfying the constraints
- $|q|$ is strictly bounded by one on the off-support

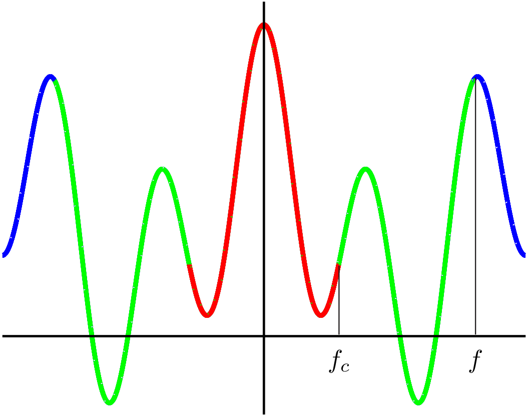
Super-resolution factor : spatial viewpoint



Super-resolution factor

$$\mathrm{SRF} = \frac{\lambda_c}{\lambda_f}$$

Super-resolution factor : spectral viewpoint



Super-resolution factor

$$\text{SRF} = \frac{f}{f_c}$$

Noise model

$$\mathcal{F}_n\, x(k) = \int_0^1 e^{-i2\pi kt} x\,(\mathrm{d}t)\,, \quad |k| \leq f_c$$

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Recovery algorithm

$$\min_{\tilde{x}} \; ||\tilde{x}||_{\mathrm{TV}} \quad \text{subject to} \quad ||\mathcal{P}_n \tilde{x} - s||_{L_1} \leq \delta$$

Robust recovery

Let ϕ_{λ_c} be a kernel with width λ_c and cut-off frequency f_c

If $||z||_{L_1} \leq \delta$, then

$$||\phi_{\lambda_c} * (x_{\text{est}} - x)||_{L_1} \approx ||z||_{L_1} \leq \delta,$$

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Theorem [Candès, F. 2012]

If $\Delta(T) \geq 2/f_c$ then the solution x_{est} to the TV-norm minimization problem satisfies

$$\left|\left|\phi_{\lambda_f} * (x_{\text{est}} - x)\right|\right|_{L_1} \lesssim \text{SRF}^2\,\delta,$$

Practical implementation

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- Dual :

$$\max_{u \in \mathbb{C}^n} \mathrm{Re}\left[y^* u\right] \quad \text{subject to} \quad ||\mathcal{F}_n^* u||_\infty \leq 1,$$

Second option : Recast as semidefinite program

Semidefinite representation

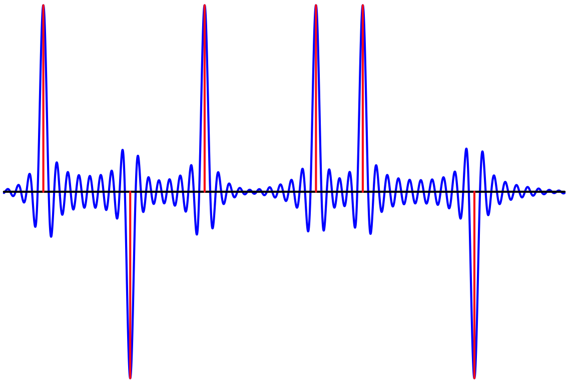
$$||\mathcal{F}_n^* \, u||_\infty \leq 1$$

is equivalent to

There exists a Hermitian matrix $Q \in \mathbb{C}^{n \times n}$ such that

$$\begin{bmatrix} Q & u \\ u^* & 1 \end{bmatrix} \succeq 0, \qquad \sum_{i=1}^{n-j} Q_{i,i+j} = \begin{cases} 1, & j = 0, \\ 0, & j = 1, 2, \ldots, n-1. \end{cases}$$

Support detection



By strong duality, $\mathcal{F}_n^* \, \hat{u}$ interpolates the sign of $\hat{x}$

Experiment

Support recovery by solving the SDP

f_c	25	50	75	100
Average error	$6.66\,10^{-9}$	$1.70\,10^{-9}$	$5.58\,10^{-10}$	$2.96\,10^{-10}$
Maximum error	$1.83\,10^{-7}$	$8.14\,10^{-8}$	$2.55\,10^{-8}$	$2.31\,10^{-8}$

For each f_c, 100 random signals with $|T| = f_c/4$ and $\Delta(T) \geq 2/f_c$

Related work

- Super-resolution of spike trains by ℓ_1 norm minimization in seismic prospecting [Claerbout, Muir 1973 ; Levy, Fullagar 1981 ; Santosa, Symes 1986]

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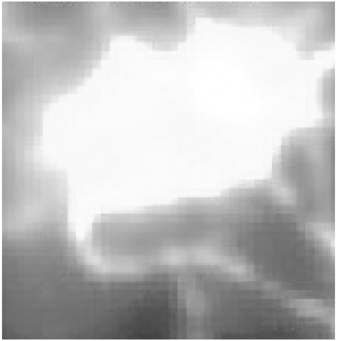
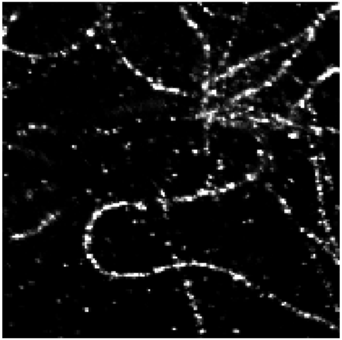
- Random undersampling of low-frequency coefficients [Tang et al 2012]

Conclusion





Stable super-resolution is possible via tractable non-parametric methods based on convex optimization

Note on single-molecule imaging : Joint work with E. Candès, V. Morgenshtern and the Moerner lab at Stanford

Thank you