Towards a Mathematical Theory of Super-resolution

Carlos Fernandez-Granda

www.stanford.edu/~cfgranda/

Information Theory Forum, Information Systems Laboratory, Stanford

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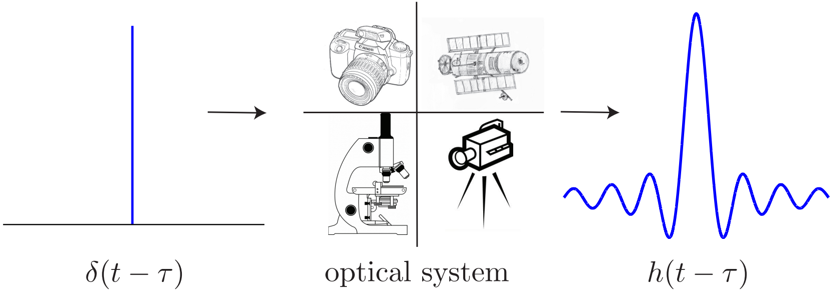
Acknowledgements

- This work was supported by a Fundación La Caixa Fellowship and a Fundación Caja Madrid Fellowship
- **Collaborator** : Emmanuel Candès (Department of Mathematics and of Statistics, Stanford)

Motivation : Limits of resolution in imaging

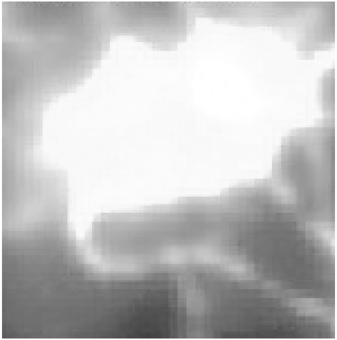
The resolving power of lenses, however perfect, is limited (**Lord Rayleigh**)

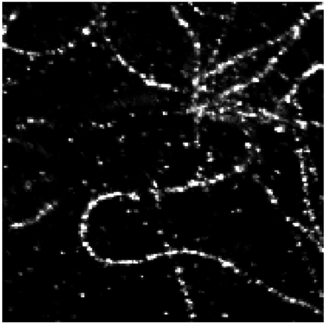


Diffraction imposes a fundamental limit on the resolution of optical systems

Motivation

Similar problems arise in electronic imaging, signal processing, radar, spectroscopy, medical imaging, astronomy, geophysics, etc.





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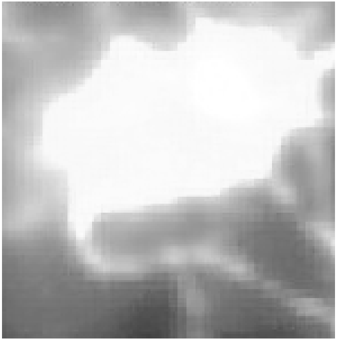
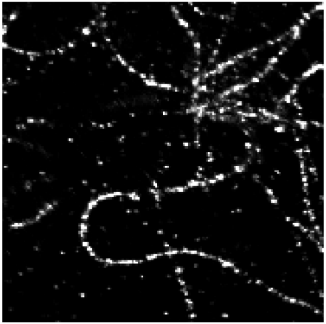
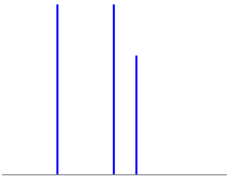
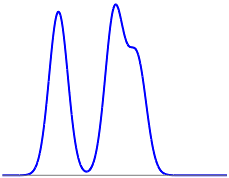




Signals of interest are often point sources : celestial bodies (astronomy), line spectra (signal processing), molecules (fluorescence microscopy), ...

Super-resolution

Aim : estimating fine-scale structure from low-resolution data

Super-resolution

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Equivalently, extrapolating the high end of the spectrum

Mathematical model

- Signal : superposition of Dirac measures with support T

$$x = \sum_j a_j \delta_{t_j} \qquad a_j \in \mathbb{C},\ t_j \in T \subset [0,1]$$

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$$x = \sum_j a_j \delta_{t_j} \qquad a_j \in \mathbb{C},\ t_j \in T \subset [0,1]$$

- Measurements : low-pass filtering with cut-off frequency f_c

$$y = \mathcal{F}_c\, x \quad \text{(vector of low-pass Fourier coefficients)}$$

$$y(k) = \int_0^1 e^{-i2\pi kt} x\,(\mathrm{d}t) = \sum_j a_j e^{-i2\pi k t_j}, \quad k \in \mathbb{Z},\ |k| \le f_c$$

Equivalent problem : line-spectra estimation

Swapping time and frequency

- Signal : superposition of sinusoids

$$x(t) = \sum_j a_j e^{i2\pi f_j t} \qquad a_j \in \mathbb{C},\ f_j \in T \subset [0,1]$$

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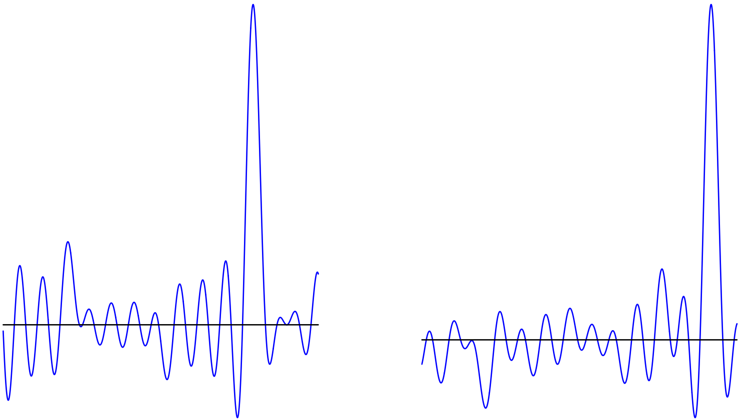
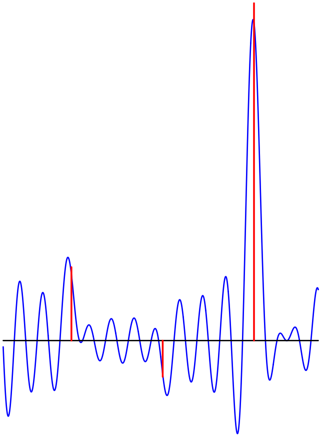
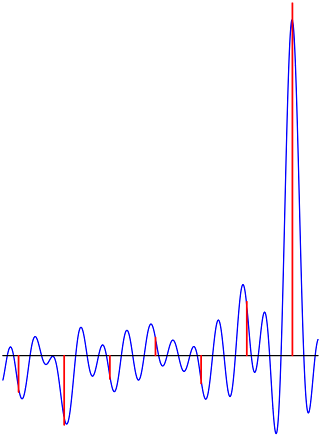
- Classical problem in signal processing

Can you find the spikes?



Can you find the spikes?





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- Non-parametric estimation

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- Non-parametric estimation
- Provably stable in the presence of noise
- Flexible variational framework based on convex programming

Outline of the talk

Sparsity is not enough

Theory

Proof (sketch)

Implementation via semidefinite programming

Robustness to noise

Sparsity is not enough

Theory

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Implementation via semidefinite programming

Robustness to noise

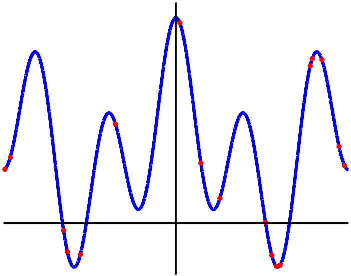
Compressed sensing vs super-resolution

Estimation of sparse signals from undersampled measurements suggests connections to compressed sensing

Compressed sensing vs super-resolution

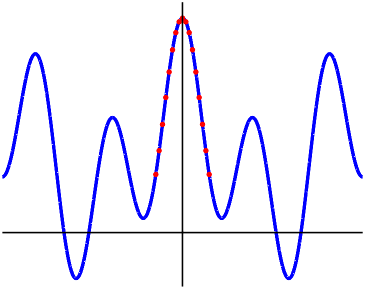
Estimation of sparse signals from undersampled measurements suggests connections to compressed sensing

Compressed sensing



spectrum interpolation

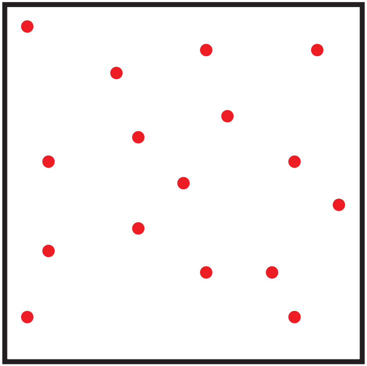
Super-resolution



spectrum extrapolation

Compressed sensing vs super-resolution

Compressed sensing



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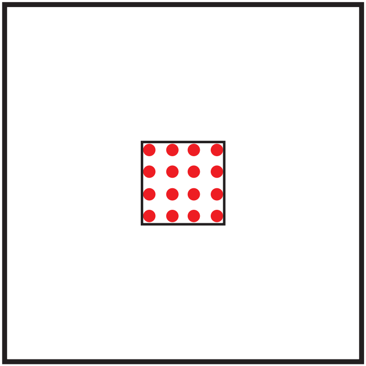

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Compressed sensing

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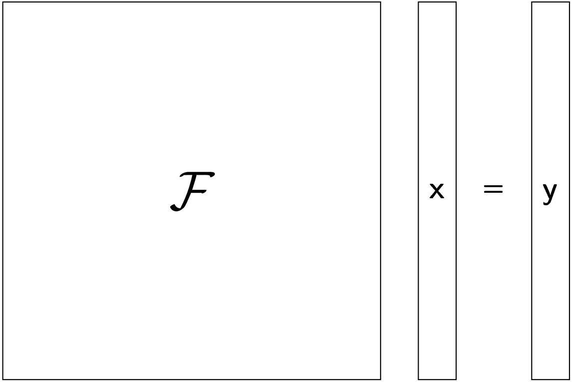
Compressed sensing

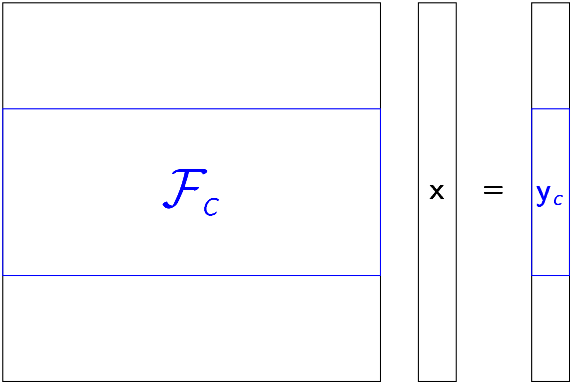
- Compressed sensing theory establishes robust recovery of spikes from random Fourier measurements [Candès, Romberg & Tao 2004]
- **Crucial insight** : measurement operator is well conditioned when acting upon sparse signals
- Equivalently, the energy of all sparse signals is preserved by the randomized measurements (restricted isometry property)
- This is a necessary condition for stable estimation, but is it the case in super-resolution ?

Simple experiment



Discretize support to lie on a grid with $N = 4096$ points

Simple experiment



Measure n low-pass DFT coefficients, super-resolution factor (SRF) : N/n

Simple experiment

$$\mathcal{F}_c \, \mathbf{x} = \mathbf{y}_c$$

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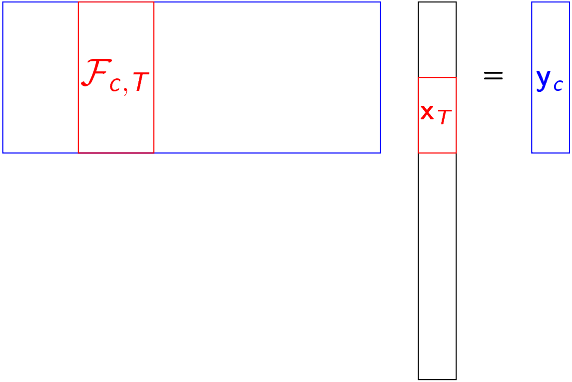


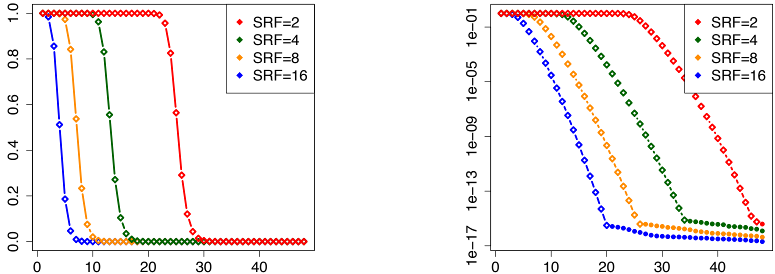
Restrict support of the signal to an interval of 48 contiguous points

Simple experiment

$$\mathcal{F}_{c,T} \, \mathbf{x}_T = \mathbf{y}_c$$

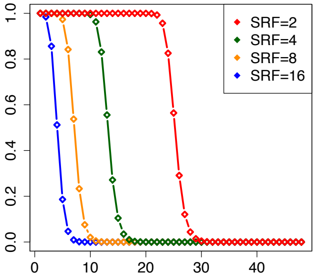
Compute singular values of resulting linear operator

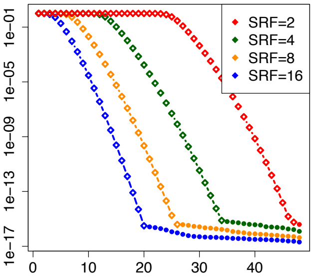
Simple experiment



For SRF $= 4$ measuring any unit-normed vector in a subspace of dimension 24 results in a signal with norm less than 10^{-7}

Simple experiment





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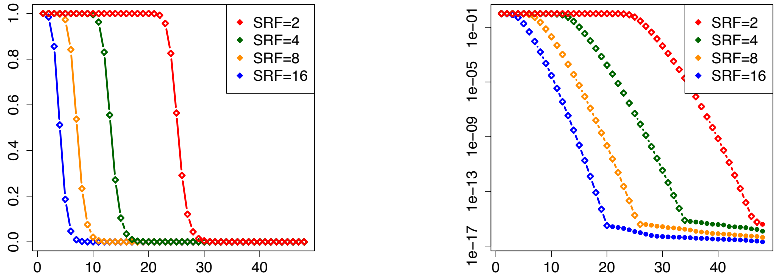

For SRF $= 4$ measuring any unit-normed vector in a subspace of dimension 24 results in a signal with norm less than 10^{-7}

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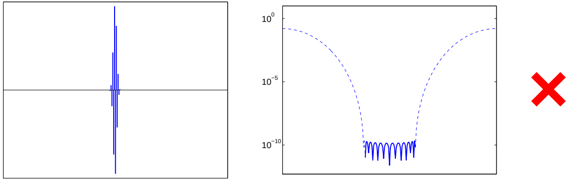
This phenomenon is characterized asymptotically by Slepian's seminal work on prolate spheroidal sequences

Conclusion

Sparsity is not enough

Signal　　　Spectrum



Conclusion

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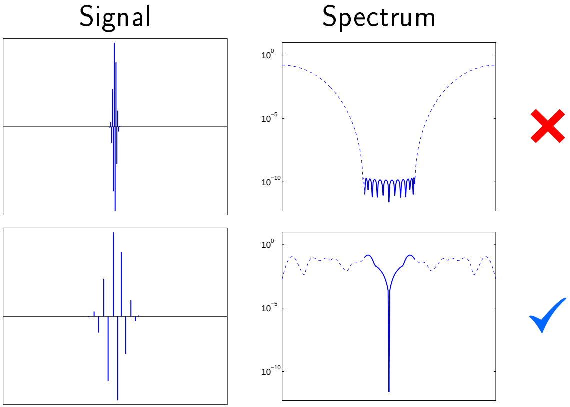


Additional conditions are necessary to restrict our signal model

Sparsity is not enough

Theory

Proof (sketch)

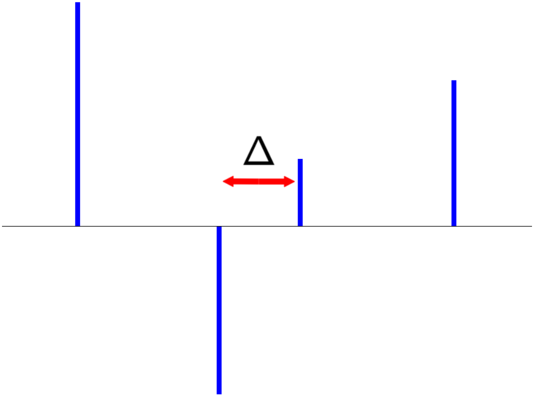
Implementation via semidefinite programming

Robustness to noise

Minimum separation

To exclude highly-clustered signals from our model, we control the minimum separation Δ of the support T

$$\Delta = \inf_{(t,t') \in T \,:\, t \neq t'} |t - t'|$$



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- Not the total variation of a piecewise-constant function
- **Formal definition :** For a complex measure ν

$$||\nu||_{\text{TV}} = \sup \sum_{j=1}^{\infty} |\nu(B_j)|,$$

(supremum over all finite partitions B_j of $[0,1]$)

Recovery via convex programming

In the absence of noise, i.e. if $y = \mathcal{F}_c\, x$, we solve

$$\min_{\tilde{x}} ||\tilde{x}||_{\mathrm{TV}} \quad \text{subject to} \quad \mathcal{F}_c\, \tilde{x} = y,$$

over all finite complex measures $\tilde{x}$ supported on $[0, 1]$

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Theorem [Candès, F. 2012]

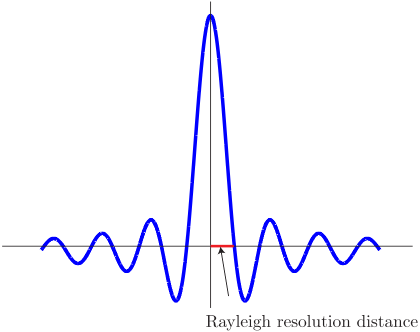
If the minimum separation of the signal support T obeys

$$\Delta \geq 2\,/f_c := 2\,\lambda_c,$$

then recovery is exact

Minimum-distance condition

- $\lambda_c/2$ is the Rayleigh resolution limit (half-width of measurement filter)



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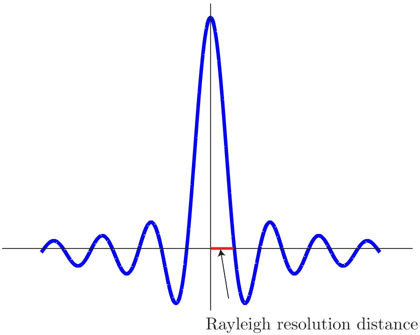

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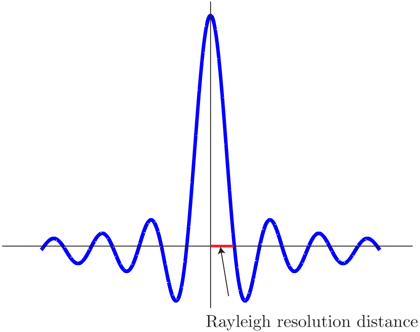

- Numerical simulations show that TV-norm minimization fails if $\Delta < \lambda_c$
- If $\Delta < \lambda_c/2$ some signals are *almost* in the nullspace of the measurement operator (no method can achieve stable estimation)

Sparse recovery in overcomplete dictionaries

If we discretize the support

- Sparse recovery via ℓ_1-norm minimization in an overcomplete Fourier dictionary

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- If $N = 20000$ and $n = 1000$, how many spikes can we recover ?
  Previous theory [Dossal 2005] : **3 spikes**
  Our result : $n/4 =$ **250 spikes**

Higher dimensions

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- Measurements : low-pass 2D Fourier coefficients

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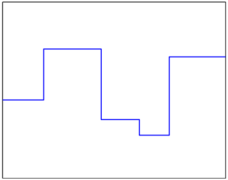
In dimension d, $\Delta \geq C_d \, \lambda_c$, where C_d only depends on d

Extensions

- Signal : piecewise-constant function
- Measurements : low-pass Fourier coefficients



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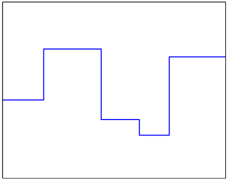

Corollary

$$\text{Solving } \min \|\tilde{x}^{(1)}\|_{\text{TV}} \quad \text{subject to} \quad \mathcal{F}_c \tilde{x} = y$$

yields exact recovery if $\Delta \geq 2\,\lambda_c$

Similar result for cont. differentiable piecewise-smooth functions

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Optimality conditions

Consider the problem

$$\min_{\tilde{x}} f(\tilde{x}) \quad \text{subject to} \quad A\,\tilde{x} = y,$$

where f is convex

Lemma

If there exists a subgradient $g(x)$ of f at a feasible point x such that $g(x) = A^* v$ for some v, then x is a solution

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For any h such that $Ah = 0$

$$\begin{aligned} f(x+h) &\geq f(x) + \langle g(x), h \rangle \quad \text{by definition of subgradient and convexity of } f \\ &= f(x) + \langle A^* v, h \rangle \\ &= f(x) + \langle v, Ah \rangle \\ &= f(x) \end{aligned}$$

Certificate of optimality

q is a subgradient of the total-variation norm at $x = \sum_{j \in T} |a_j| \, e^{i\phi_j} \delta_{t_j}$ if

$$\begin{cases} q(t_j) = e^{i\phi_j}, & t_j \in T \\ |q(t)| \leq 1, & t \in [0,1] \setminus T \end{cases} \tag{1}$$

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If (1) is strengthened to

$$|q(t)| < 1, \quad t \in [0,1] \setminus T$$

then x is the unique solution

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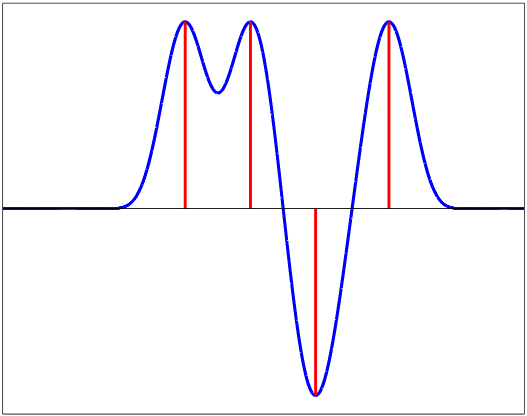
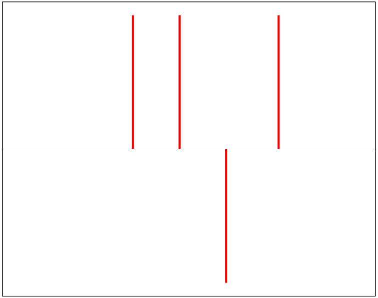


Construction of the certificate



1st idea : interpolation with a low-frequency fast-decaying kernel K

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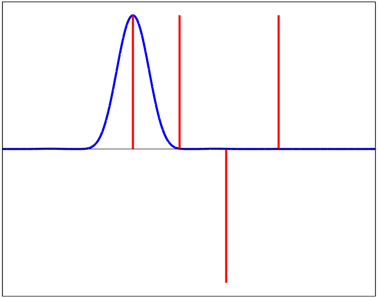

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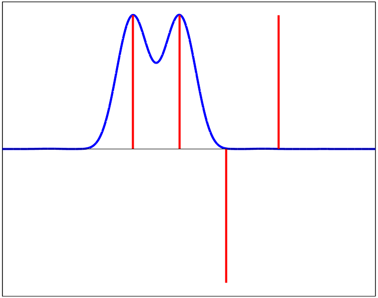

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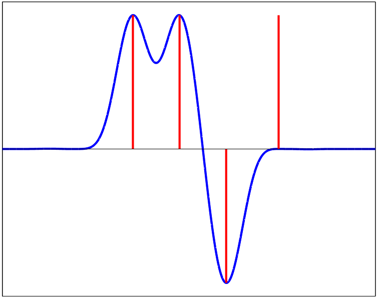

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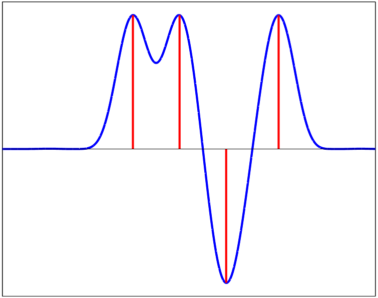

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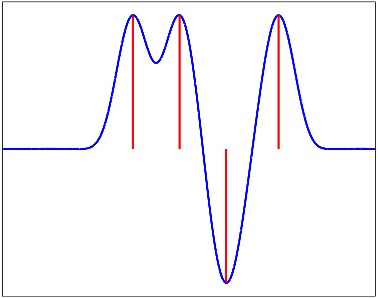

Problem : magnitude of polynomial locally exceeds 1

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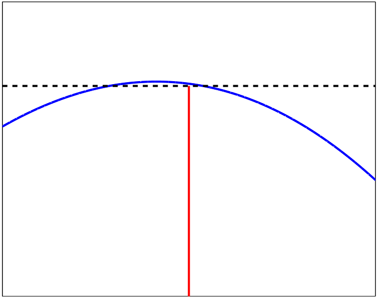

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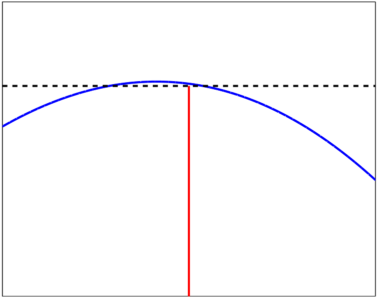

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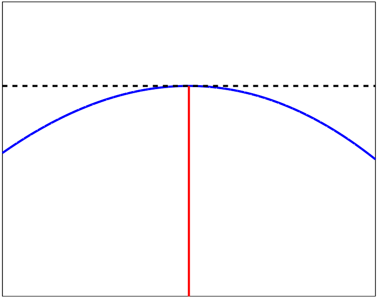

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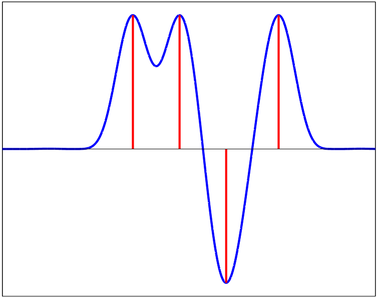

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Sparsity is not enough

Theory

Proof (sketch)

Implementation via semidefinite programming

Robustness to noise

Practical implementation

- **Primal problem** :

$$\min_{\tilde{x}} ||\tilde{x}||_{\mathrm{TV}} \quad \text{subject to} \quad \mathcal{F}_c\, \tilde{x} = y,$$

Infinite-dimensional variable x (measure in $[0, 1]$)

First option : Discretize domain and apply ℓ_1-norm minimization

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- **Dual problem** :

$$\max_{u \in \mathbb{C}^{n}} \operatorname{Re}\left[y^* u\right] \quad \text{subject to} \quad ||\mathcal{F}_c^*\, u||_\infty \leq 1, \quad n := 2f_c + 1$$

Finite-dimensional variable u, but infinite-dimensional constraint

$$\mathcal{F}_c^*\, u = \sum_{k \leq |f_c|} u_k\, e^{i2\pi kt}$$

Second option : Recast dual problem as semidefinite program

Lemma : Semidefinite representation

The Fejér-Riesz Theorem and the semidefinite representation of polynomial sums of squares imply that

$$||\mathcal{F}_c^* u||_\infty \leq 1$$

is equivalent to

There exists a Hermitian matrix $Q \in \mathbb{C}^{n \times n}$ such that

$$\begin{bmatrix} Q & u \\ u^* & 1 \end{bmatrix} \succeq 0, \qquad \sum_{i=1}^{n-j} Q_{i,i+j} = \begin{cases} 1, & j = 0, \\ 0, & j = 1, 2, \ldots, n-1. \end{cases}$$

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We can solve the dual problem, but how do we retrieve a primal solution ?

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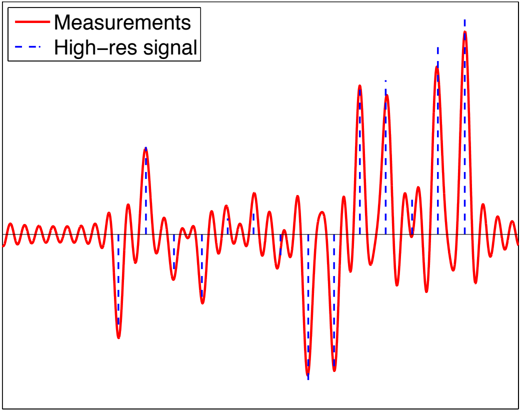
Idea : Use the polynomial to locate the support of the signal and then estimate the amplitudes by least squares

Super-resolution via semidefinite programming



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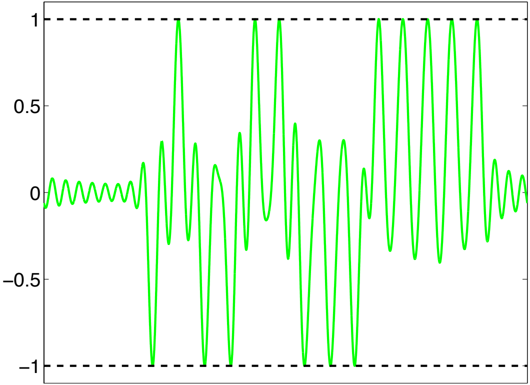
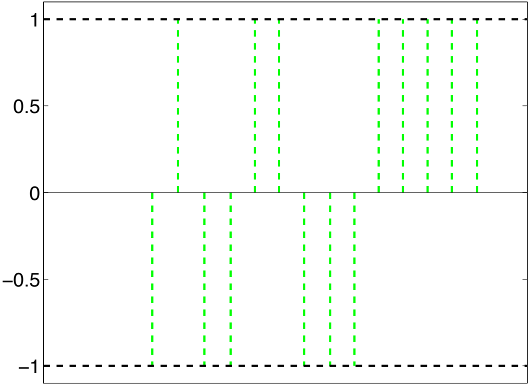


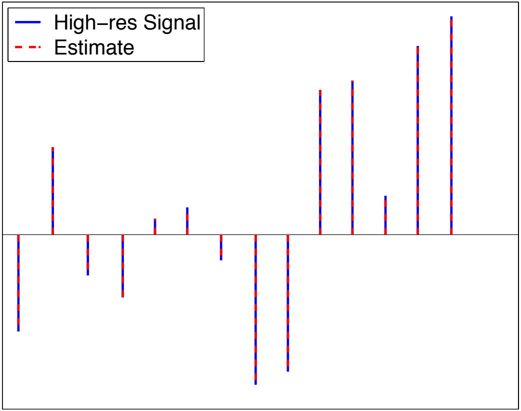
1. Solve semidefinite program to obtain dual solution

Super-resolution via semidefinite programming



2. Locate points at which corresponding polynomial has unit magnitude

Super-resolution via semidefinite programming



3. Estimate amplitudes via least squares

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Estimation from noisy data

- Without noise, we achieve perfect precision, i.e. infinite resolution

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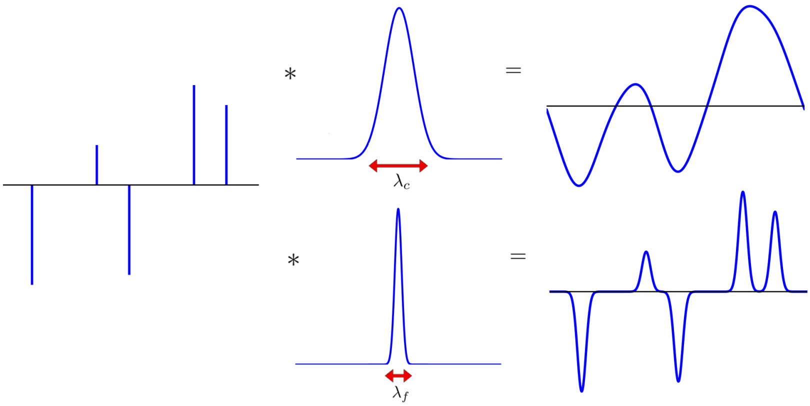
  Understanding the performance in a noisy setting is crucial for applications
- Metrics :
  1. Approximation error at a higher resolution
  2. Support-detection error

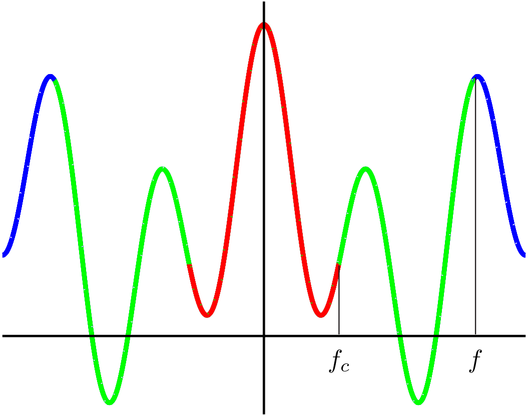
Super-resolution factor : spatial viewpoint



Super-resolution factor

$$\mathrm{SRF} = \frac{\lambda_c}{\lambda_f}$$

Super-resolution factor : spectral viewpoint



Super-resolution factor

$$\mathsf{SRF} = \frac{f}{f_c}$$

Approximation at a higher resolution

Resolution at scale λ is quantified by convolution with kernel ϕ_λ of width λ

At the resolution of the measurements

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How does the estimate degrade at a higher resolution ?

Theorem [Candès, F. 2012]

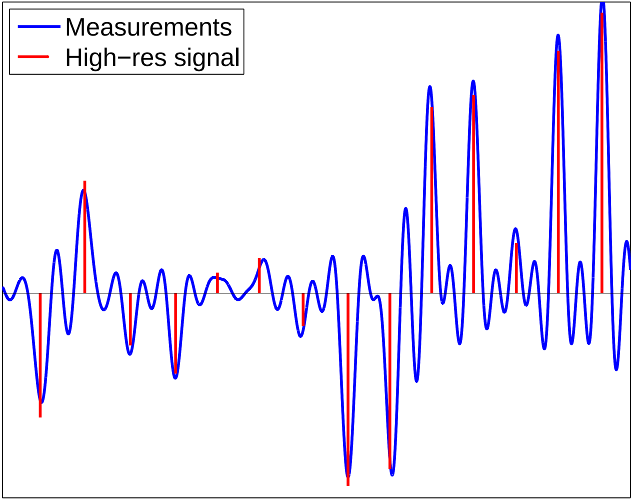
If $\Delta \geq 2/f_c$ then the solution $\hat{x}$ to

$$\min_{\tilde{x}} ||\tilde{x}||_{\text{TV}} \quad \text{subject to} \quad ||\mathcal{F}_c\, \tilde{x} - y||_2 \leq \delta,$$

$$\text{satisfies} \quad \left|\left|\phi_{\lambda_f} * (\hat{x} - x)\right|\right|_{L_1} \lesssim \text{SRF}^2\, \delta$$

Example

SNR : 25 dB



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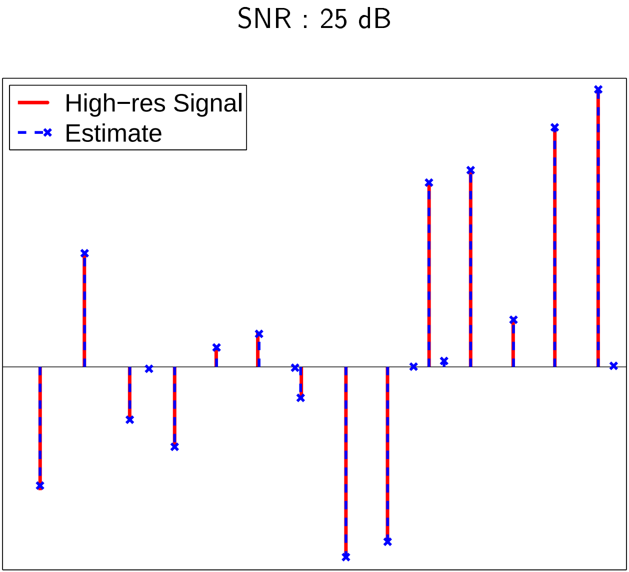

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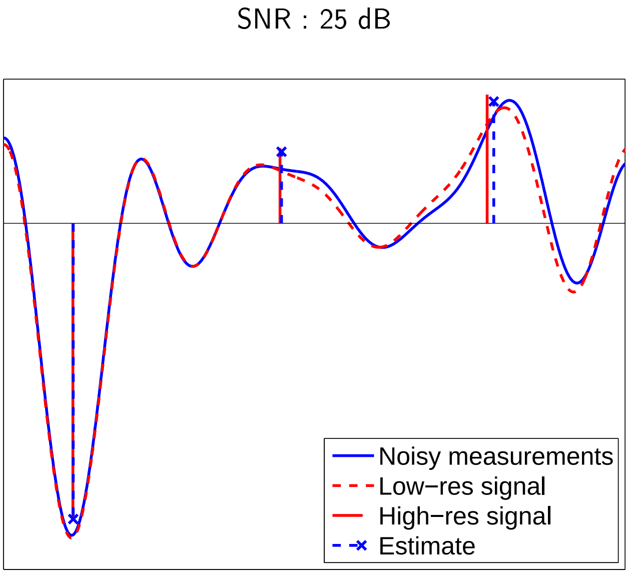
SNR : 25 dB
Noisy measurements
Low−res signal
High−res signal
Estimate

Support detection

- Original signal, support T

$$x = \sum_j a_j \delta_{t_j} \qquad a_j \in \mathbb{C},\ t_j \in T \subset [0,1]$$

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How accurately can we detect the support at a certain noise level δ?

Support-detection accuracy

Theorem [F. 2013]

For any $t_i \in T$, if $|a_i| > C_1\delta$ there exists $\hat{t}_i \in \widehat{T}$ such that

$$\left|t_i - \hat{t}_i\right| \leq \frac{1}{f_c}\sqrt{\frac{C_2\delta}{|a_i| - C_1\delta}}$$

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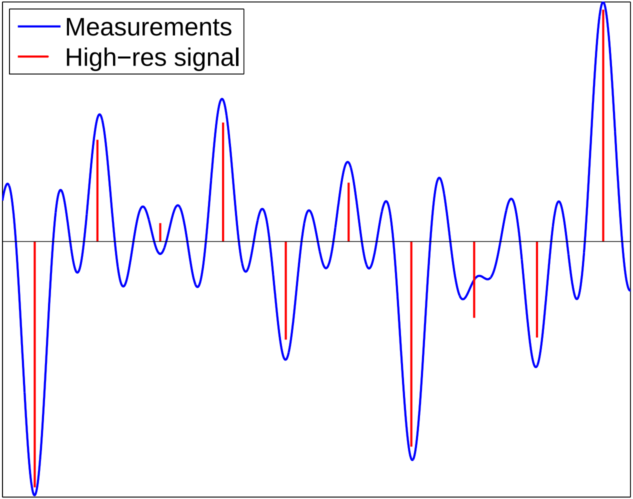
The support-detection accuracy is not affected by aliasing
(no dependence on the amplitude of the signal at other locations)

Consequence

Robustness of the algorithm to high dynamic ranges



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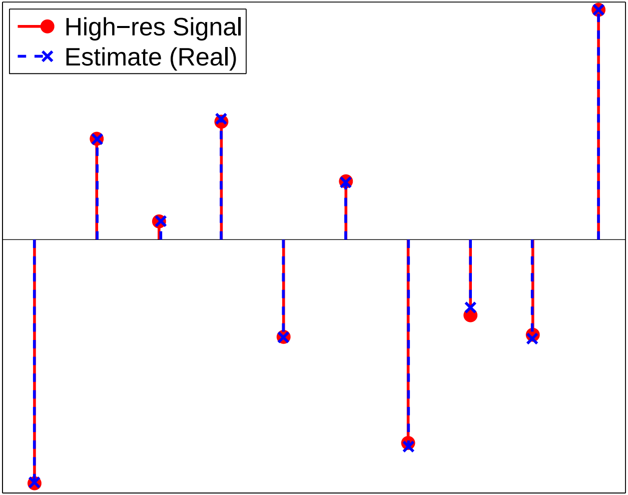

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- Research directions :
  - Super-resolution of images with sharp edges
  - Developing fast solvers to solve sdp formulation
  - Extending results to other overcomplete dictionaries

For more details

- **Towards a mathematical theory of super-resolution**. E. J. Candès and C. Fernandez-Granda. *Comm. on Pure and Applied Math.*
- **Super-resolution from noisy data**. E. J. Candès and C. Fernandez-Granda. *Journal of Fourier Analysis and Applications*
- **Support detection in super-resolution**. C. Fernandez-Granda. *Proceedings of SampTA 2013*
- **Prolate spheroidal wave functions, Fourier analysis, and uncertainty V - The discrete case**. D. Slepian. *Bell System Technical Journal*, 57 :1371-1430, 1978