Complex Analysis Oral Exam Notes 2008

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1 Analytic functions

1.1 Definitions

analytic function, conformal (angle-preserving), conformal (scaling), conjugate-harmonic, Cauchy-Riemann equations, analytic antiderivative

1.2 Useful Theorems

Theorem 1. (Alternative definitions of analytic function)
Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be C^1. Then:

1. f has a complex derivative $f'(z)$ $\forall z \in U$ $\Leftrightarrow$
2. f satisfies the CR equations $\Leftrightarrow$
3. f is conformal in the angle-preserving sense, i.e. if $\gamma : [0,1] \to \mathbb{C}$ is a smooth curve carried to the image curve Γ under f, then $\forall t \in (0,1)$, $\arg Tgnt_\Gamma(f(\gamma(t))) - \arg Tgnt_\gamma(\gamma(t))$ is independent of γ. $\Leftrightarrow$
4. f is conformal in the scaling sense (and does not reverse angles), i.e. $\lim_{z \to z_0} \dfrac{|f(z) - f(z_0)|}{|z - z_0|}$ exists and is independent of the direction in which $z \to z_0$. $\Leftrightarrow$
5. $\frac{d}{d\bar{z}} f = 0$ $\forall z \in U$

Depends on:

Proof idea:

1. (1) $\Rightarrow$ (2) by writing $f = u + iv$ by evaluating $\lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ where $z \to z_0$ in the horizontal and vertical directions and equating them.
2. (2) $\Rightarrow$ (1) by taking a 1st-order approximation of $u(x,y)$ and $v(x,y)$ and plugging into the complex derivative formula and applying CR. Evaluating the limit gives $f'(z) = u_x + iv_x$.
3. (1) $\Rightarrow$ (3): Clearly $\arg Tgnt_\Gamma(f(\gamma(t))) - \arg Tgnt_\gamma(\gamma(t)) = \arg f'(\gamma(t))\gamma'(t) - \arg \gamma'(t) = arg f'(\gamma(t))$ which is independent of the curve γ.

4. (1) $\Rightarrow$ (4): Obvious since the limit in question is $|f'(z_0)|$.

5. (3) $\Rightarrow$ (5): Apply the chain rule: $\frac{d}{dt}f(\gamma(t_0)) = \frac{df}{dz}\gamma'(t_0) + \frac{df}{d\bar{z}}\overline{\gamma'(t_0)}$ But since arg $(\frac{\frac{d}{dt}f(\gamma(t_0))}{\gamma'(t_0)})$ is independent of $\gamma'(t_0)$, dividing by $\gamma'(t_0) \Rightarrow \frac{df}{d\bar{z}} + \frac{df}{d\bar{z}}\frac{overline\gamma'(t_0)}{\gamma'(t_0)}$ has a constant argument w.r.t. $\gamma'(t_0)$. But this is just a circle with center $\frac{df}{d\bar{z}}$ and radius $\frac{df}{d\bar{z}}$ so the radius must be constant, thus yielding the CR equations.

6. (4) $\Rightarrow$ (5): Using the same equation as above, the scaling property implies that the modulus is constant along the circle which means either $\frac{df}{dz} = 0$ or $\frac{df}{d\bar{z}} = 0$ (i.e. angles are reversed).

7. (5) $\Rightarrow$ (2): Take the transformation $x \leftarrow \frac{z+\bar{z}}{2}$ and $y \leftarrow \frac{i(z-\bar{z})}{2}$ which gives $\frac{df}{dz} = \frac{1}{2}(\frac{d}{dx} - i\frac{d}{dy})f$ and $\frac{df}{d\bar{z}} = \frac{1}{2}(\frac{d}{dx} + i\frac{d}{dy})f$. So $\frac{df}{d\bar{z}} = 0 \Rightarrow$ CR equations.

$\square$

2 Complex integration

2.1 Definitions

complex line integral, path-independence, winding number

2.2 Useful Theorems

Theorem 1. (functions have antiderivatives iff they are path independent) Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ be continuous. Then:

1. $\exists F : D \to \mathbb{C}$ s.t. $\forall z \in D$, $F'(z) = f(z) \Leftrightarrow$

2. For any C^1 path $\gamma \subseteq D$, $\displaystyle\int_\gamma f(z)dz$ depends only on endpts of $\gamma \Leftrightarrow$

3. For any closed C^1 contour $\gamma \subseteq D$, $\displaystyle\oint_\gamma f(z)dz = 0$.

Depends on: dependencies...

Proof idea: 3 steps:

1. (1) $\Rightarrow$ (2): $\displaystyle\int_\gamma f(z)dz = \int_a^b \frac{d}{dt}F(z(t)dt = F(z(b)) - F(z(a))$ by FTOC (extends to complex case if you just separate real and imaginary components).

2. (2) $\Rightarrow$ (1): Since contour integrals are path-independent, the function $F(z) = \displaystyle\int_{z_0}^{z} f(\phi)d\phi$ is well defined and $\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\displaystyle\int_z^{z+\Delta z}(f(\phi)-f(z))d\phi \leq \frac{\epsilon\Delta z}{\Delta z}$ for arbitrary ϵ if Δz small enough.

3. (2) $\Leftrightarrow$ (3) is obvious by splitting an integral along a closed curve into 2 integrals along paths with the same endpoints but opposite orientation and vice versa.

□

Theorem 2. Cauchy's Theorem (NOT Cauchy-Goursat) for C^1 analytic fns
Let $f : D \to \mathbb{C}$ be analytic and ***C^1***. Let $\gamma : [0,1] \to D$ be a simple closed piecewise C^1 curve. Then $\displaystyle\oint_{dR} fdz = 0$.

Depends on: Greene's theorem

Proof idea: Writing $f = u + iv$, $\displaystyle\oint_\gamma fdz = \oint_\gamma udx - vdy + i\oint_\gamma vdx + udy$. Apply Greene's theorem and then use the CR equations to show that integrand is 0. □

Remark: The assumption that f be C^1 is crucial since it is needed to apply Greene's theorem. Without it, the result has to be proved from scratch via Cauchy-Goursat. In the end it doesnt matter since analytic functions turn out to be C^∞.

Theorem 3. Cauchy-Goursat theorem on a rectangle
Let $R \subseteq D$ be a rectangle with boundary dR. Suppose $f : D \to \mathbb{C}$ is analytic. Then $\displaystyle\oint_{dR} fdz = 0$.

Depends on: $\oint 1dz = 0$ and $\oint zdz = 0$ (since they have antiderivatives)

Proof idea: Use bisection argument. Let $R_0 = R$ and let R_j be s.t. $|\displaystyle\oint_{R_j} fdz| \geq 4^{-n}|\oint_R fdz|$, $\forall j \geq 1$. Let $z^* = \lim R_j$ and use the fact that $|\displaystyle\oint_{R_j} f(z)dz| =$

$$|\oint_{R_j}(f(z)-f(z^*)-(z-z^*)f'(z^*))dz| \le \oint_{R_j}\epsilon|z-z^*|dz \le 4^{-j}C\epsilon.$$
for $|z-z^*|<\delta$,

$\square$

Theorem 4. Cauchy-Goursat thm on a rectangle with finite singularities
Let $R \subseteq D$ be a rectangle with boundary dR. Suppose $f : D \to \mathbb{C}$ is analytic except for finitely many points $\{\xi_j\}$ at which $\lim_{z\to\xi_j}(z-\xi_j)f(z) = 0$. Then $\oint_{dR} f dz = 0$.

Depends on: Theorem without the singularities

Proof idea: By previous theorem, we can reduct the integral around R to that along small rectangles R_j around each singularity (WLOG assume there is only 1). Choose the rectangles so small s.t. that $|f(z)| \le \frac{\epsilon}{|z-\xi_j|}\forall z \in dR_j$. Bound the integral above by 8ϵ using elementary estimates. $\square$

Theorem 5. Cauchy-Goursat thm on a disk with finite singularities
Let $D(P,r) \subseteq U$ be a disk about $P \in \mathbb{C}$ of radius r. Suppose $f : U \to \mathbb{C}$ is analytic except for finitely many points $\{\xi_j\}$ at which $\lim_{z\to\xi_j}(z-\xi_j)f(z) = 0$. Then $\oint_{dD(P,r)} f dz = 0$.

Depends on: Theorem for rectangles with finite singularities

Proof idea: Construct an analytic antiderivative $F(z) = \int_{z_0}^{z} f(z)dz$ where the integral is taken along block paths (not going through any singularities) from $z_0 \to z$. Show that $F'(z) = f(z)$ as before. For F to be analytic we need the defining integral to be well-defined (follows from Theorem on rectangles).

$\square$

Theorem 6. The winding number is a multiple of $2\pi i$
Let $\gamma : [\alpha,\beta] \to \mathbb{C}$ be a piecewise smooth closed curve not passing through $a \in \mathbb{C}$. Then $\eta_\gamma(a) = \int_\gamma \frac{dz}{z-a}$ is a multiple of $2\pi i$.
Depends on:

Proof idea: Let $h(t) = exp(-\int_\alpha^t \frac{\gamma'(s)ds}{\gamma(s)-a})(\gamma(t)-a)$. Then $h'(t) = 0 \Rightarrow h(t) = h(\alpha) = \gamma(\alpha) - a \Rightarrow g(t) = exp(\int_\alpha^t \frac{\gamma'(s)ds}{\gamma(s)-a}) = \frac{\gamma(t)-a}{\gamma(\alpha)-a} \Rightarrow g(\beta) =$

$$exp(\int_\gamma \frac{dz}{z-a}) = \frac{\gamma(\beta)-a}{\gamma(\alpha)-a} = 1 \qquad \square$$

Theorem 7. Cauchy's integral formula
Let $D(P,r) \subseteq U$ be a disk about $P \in \mathbb{C}$ of radius r. Suppose $f : U \to \mathbb{C}$ is analytic except for finitely many points $\{\xi_j\}$ *inside* $D(P,r)$ s.t. $\lim_{z\to\xi_j}(z-\xi_j)f(z) = 0$. Then $\forall z \in D(P,r)\backslash\{\xi_j\}$, $f(z) = \frac{1}{2\pi i}\oint_{dD(P,r)} \frac{f(\xi)}{\xi - z} d\xi$.

Depends on: Cauchy-Goursat on disk with singularities

Proof idea: Apply Cauchy-Goursat thm on disk with singularities to $F(z) = \frac{f(z)-f(a)}{z-a}$ and use the fact that $\eta_{dD(P,r)}(a) = 1$. $\square$

Remark: The proof yields the more general fact that $f(a)\eta_{dD(P,r)}(a) = \frac{1}{2\pi i}\oint_{dD(P,r)} \frac{f(\xi)}{\xi - z} d\xi$

Theorem 8. (analytic functions are infinitely smooth)
Let $U \subseteq \mathbb{C}$ be a region and suppose $f(z)$ analytic on U. Then $f \in C^\infty(U)$ and $\overline{D(P,r)} \subseteq U \Rightarrow \forall k \geq 0$, $f^{(k)}(z) = \frac{k!}{2\pi i}\oint_{dD} \frac{f(\xi)d\xi}{(\xi - z)^{k+1}}$

Depends on: cauchy integral formula, differentiation under integral sign

Proof idea: Write $f(z)$ using cauchy integral formula and differentiate under integral sign. The procedure is justified because the limit defining $\frac{d}{dz}(\frac{f(\xi)}{\xi-z}$ converges *uniformly* on the compact set $\{\xi : |\xi - P| = r\}$ and $|\xi - z|$ is bounded below by $r - |z-P| > 0$. $\square$

2 important consequences:

1. The derivative of an analytic function is analytic (just take second derivatives and show that CR equations hold)

2. If we have a continuous function $\phi(\xi)$ defined $dD(P,r)$ then we can *construct* an analytic function in $D(P,r)$ via $f(z) = \frac{1}{2\pi i}\oint_{dD} \frac{\phi(\xi)d\xi}{\xi - z}$ (since $f'(z)$ exists in $D(P,r)$ and is given by above theorem).

Theorem 9. Cauchy estimates
Suppose $U \subseteq \mathbb{C}$ is a region and $f(z)$ analytic on U and that $\overline{D(P,r)} \subseteq U$. Let $M = \sup_{z\in\overline{D(P,r)}} |f(z)|$. Then $\forall k \geq 1$, $|f^{(k)}(P)| \leq \frac{Mk!}{r^k}$.
Depends on: cauchy integral formula

Proof idea: Take the integral formula for the k^{th} derivative and estimate in the obvious way. $\square$

Remark: This estimate also gives the radius of convergence for the power series $\sum \frac{f^{(k)}(P)}{k!}(z-P)^k$ to be $\geq r$ via the root test

Theorem 10. Liouiville's theorem (entire bnded anal. fns are constant)
Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic on $\mathbb{C}$ and is bounded. Then f is constant
Depends on: cauchy estimates

Proof idea: Apply cauchy estimate and take $\lim\limits_{r\to\infty}$. $\square$

Remark: If assumption is weakened to $|f(z)| \leq C|z|^k \ \forall |z| > 1$ for some $k > 0$ then f is a polynomial of degree $\leq k$ (show that all derivatives at 0 of order more than k are 0 by cauchy estimates).

Theorem 11. Morera's theorem (path independence is suff. for analyticity)
Suppose $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is continuous U and that for every closed piecewise C^1 curve $\gamma : [0,1] \to U$, $\oint_\gamma f(\xi)d\xi = 0$. Then f is analytic on U.
Depends on: derivative of an analytic function is analytic

Proof idea: Construct an analytic antiderivative $F(z) = \displaystyle\int_{P_0}^{z} f(\xi)d\xi$. Then F is well defined by the path independence of f and $F'(z) = f(z)$. Thus F analytic $\Rightarrow f$ analytic. $\square$

Theorem 12. (Power series expansion for an analytic function)
Suppose $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is analytic on U and that $D(P, r) \subseteq U$. Then the series $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(P)}{k!}(z-P)^k$ has radius of convergence at least r and the converges to $f(z)$ on $D(P, r)$.
Depends on: cauchy integral formula

Proof idea: Express $f(z)$ using the cauchy integral formula on a *slightly smaller disk* $D(P, r') \subseteq \overline{D(P, r')} \subseteq D(P, r)$. Rewrite the denominator using the trick: $f(\xi)(\xi - z)^{-1} = f(\xi)(\xi - P)^{-1}(1 - \frac{z-P}{\xi-P})^{-1}$. Expand the last term as a geomtric series and interchange sum and integral (since sum converges uniformly on $dD(P, r')$. $\square$

Theorem 13. (Convergent power series are analytic functions)

Suppose the power series $\displaystyle\sum_{j=0}^{\infty} a_j(z-P)^j$ converges on $D(P, r)$, then the series defines a C^∞ function $f(z)$ on $D(P,r)$. Furthermore, $f^{(k)}(z)$ is the term-wise kth derivative $\sum_{j=k}^{\infty} j(j-1)...(j-k+1)a_j(z-P)^{j-k}$

Depends on: root test, Weierstrass M-test

Proof idea: Compute the complex derivative of the series as a limit $\lim_{h\to 0}\sum_{j=0}^{\infty}\frac{a_j(z+h)^j - a_j z^j}{h}$. Express the numerator as an integral using FTOC and derive an upper bound of the summand:
$j|a_j|(|z|+|h|)^{j-1} \leq j|a_j|(0.5(r+|z|))^{j-1}$ which converges by the root test. Apply Weierstrass M-test to get the result to get uniform convergence so that we can switch limit and infinite sum. $\square$

Remark: As a result, any two convergent power series defining the same analytic function $f(z)$ on the same region must have exactly the same coefficients a_j.

Theorem 14. Weierstrass theorem: (Uniform limit of analytic functions) Suppose $f_j : U \subseteq \mathbb{C} \to \mathbb{C}$ are analytic on U and that $\forall E \subseteq U$ compact, $f_j|_E \to f|_E$ uniformly. Then f is analytic on U.

Depends on: cauchy integral formula

Proof idea: Take any closed disk $\overline{D(P,r)} \subseteq U$ on which $f_j \to f$ uniformly. Then f is necessarily continuous on $\overline{D}$ and writing $f(z) = \lim_{j\to\infty} f_j(z)$ and expanding $f_j(z)$ using cauchy integral formula on $\overline{D}$, we can interchange limit and integral, again because $\frac{f_j(\xi)}{\xi - z} \to \frac{f(\xi)}{\xi - z}$ uniformly on $dD(P,r)$. Thus, the cauchy integral formula holds for f, and so f is analytic (see remark of theorem showing analytic functions are C^∞). $\square$

Remark:

1. This can also be proved using Morera's theorem by showing that the integral of f along any closed curve in U has to be 0 since we can exchange limit and integral by unif. cvgnce

2. If the same assumptions hold, one can also show that the k_{th} derivatives converge uniformly on compact sets (apply Cauchy estimates to $|(f_m(z) - f_n(z))^{(k)}|$ to show that this is a uniformly cauchy sequence).

2.3 Important examples

1. The function need not have an analytic antiderivative (and therefore its integral need not be 0) if it not analytic on the whole region: consider $f(z) = 1/z$ on $D(0,1)\backslash\{0\}$. The integral along the boundary of the disk is in fact $2\pi i$.

2. Liouville's theorem does not hold in $\mathbb{R}$. Consider the $f(x) = \frac{1}{x^2+1}$, which is real-analytic on $\mathbb{R}$ and nonconstant. This example also shows that a real-analytic function may not be extendable to the entire complex plane (it is undefined at $\pm i$).

3. Weierstrass's theorem does not have an analog in $\mathbb{R}$ - the uniform limit of real-analytic functions is necessarily continuous but need not be real-analytic or even C_∞. Consider $f_n(x) = |x|^{1+1/n} \to |x|$ uniformly on $[-1,1]$ but the limit is nonsmooth. The key is that with complex-analytic functions, uniform cvgnce $\Rightarrow$ uniform cvgnce of all derivatives (see Remark 2), unlike in $\mathbb{R}$.

3 Zeros

3.1 Definitions

isolated 0, argument principle, open mapping, preimage, multiplicity (of a root)

3.2 Useful Theorems

Theorem 1. Zeroes of analytic functions are isolated
Suppose f is analytic on $U \subseteq \mathbb{C}$ open, connected. If there is a sequence of points $(z_j) \to z_0$ in U s.t. $\forall j, f(z_j) = 0$ and $f(z_0) = 0$, then $f \equiv 0$ on U.

Depends on: Power series expression and relation with higher-order derivatives, clopen subsets of connection regions

Proof idea: Show that all derivatives of f are 0 in U. We can do this by showing that the set $E = \{z \in U : f^{(k)}(z) = 0 \forall k \geq 0\}$ is:

1. Nonempty: Show that $z_0 \in E$ by contradiction. Take the first nonzero derivative $f^{(j)}(z_0)$. Then using the power series expansion of f about z_0, we can define $g(z) = f(z)(z - z_0)^{-j}$ on a small disk around z_0 and by def., $g(z_0) \neq 0$. But $g(z_j) = 0 \forall j$, contradicting continuity of g.

2. Open: Follows from the power series expansion around a point where all derivatives vanish.

3. Closed (wrt U): E is the countable intersection of preimages of the point $\{0\}$ under continuous functions $f^{(k)}(z)$.

Therefore, $E = U \Rightarrow f \equiv 0$. $\square$

Remark: Several important consequences follow:

1. If f is 0 on a disk $D \subseteq U$, $f \equiv 0$.

2. If f, g are entire and agree on all of $\mathbb{R}$, then they agree on all of $\mathbb{C}$. Can use this to check identities (e.g., since $sin^2(x) + cos^2(x) = 0$ on $\mathbb{R}$, it extends to $\mathbb{C}$).

3. If there is a point in U at which all derivatives vanish, then $f \equiv 0$.

Theorem 2. Argument principle
Suppose f is analytic on a neighborhood of $\overline{D(P,r)}$ and has finitely many zeros $\{\xi_j\} \subseteq D(P,r)$. Then for any closed curve $\gamma \subseteq D(P,r)\backslash\{\xi_j\}$, $\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)dz}{f(z)} = \sum_j \eta_\gamma(\xi_j)$

In particular, the integral is $\eta_\Gamma(0) = \frac{1}{2\pi i}\oint_\Gamma \frac{dw}{w}$ where $\Gamma = f(\gamma)$ is the image curve.
Depends on: power series representation, log-derivative

Proof idea: Let $g(z) = f(z)/(\prod(z-\xi_j))$. Then we can see that since g is in fact analytic on $D(P,r)$ if we define the values $g(\xi_j) \equiv \frac{f^{(n_j)}(xi_j)}{n_j!} \neq 0$ where n_j is the order of the zero at xi_j (i.e. the first nonzero derivative in the taylor expansion about xi_j). In fact $g(z) \neq 0$ on $D(P,r)$. Then:
$\forall z \neq \xi_j$, $\frac{f'(z)}{f(z)} = \sum_{j=1}^{n} \frac{1}{z-\xi_j} + \frac{g'(z)}{g(z)}$ Integrating both sides and noting that the last term integrates to 0 (since $g \neq 0$ and is analytic), gives the result. $\square$

Remark: In the case of infinite zeros (but $f \neq 0$), we can apply the theorem to a slightly smaller disk $D(P,r')$ in whose closure there must be only finitely many zeros since otherwise there would be a convergent subsequence of zeros, contradicting previous theorem.

Theorem 3. The preimage of the neighborhood of an order-n zero
Suppose f is analytic at z_0 with $f(z_0) = w_0$ and suppose $f(z) - w_0$ has a zero of order n at $z = z_0$. Then $\forall \epsilon > 0, \exists \delta > 0$ s.t. $|w - w_0| < \delta \Rightarrow f(z) = w$

has exactly n roots in $\{|z - z_0| < \epsilon\}$.

Depends on: Argument principle (for zeros only)

Proof idea: By a change of variables we know that $2\pi i \eta_\Gamma(w_0) = \displaystyle\int_\Gamma \frac{1}{w - w_0} dw = \int_\gamma \frac{f'}{f - w_0} dz$. In other words the multiplicity of w_0 at a point z_0 is equal to the winding number of the image curve $\Gamma = f(dD(z_0, \epsilon))$ about 0. For ϵ suff. small, we know that $f \neq 0$ on $D(z_0, \epsilon) \backslash \{z_0\}$. Clearly, there is a small disk in the w-plane, $D(w_0, \delta)$ in which $\eta_\Gamma(w) = \eta_\Gamma(w_0) = n$. But $\eta_\Gamma(w) = \frac{1}{2\pi i} \oint \frac{f'}{f - w} dz$.

□

Remark: If we choose ϵ small enough, we can guarantee that the roots are all simple. Also, if $f'(z_0) \neq 0$, then each point in $D(w_0, \delta)$ has exactly 1 preimage (why? because if a point had $p > 1$ preimages , making f 'locally invertible' and its local inverse f^{-1} has derivative $1/f'(f^{-1}(w))$ on $D(w_0, \delta)$.

Theorem 4. Open mapping theorem
Suppose $f : U \to \mathbb{C}$ is nonconstant and analytic on an open connected set $U \subset \mathbb{C}$. Then $f(U)$ is open.

Depends on: N roots theorem

Proof idea: This follows immediately from the previous theorem. In other words we can take any point z_0 with $f(z_0) = w_0$ and define $g(z) = f(z) - w_0$. Then the function $N(z) = \frac{1}{2\pi i} \displaystyle\oint_{dD(P,\epsilon)} \frac{f'(\varphi) d\varphi}{f(\varphi) - z}$ counts the number of roots of g(z) in $D(P, \epsilon)$ and is constantly equal to $N(P) = n$. □

Theorem 5. Rouche's theorem
Suppose $f, g : U \to \mathbb{C}$ are analytic on U and suppose that $\overline{D}(P, r) \subseteq U$ and that $|f(\varphi) - g(\varphi)| < |f(\varphi)| + |g(\varphi)|$ on $dD(P, r)$.Then:

$$\frac{1}{2\pi i} \oint_{dD(P,r} \frac{f'(\varphi) d\varphi}{f(\varphi)} = \frac{1}{2\pi i} \oint_{dD(P,r} \frac{g'(\varphi) d\varphi}{g(\varphi)}.$$

In other words, g and f have the same number of roots inside $D(P, r)$, counting multiplicities.

Depends on: Argument principle for zeros

Proof idea: One can show that the hypothesis implies that $f, g \neq 0$ on $dD(P, r)$. The basic idea is to show that the image of $dD(P, r)$ under the

map $f(\varphi)/g(\varphi)$ does not completely go around 0. The hypothesis implies that this image curve never intersects the negative real axis, because if it did, then:
$f(\varphi)/g(\varphi) = \lambda \leq 0$, then $|f(\varphi)/g(\varphi) - 1| = -\lambda + 1 = |f(\varphi)/g(\varphi)| + 1 \Rightarrow |f(\varphi) - g(\varphi)| = |f(\varphi)| + |g(\varphi)|$.

More rigorously, if define $f_t = tf + (1-t)g$, we can show that $\forall t$, $f_t \neq 0$ on $dD(P,r)$ and that $I_t = \dfrac{1}{2\pi i}\displaystyle\oint_{dD(P,r)} \dfrac{f_t'(\varphi)d\varphi}{f_t(\varphi)}$ is continuouts in t and integer valued $\Rightarrow I_0 = I_1$. $\square$

Theorem 6. Hurwitz Thm (uniform limit of nonvanishing anal. fns)
Suppose $U \subseteq \mathbb{C}$ is open/connected and (f_j) is a sequence of nowhere vanishing analytic functions converging uniformly to f on every compact subset of U. Then either $f \equiv 0$ or f is nowhere vanishing on U.

Depends on: Argument principle for zeros

Proof idea: This follows from the fact that we can exchange the uniform limit with the integral $\dfrac{1}{2\pi i}\displaystyle\oint_{dD(P,\epsilon)} \dfrac{f_j'(\varphi)d\varphi}{f_j(\varphi} = 0$ $(\forall j)$ for any point P s.t. $f(P) = 0$.

$\square$

3.3 Important examples

1. It is possible that a sequences of zeros of an analytic function f on U converges to a point *on the boundary* of U. Consider $f(z) = sin(1/(1-z))$ on the open unit disk which vanishes on $\{1 - (\pi n)^{-1} : n \in \mathbb{N}\}$.

2. Hurwitz's theorem fails in $\mathbb{R}$. Consider $f_n(x) = x^2 + 1/n \neq 0 \to f(x) = x^2$ on $\mathbb{R}$ which has a double 0 at $x = 0$. If we 'complexify' the functions the f_j are no longer always nonzero (they each have 2 complex roots).

4 Singularities

4.1 Definitions

removable singularity, pole, essential singularity, meromorphic, residue

4.2 Basic properties of singularity types

Let f be a analytic on the disk $D(P,r)\backslash\{P\}$. The possibilities for the behavior of $f(z)$ as $z \to P$ are as follows:

1. Removable singularities:

   (a) f is well behaved near P i.e. it approaches a finite limit, which can be used to extend f to a holomorphic function on $D(P,r)$.

   (b) $f(z)$ has a (Taylor) power series expansion about P

2. Pole: $\lim_{z\to P} |f(z)| = \infty$

   (a) for some $k > 0$, $(z-P)^k f(z)$ is bounded near P but $(z-P)^{k-1} f(z)$ is not. We can write $f(z) = (z-P)^{-k} g(z)$ where $g(z)$ is analytic on $D(P,r)$. k is called the order of the pole

   (b) We can also interpret the order of the pole as the order of the zero of the function $g(z) = 1/f(z)$, which has a removable singularity at P.

   (c) f has a Laurent series expansion with exactly k nonzero negative-power terms.

3. Essential singularity: neither removable nor a pole. The image is dense in $\mathbb{C}$ and the Laurent series expansion has infinitely many nonzero negative-power terms.

4.3 Useful theorems

Theorem 7. Removable singularities theorem
Suppose f is analytic on $D(P,r)\backslash P$. Then $\exists$ an extension $\tilde{f}$ of f that is analytic on $D(P,r) \Leftrightarrow \lim_{z\to P}(z-P)f(z) = 0$.
Depends on: Cauchy's formula with finite singularities

Proof idea: 1. $\Rightarrow$: Trivial by continuity of the extension $\tilde{f}$.

2. $\Leftarrow$: Apply CIF to f to get an integral expression for $f(z)$ in terms of the values of f on $dD(P,r)$, $\forall z \neq P$. This expression can be naturall extended to P, and clearly the extension $\tilde{f}$ is analytic by construction (for instance, see Remark 2 in the theorem for infinite derivatives).

$\square$

Remark: By repeated application of this theorem we can derive a 'Taylor-expansion' for analytic functions. Since the theorem applies to $F(z) = \frac{\tilde{f}(z)-\tilde{f}(P)}{z-P}$, we have an extension $F_1(z)$ that is $\tilde{f}'(P)$ at $z = P$. Repeating the procedure with F_1 instead of $\tilde{f}$ and so on, we get an analytic sequence $f_j(z)$ satisfying $f_{j-1}(z) = f_{j-1}(P) + (z-P)f_j(P)\ \forall 1 \leq j \leq n$. Recursively expanding the expression for $f(z)$ gives:
$f(z) = f(P) + (z-P)f_1(P) + (z-P)^2 f_2(P) + ... + (z-P)^n f_n(P)$ and one can show by differentiating n times at $z = a$ that $f^{(j)}(P) = j! f_j(P)$.

Theorem 8. Casorati-Weierstrass theorem
Suppose f is analytic on the punctured disk $D(P,r)\backslash\{P\}$ and has an essential singularity at P. Then $f(D(P,r)\backslash\{P\})$ is dense in $\mathbb{C}$.

Depends on: removable singularities theorem

Proof idea: Suppose not i.e. $\exists z_0 \in \mathbb{C}, \epsilon > 0$ s.t. $|f(z) - z_0| \geq \epsilon$ on $D(P,r)\{P\}$. Then we can apply removable singularities to the (bounded) function $g(z) = (f(z) - z_0)^{-1}$ and get an extension $\hat{g}(z)$ on $D(P,r)$. But then $f(z) = z_0 + \hat{g}(z)^{-1}$. If $g(\hat{P}) = 0$, f as a pole at P, otherwise, it has a removable singularity at P, both being contradictions. □

Theorem 9. Residue theorem
Let $U \subseteq \mathbb{C}$ be a simply connected region (for our purposes we only need the fact that any analytic function on U has an analytic antiderivative on U and so the integral on any closed curve is 0). Suppose f is analytic on U except at a finitely many singularities $\{\xi_j\}$. Then for any closed curve $\gamma \subseteq U\backslash\{\xi_j\}$:
$\frac{1}{2\pi i}\oint_\gamma f(z)dz = \sum_{j=0}^{n} Res_f(\xi_j)\eta_\gamma(\xi_j)$ where $Res_f(\xi_j) = \frac{1}{2\pi i}\oint_{dD(\xi_j,\delta_j)} f(\varphi)d\varphi$ is the unique number in $\mathbb{C}$ s.t. $f(z) - \frac{Res_f(\xi_j)}{z-\xi_j}$ has an analytic antiderivative in U.

Depends on: Cauchy's theorem for general curves

Proof idea: In the case of finite singularities, the proof is an immediate consequence of Cauchy's theorem for general regions/curves, since the integral along γ is the same as the sum of integrals around little circles about each singularity, weighted by $\eta_\gamma(\xi_j)$. The fact that $f(z) - \frac{Res_f(\xi_j)}{z-\xi_j}$ has an analytic antiderivative follows from the fact that $\oint_{dD(\xi_j,\delta_j)}[f(\varphi) - \frac{Res_f(\xi_j)}{\varphi - \xi_j}]d\varphi = 0$.
□

Remark:

1. Cauchy's integral formula is a special case of this theorem with $f = g(z)/(z-a)$ and $Res_f(a) = g(a)$.

2. Another special case is the argument principle for zeros - here we apply the theorem to $f'(z)/f(z)$ which has singularities at the zeros of f $\{\xi_j\}$, each with order a_j. Thus the residue at each ξ_j is a_j.

3. In practice, residues can be calculated easily in the case where the singularities are poles of finite order. In this case we calculate a_{-1} in the Laurent series expansion about the pole, which is $[\frac{1}{(k-1)!}f^{(k-1)}((z-P)^k f(z))]_{z=\xi}$ (see next section). For simple poles, $Res_f(\xi) = (z-\xi)f(z)|_{z=\xi}$.

Theorem 10. General argument principle (for zeros and poles)
Suppose f is meromorphic on a simply connected region $\Omega \subseteq \mathbb{C}$ with zeros $\{z_j\}$ and poles $\{\xi_j\}$ (counted according to multiplicity). Then for any closed curve $\gamma \subseteq \Omega \backslash (\{\xi_j\} \cup \{z_j\})$, $\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)dz}{f(z)} = \sum_j \eta_\gamma(a_j) - \sum_k \eta_\gamma(\xi_k)$.

Depends on: Residue theorem

Proof idea: We write $f = [\prod_j (z-z_j)][\prod_k \frac{1}{z-\xi_k}]g(z)$ where g(z) is nonzero and analytic on Ω. Taking the logarithmic derivative (can factor iteratively) as in the argument principle before and noting that g'/g integrates to 0, gives the result. $\square$

4.4 Important examples

1. Removable singularities theorem does not necessarily hold if f is C^∞ but not analytic. Consider $f(z) = sin(1/|z|)$. Then f is bounded around 0 but cannot even be continuously extended to 0.

2. An essential singularity: $f(z) = e^{1/z}$ on $D(0,1)\backslash\{0\}$.

3. Calculus of residues: $\int_{-\infty}^{\infty} \frac{\sin x dx}{x}$ can be evaluated by taking γ to be a semicircle with base as the x-axis and a downward bump near the origin. Let $g(z) = e^{iz}/z$ and apply the residue theorem. The desired integral is the imaginary portion on the x-axis.

5 Laurent series

5.1 Definitions

5.2 Useful theorems

Theorem 11. (Existence of Laurent expansion)
Let f be analytic on the region $U = D(P,R)\backslash\overline{D(P,r)}$ with $0 \leq r < R$. Then $f(z) = \sum_{-\infty}^{\infty} a_j(z-P)^j$ on U. The series converges absolutely and uniformly on slightly smaller annuli.
Depends on: Cauchy integral formula, Cauchy's theorem, power series expansion for analytic functions

Proof idea: We find 2 analytic functions f_1 on $\{|z| < R\}$ and f_2 on $\{|z| > r\}$ s.t. $f(z) = f_1(z) + f_2(z)$ and their power series expansions correspond to the positive powers and negative powers in the above series, respectively:

1. Let $f_1(z) = \frac{1}{2\pi i}\int_{|\xi-P|=R'} \frac{f(\xi)d\xi}{\xi - z}$ (well defined for $|z-P| \in [0, R')$ for any $R' < R$). Thus $f_1(z)$ is analytic on $D(P,R)$ and has a power series expansion $\sum_{n=0}^{\infty} A_n(z-P)^n$ in this region.

2. Let $f_2(z) = -\frac{1}{2\pi z}\int_{|\xi-P|=r'} \frac{f(\xi)d\xi}{\xi - z}$ (well-defined $|z-P| \in (r', \infty)$ for any $r' > r$). Making the change of variables $z' \leftarrow 1/(z-P)$, $\xi' \leftarrow 1/(\xi-P)$, we can show that use the same tricks (expand geometric series inside integral) to get an expansion $\sum_{n=1}^{\infty} B_n z'^n$.

3. To show $f = f_1 + f_2$ we use Cauchy's theorem. To do so rigorously, we apply for each fixed z, Cauchy's theorem to the integral along a slightly smaller annuli of the function $g_z(\xi) = \frac{f(\xi)-f(z)}{\xi - z}$ with the hole at z filled in by $f'(z)$.

$\square$

Remark: Conversely if we have a doubly infinite series that is convergent, by definition this means that the nonnegative power series and negative power series each converge separately. Thus they define analytic functions f_1 on a disk $D(P,R)$ and f_2 on the region $|z| > r$. Thus if $r < R$ we have an analytic function in the annulus.

Theorem 12. (Relationship between singularity type and Laurent exp.) Let f be analytic on $U = D(P,r)\backslash\{P\}$. Then f has a unique Laurent expansion $f(z) = \sum_{-\infty}^{\infty} a_j(z-P)^j$ converging absolutely on U with $a_j = \frac{1}{2\pi i}\oint_{dD(P,s)} \frac{f(\xi)d\xi}{(\xi - P)^{j+1}}$, for $0 < s < r$. Furthermore:

1. f has a removable singularity at $P \Leftrightarrow a_j = 0 \forall j < 0$
2. f has a pole at $P \Leftrightarrow a_j = 0 \ \forall -\infty < j < -k$ for some $k > 0$.
3. f has an essential singularity at $P \Leftrightarrow$ (1) and (2) do not hold.

Depends on: Previous theorem)

Proof idea: The formula for a_j follows from the expansions of the component functions f_1 and f_2.

1. $\Rightarrow$: If f has a removable singularity, the extension $\hat{f}$ has a regular power series expansion. By uniqueness, this must also be the Laurent expansion of f. $\Leftarrow$: The power series with no negative-power terms defines an analytic function on $D(P,r)$. So it has a (finite) value at $z = P \Rightarrow f$ has a removable singularity.
2. $\Rightarrow$: Since f is necessarily nonzero in a neighborhood of P, we can apply removable singu

   $\Leftarrow$:
3.

$\square$

5.3 Important examples

1.

6 Conformal mappings

6.1 Definitions:

6.2 Useful theorems:

Theorem 1. (Maxmum modulus principle)
Suppose f is analytic and nonconstant on a region Ω. Then $|f(z)|$ cannot achieve a maximum in Ω.

Depends on: Open mapping theorem

Proof idea: Follows directly from the fact that for any point $w_0 \in f(\Omega)$, there is a small δ-ball around w_0 also contained in the image, in which we can always find one point with greater modulus than w_0. □

Remark: The above theorem can be restated by saying that if f is analytic on a closed bounded set E than the maximum is attained on the boundary.

Theorem 2. Schwarz's Lemma
Let $f(z)$ be analytic on $D(0,1)$ satisfying $|f(z)| \leq 1$, $f(0) = 0$. Then $|f(z)| \leq |z|$ and $|f'(0)'| \leq 1$. Furthermore, if $|f(z)| = |z|$ for any $z \neq 0$ or if $|f'(0)| = 1$, then $f(z) = cz$ for some constant c s.t. $|c| = 1$.

Depends on: Maximum modulus principle, Removable singularities

Proof idea: Apply maximum modulus to $g(z) = f(z)/z$ with $g(0) \equiv f'(0)$ (which is necessarily analytic by removable singularities) on the slightly smaller disk $D(0, 1-\epsilon)$. Since ϵ is arbitrary, this gives $|g| \leq 1$, giving the result. If either $|f(z)| = |z|$ or $|f'(0)| = 1$, then g attains its maximum (of 1) in the interior, thus yielding it constant. □

Remark: We can generalize this result in two stages:

1. If f analytic on $D(0,R)$ and satisfies $|f| \leq M$, we can apply the original Schwarz lemma to $f(Rz)/M$ to get that $|f(z)| \leq M|z|/R$.

2. If $f(0) = 0$ is replaced by $f(z_0) = w_0$ with $|z_0| < R$, $|w_0| < M$, then we can definte T to be the lnear transformation taking $D(0,R) \to D(0,1)$ and sending $z_0 \mapsto 0$, and likewise S takes $D(0,M) \to D(0,1)$ and sending $w_0 \mapsto 0$. Then we can apply Schwarz to SfT^{-1} to get $|Sf(z)| \leq |Tz|$.

Theorem 3. (The biholomorphic maps of the plane are the nonconstant linear maps)
Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic, 1-1, and onto. Then $f(z) = az + b$ for some $a \neq 0, b \in \mathbb{C}$.

Depends on: (Heine Borel, Removable singularities, Generalized Liouville

Proof idea: Clearly every linear map is a biholomorphic map of the plane. Show the converse in 3 steps:

1. Show that $\lim_{|z|\to\infty} |f| = \infty$: the preimage of the closed disk $f^{-1}(\overline{D(0, 1/\epsilon)})$ is compact and thus bounded by some radius R. Therefore, outside $D(0, R)$, $|f(z)| > 1/\epsilon \to \infty$.

2. Show that on $\{z : |z| > D\}$, $|f(z)| < B|z|$ for some numbers $B, D > 0$: Look at the function $g(z) = 1/(f(1/z))$ outside a disk of radius R where $|f| \geq 1$. Then we have $|g(z)| \leq 1$ on $\{z : 0 < |z| < 1/C\}$. By removable singularities and previous step we can extend g to be 0 at $z = 0$. Since g is 1-1 $\Rightarrow g'(0) \neq 0 \Rightarrow |g(z)| \geq A|z|$ for some constant A for $|z|$ small. This implies that $|f| \leq A|z|$ for $|z|$ large.

3. Apply Liouville generalized and use the fact that f cannot be constant.

□

Remark: The line of reasoning in steps 2-3 above can be used to show that any entire function that has a pole at ∞ is a polynomial. Specifically, we define $g(z)$ as above. Then we know we extend to $g(0) = 0$ and since g is nonconstant, we can say that $|g(z)| \geq A|z|^n$ for some $n \geq 1$. Writing this in terms of $f(z)$ and applying Generalized Liouville gives that f is a polynomial of degree $\leq n$.

Theorem 4. (Biholomorphic self-maps of the disk that fix 0 are rotations)
Let $D = D(0, 1) \subseteq \mathbb{C}$. Then $f : D \to D$ is analytic,1-1, and onto with $f(0) = 0 \Leftrightarrow f(z) = wz$ for some $|w| = 1$.

Depends on: Schwarz lemma

Proof idea: Clearly all rotations satisfy the conditions. Conversely, we can apply the Schwarz lemma to both f and f^{-1} to get that $|f'(0)| \leq 1$ and $1/|f'(0)| \leq 1 \Rightarrow |f'(0)| = 1$. By uniquness clause of Schwarz lemma, we have that $f(z) = (f'(0))z$ on D. □

Theorem 5. (Biholomorphic self-maps of the disk are mobius transforms) Let $D = D(0,1) \subseteq \mathbb{C}$ and let $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$ be the conformal self-map of D taking $a \mapsto 0$. Then $f : D \to D$ is analytic,1-1, and onto $\Leftrightarrow f(z) = w\phi_a(z)$ for some $|w| = 1$ and $|a| < 1$.

Depends on: Previous theorem

Proof idea: Clearly all rotated mobius transforms are biholomorphic self-maps of D. Conversely, letting $b = f(0)$ we can apply the previous theorem to show that the function $(\phi_b f)$, which fixes 0, is a rotation wz for some $|w| = 1$. Noting that $\phi_b^{-1} = \phi_{-b}$, we have that $f(z) = \phi_{-b}(wz) = w\phi_{-b/w}(z)$ (manipulate). $\square$

Remark: The set of Mobius transformations forms a group under composition.

Theorem 6. (Biholomorphic self-maps of $\mathbb{C} \cup \{\infty\}$ are LFT's) Let $\phi : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$. Then ϕ is conformal, 1-1, and onto $\Leftrightarrow$ $\exists a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ s.t. $\phi(z) = \frac{az+b}{cz+d}$.

Depends on: Conformal self-maps of the plne

Proof idea: Clearly every LFT is a conformal self-map of the extended plane. To show the converse, compose ϕ with an LFT taking $\phi(\infty)$ to ∞. Then the composition $f(z) : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ fixes the point ∞, and so its restriction to the plane $\mathbb{C}$ is a nonconstant linear map by the previous theorem. Composing with the inverse of the LFT gives another LFT. $\square$

Theorem 7. (Conformal maps on the Riemann sphere are exactly the rational functions)
Let $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$. Then f is analytic $Leftrightarrow$ $f = P(z)/Q(z)$ for some polynomials P, Q.

Depends on: Compactness of the Riemann sphere, poles at infinity, removable singularities, Liouville theorem, Laurent expansion

Proof idea: 2 steps:

1. First show that, except for a possible pole at ∞, the poles of f are contained in a finite disk $D(0, R)$: If it has a pole at ∞ then it is isolated and so all other poles are in a finite disk. There can only be finitely many in this disk since otherwise there would be a convergent sequence of poles in the fintie plane. If there is not a pole at ∞ then f

is finite-valued outside some big disk and so the only poles are inside this disk - and the same compactness argument holds. Factoring out these poles it follows that $f(z)(z-P_1)^{n_1}...(z-P_k)^{n_k} = F(z)$ where F is analytic except for removable singularities in $\mathbb{C}$.

2. If F has a pole or removable singularity at ∞ then by Louiville it is a polynomial or constant. But F cannot have an essential singularity at ∞ since then the Laurent expansion of f would contain infintiely many negative-power terms (it wouldnt be meromorphic).

□

Theorem 1. (Montel's theorem: sufficient conditions for normal families) Suppose $U \subseteq \mathbb{C}$ be openand let $\mathcal{F}$ be a family of analytic functions on U.Suppose for each compact $K \subseteq U$ $exists M_K$ s.t. $\sup_{f\in\mathcal{F}} \sup_{z\in K} |f(z)| \leq M_K$. Then $\mathcal{F}$ is a normal family, i.e. every sequence (f_j) in $\mathcal{F}$ has a subsequence converging uniformly on compact subsets of U.

Depends on: Cauchy estimates, Arzela-Ascoli theorem, diagonalization

Proof idea: 2 steps:

1. For a fixed $K \subseteq U$, show that each $f \in \mathcal{F}$ satisfies a Lipschitz condition (uniform in f and z):
   (a) Take $L \supseteq K$ compact and use Cauchy estimates to show that $\forall f \in \mathcal{F}$, $\sup_{z\in L} |f'(l)| \leq M_K/R = C$ where R is chosen so that the ball of radius R around any point in L is contained in U.
   (b) Take η s.t. the line segment joining 2 pts $z, w \in K$ less than η from each other is contained in L. Bound $|f(z)-f(w)|$ by $C|z-w|$ by integrating along the line segment.
2. Since the Lipschitz condition implies equicontinuity of $\mathcal{F}$, we can apply Arzela-Ascoli to get a uniformly convergent subsequence f_{n_j} on K.
3. Take a sequence of compact sets K_j increasing to U. Then use diagonalization to get a subsequence f_{n_j} from $\mathcal{F}$ that converges on all K_j and thus on all compact subsets of U.

□

Theorem 2. (Riemann mapping theorem - when are regions biholomorphically equiv. to D(0,1)?)
Let $U \subset \mathbb{C}$ be open (and not all of $\mathbb{C}$). Suppose also that U is 'topologically equivalent' i.e. homeomorphic to $D(0,1)$. In other words $\exists$ a 1-1 onto continuous function from U to $D(0,1)$. Then for any $z_0 \in U$ there is a *unique* conformal 1-1 onto function $f : U \to D(0,1)$ s.t. $f(z_0) = 0$ and $f'(z_0) > 0$ for any $z_0 \in U$.

Depends on: Cauchy estimates, single-valued square-root, open mapping theorem, Montel's theorem, maximum modulus, Hurwitz's theorem, Mobius transforms

Proof idea: Basic outline:

1. Fix $z_0 \in U$ and define the family $\mathcal{F} = \{g : U \to D(0,1) : g(z_0) = 0, g'(z_0) > 0, g \text{ is 1-1}\}$.

2. Show that $\mathcal{F}$ is nonempty by construction: Take a point $w \notin U$ and define a single-valued, 1-1 square-root function (this requires the simply connected assumption) $h(z)$ on U s.t. $h(z)^2 = z - w$. By the open mapping theorem, the image of h contains a disk $D(P, r) \subseteq \mathbb{C}$ and since h cannot take opposite-signed values since $h^2 = z - w$ is 1-1, this disk is disjoint from $D(-P, r)$. Thus we can define $f(z) = \frac{r}{2(h(z)-(-P))}$, compose it with a Mobius transform to take z_0 to 0, verify that the composed fn is $\in \mathcal{F}$.

3. Show that $\exists f \in \mathcal{F}$ with maximal derivative $f'(z_0)$:

   (a) Recall that $|f'|$ has a uniform (over $f \in \mathcal{F}$) bound C. Thus $\exists$ a sequence (f_j) with $|f_j'(z_0)| \uparrow C$.

   (b) Since all $f \in \mathcal{F}$ are bounded by 1 ,apply Montel's theorem to get a subsequence (f_{n_j}) converging normally on U to a function f_0

   (c) By Cauchy's estimate $|f_0'(z_0)| = C$. Also, if f_0's image exceeds the open disk then f_0 is constant by max modulus thm, so f_0 maps U into the disk.

   (d) Show that f_0 is 1-1 using Hurwitz's theorem on the functions $g_j(z) = f_{n_j}(z) - f_{n_j}(b) \neq 0$ on $D \backslash \{b\}$, for each $b \in D$.

4. Show that f_0 satisfies the desired properties (i.e. it is onto) by contradiction: suppose $R \in D(0,1)$ is not in the image of f_0.Let $\phi(z)$ be the composition of f_0 with the Mobius transform sending R to 0.

Then since $f_0 \neq R$, $\phi \neq 0$ on U. Then we can define a single-valued square-root ψ of ϕ on U (again using simple connectivity). Although $\psi \notin \mathcal{F}$ since it is nonvanishing, we can modify it by composing it with Mobius transform sending $\psi(z_0)$ to 0 to get a final function $\Gamma \in \mathcal{F}$. Through some manipulation one can show that $|\Gamma'(z_0)| \geq |f_0'(z_0)| = C$, a contradiction.

□

Theorem 8. (Schwarz-Christoffel theorem)
Consider a convex polygon D with angles $\alpha_k \pi$ for $k = 1, ..., n$. The analytic functions that map the open unit disk biholomorphically onto D are of the form $F(z) = C \int_0^z \prod (z - z_k)^{\alpha_k - 1} dz + C'$ ***Depends on:*** dependencies...

Proof idea: Proof here □

6.3 Important examples

1. Montel's theorem does not have an analogue in $\mathbb{R}$. Consider the family $\{sin(kx)\}_{k=1}^{\infty}$. This sequence does not even have a pointwise-convergent subsequence.

7 Harmonic functions

Theorem 1. (Harmonic conjugates)
Let $\Sigma \subseteq \mathbb{C}$ be simply connected and let $u : \Sigma \to \mathbb{R}$ be a harmonic function. Then $\exists v : \Sigma \to \mathbb{R}$ harmonic s.t. $f = u + iv$ is analytic on Σ, i.e. u is the real part of an analytic fn on Σ.

Depends on: CR equations, analytic antiderivatives

Proof idea: We can verify that the function $\tilde{f}(z) = u_x - iu_y$ is analytic via the CR equations and the assumption on u. Thus $\tilde{f}$ has an analytic antiderivative F in Σ. One can show that the partial derivatives of $\Re F$ and u must agree and so they can only differ by a constant. □

Theorem 2. (Max principle for harmonic fns)
Let $\Sigma \subseteq \mathbb{C}$ be simply connected and let $u : \Sigma \to \mathbb{R}$ be a harmonic function. Then if the maximum value of u is attained in the interior of Σ, u is constant. In other words, the maximum value of u is attained on the boundary.

Depends on: harmonic conjugates, maxmod for analytic fns

Proof idea: We show that the set of points in Σ at which the maximum is achieved is clopen. It is obviously closed. To show it's open, take any such $z_0 \in \Sigma$ and take a fn e^{u+iv} analytic on some $D(z_0, \epsilon)$. By the max. principle for analytic fns, we have that f is constant on this disk and thus so is u. $\square$

Theorem 3. (Mean value property of harmonic fns)
Let $\Sigma \subseteq \mathbb{C}$ be simply connected and let $u : \Sigma \to \mathbb{R}$ be a harmonic function. For any $P \in \Sigma$ with $D(P, R) \subseteq \Sigma$, we have that $u(P) = \frac{1}{2\pi} \int_0^{2\pi} u(P+Re^{i\theta})d\theta$, i.e. $u(P)$ is the average value of the values of u on a circle around P.

Depends on: harmonic conjugates,CIF for analytic fns

Proof idea: We take an analytic fn $f = u + iv$ on a neighborhood of $\overline{D(P, R)} \subseteq \Sigma$. Apply the CIF to f and separate the integral into real and imaginary parts, and equate the real parts on each side.2 $\square$

Theorem 4. (Poisson integral formula for harmonic fns)
Let $\Sigma \subseteq \mathbb{C}$ be simply connected containing $\overline{D(0,1)}$ and let $u : \Sigma \to \mathbb{R}$ be a harmonic function. For any $P \in D(0,1)$, we have that $u(P) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \frac{1 - |P|^2}{|a - e^{i\theta}|^2} d\theta.$

Depends on: previous theorem and mobius transform conjugation, composing harmonic/anal. fns

Proof idea: First show that the composition of a harmonic fn with an analytic fn is harmonic (chain rule and computation). Then apply the previous theorem to $h(z) = u(\phi_{-a}(z))$ where $\phi_{-a}(z)$ is the Mobius transform taking 0 to a. Convert the real integral into a complex integral (multiply top/bottom by $e^{i\theta}$), and do a complex change of variable using the fact that $\phi'_{-a}(z) = \frac{1-|a|^2}{(1-\overline{a}z)^2}$. $\square$

Theorem 5. (Poisson integral formula for harmonic fns)
Let $f : \{|z| = 1\} \subseteq \mathbb{C} \to \mathbb{R}$ be cnts. Consider the problem of finding a fn u that is harmonic on $D(0,1)$, cnts on $\overline{D(0,1)}$ and agrees with f on $dD(0,1)$. The solution is exactly the one obtained by filling in interior values using the Poisson integral formula, i.e.:
$u(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{1 - |z|^2}{|z - e^{i\theta}|^2} d\theta$ on $D(0,1)$ and f on $dD(0,1)$.

Depends on: Poisson integral formula, change of variables

Proof idea: The key trick to showing that u is harmonic on $D(0,1)$ is to express:

$\frac{1-|z|^2}{|z-e^{i\theta}|^2} = \frac{e^{i\theta}}{e^{i\theta}-z} + \frac{e^{-i\theta}}{e^{-i\theta}-\overline{z}} - 1$. Then we can write $u(z)$ as the sum of 3 integrals, the first of which is the integral of a holomorphic fn (so it's zero), the second of which is the integral of a harmonic fn (so it's harmonic), and the third of which is constant (also harmonic). Showing continuity is complicated - see p215 Green and Krantz. . □

Theorem 6. (Schwarz reflection for harmonic functions)
Let u be a real-valued harmonic function on some simply connected region $\Sigma \subseteq \mathbb{C}$ whose intersection with the real line is exactly the interval (a, b). Let U be the portion of Σ above the real line and U^* conjugate of U. Suppose that $\forall x \in (a, b)$, $\lim\limits_{z \in U \to x} u(z) = 0$. Then the function $\hat{u}$ defined as u on U, 0 on (a, b), and $-u(\overline{z})$ on L is a harmonic fn on $W = U \cup (a, b) \cup U^*$ that agrees with u on U.

Depends on: MVP

Proof idea: We first note that if the MVP holds for $\hat{u}$ on any small disk in W, then we can conclude $\hat{u}$ is harmonic in W. To see this we first show that any fn satisfying the MVP on a disk also satisfies the max principle (easy). Then for a fn f that satisfies MVP on any little disk D, we can construct a harmonic fn h (via Poisson integral formula) agreeing with f on dD. Now, $f - h$ and $h - f$ both satisfy the MVP on D and are 0 on dD, so must be 0 on D. So f is harmonic on any arbitrary little disk D, so it's harmonic everywhere.

Finally, show that if we draw a small disk around any point in W, the MVP holds for $\hat{u}$. If the point is in U it follows since u is harmonic. If it's in L it follows by by def. of $\hat{u}$. If the point is in (a, b), split the integral along the circle into 2 integrals along the arc in U and L, and use the def. of $\hat{u}$ to show it vanishes. □

Theorem 7. (Schwarz reflection for analytic fns)
Let f be an analytic fn on some simply connected region $\Sigma \subseteq \mathbb{C}$ whose intersection with the real line is exactly the interval (a, b). Let U be the portion of Σ above the real line and U^* conjugate of U. Suppose that $\forall x \in (a, b)$, $\lim\limits_{z \in U \to x} \Im f(z) = 0$. Then the function $\hat{f}$ defined as f on U, $\lim\limits_{z \in U \to x} \Re f(z)$ on (a, b), and $\overline{f(\overline{z})}$ on L is an analytic fn on $W = U \cup (a, b) \cup U^*$ that agrees with f on U.

Depends on: Schwarz reflection for harmonic fns

Proof idea: The basic idea is as follows. For the upper half of any little disk D around $x \in (a, b)$, we look at $\Im f(z)$, which is harmonic. Thus it has an extension $\hat{v}(z)$to all of D by reflection. Now $\hat{v}$ has a harmonic conjugate $\hat{u}$ on D and the fn $\hat{f} = \hat{u} + i\hat{v}$ is analytic on D. But $\hat{f}$ and f must be the same on the upper half of D since their imaginary parts are the same by definition (or differ by a constant which we can subtract from $\hat{f}$). So f has an analytic extension $\hat{f}$ on D that is real-valued on $D \cap (a, b)$. One can show that $\hat{f}(z) = \overline{\hat{f}(\overline{z})}$ on D (they agree on the real axis and they're both analytic). Since the disk is arbitrary, we're done.

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