ODE Oral Exam Notes 2008

Chaitanya Ekanadham

1 Existence and uniqueness

1.1 Definitions

ODE/IVP, Lipschitz, C_1, local/global solution, method of successive approx.,

1.2 Useful Theorems

Theorem 1. Gronwall's lemma
Suppose g, β are nonneg. cnts functions and $g(t) \leq \alpha + \int_\tau^t \beta(s)g(s)ds$ on some interval $I \subseteq \mathbb{R}$. Then $g(t) \leq \alpha e^{\int_\tau^t \beta(s)ds}$ on I.

Depends on: FTOC

Proof idea: Look at the function $f(t) = exp(-\int_\tau^t \beta(s)ds)\int_\tau^t \beta(s)g(s)ds$. Since $f'(t) \leq \alpha\beta(t)exp(-\int_\tau^t \beta(s)ds)$ it follows by FTOC that $f(t) \leq \int_\tau^t \alpha\beta(s)exp(-\int_\tau^s \beta(r)dr)ds)$. Bound $\int_\tau^t \beta(s)g(s)ds = e^{\int_\tau^t \beta(s)ds}f(t)$ above by $\alpha e^{\int_\tau^t \beta(s)ds}$ using this inequality.

□

Theorem 2. Picard-Lindelof existence/uniqueness theorem
Suppose $f : (I \times U) \subseteq (\mathbb{R} \times \mathbb{R}^d) \to \mathbb{R}$ is continuous in (t, y) and uniformly Lipschitz in y. Then $\forall(\tau, \xi) \in (I \times U)$, the ODE $[\dot{y} = f(t, y)$ with $y(\tau) = \xi]$ has a unique solution on $\{t \in I : |t - \tau| < \alpha\}$ for some $\alpha > 0$.
Depends on: Completeness of $\mathbb{R}$, Gronwalls lemma (for uniqueness

Proof idea: 5 steps:

1. Choose a, b s.t. $\{t : |t - \tau| < a\} \subseteq I$ and $\{y : |y - \xi| < b\} \subseteq U$. Let $\alpha = \min a, \frac{b}{M}$ where M is an upper bound of $|f|$ on a closed set in $I \times U$ (so that f can't got outside of U on a small interval around τ).

2. Set $\varphi_0(t) = \xi$ and $\forall n \geq 0$ define $\varphi_{n+1}(t) = \xi + \int_\tau^t f(s, \varphi_n(s))ds$. The idea is that a fixed point of this iterative process must be a solution to the ODE (verify using FTOC). So it suffices to show that the sequence φ_n converges to a limit function φ that is a fixed point.

3. Show by induction that $\forall k \geq 0$, $|\varphi_{k+1}(t) - \varphi_k(t)| \leq \frac{MC^k|t-\tau|^{k+1}}{(k+1)!}$ where C is the Lipschitz constant of f. So $varphi_n(t)$ converges uniformly and absolutely to some cnts $\varphi(t)$ on $\{t \in I : |t - \tau| < \alpha\}$.

4. Show that $\varphi(t)$ is a fixed point of the process, i.e. that $|\varphi(t) - \int_\tau^t f(s, \varphi(s))ds| = 0$ by expanding $\varphi(t) = \lim \varphi_n(t)$ and using the uniform convergence and Lipschitz properties.

5. Show uniqueness by taking two solutions φ, Ψ that are fixed points of the iteration and showing that $|\varphi(t) - \psi(t)| = 0$ using Gronwall's lemma.

$\square$

Theorem 3. Cauchy-Peano existence theorem
Suppose $f : (I \times U) \subseteq (\mathbb{R} \times \mathbb{R}^d) \to \mathbb{R}$ is continuous in (t, y). Then $\forall (\tau, \xi) \in (I \times U)$, the ODE $[\dot{y} = f(t, y)$ with $y(\tau) = \xi]$ has a unique solution on $\{t \in I : |t - \tau| < \alpha\}$ for some $\alpha > 0$.

Depends on: Arzela-Ascoli theorem

Proof idea: The basic idea is to define a family of approximate solutions $\{\varphi_{\epsilon_n}(t)\}$, extract a convergent subsequence, and show that the limit function is a solution.

1. Define $\varphi_{\epsilon_n}(t) = \xi + 1_{t>\tau}[\int_\tau^t f(s, \varphi_\epsilon(s - \epsilon))ds]$.

2. Show that $\{\varphi_{\epsilon_n}(t)\}$ are uniformly bounded on some interval containing τ. Choose α small s.t. $\varphi_\epsilon(t)$ does not exit U on $\{t \in I : |t - \tau| < \alpha\}$ and also s.t. $|f| \leq M$ on $\{(t, \varphi_\epsilon(t)) : \tau \leq t \leq \tau + \alpha\}$.

3. Show that $\{\varphi_{\epsilon_n}(t)\}$ are equicontinuous by showing that they are unif. Lipschitz (because $|\varphi'_\epsilon(t)| \leq |f| \leq M$).

4. Apply Arzela-Ascoli theorem to extract a subsequence $(\varphi_{n_j}(t)) \to \varphi(t) \forall t \in [\tau, \tau + \alpha]$.

5. Show that $\varphi(t)$ is a fixed point of the iterative process using similar tricks to previous Theorem (use triangle inequality and ϵ, δ argument).

6. Do the same for $[\tau - \alpha, \alpha]$.

$\square$

1.3 Important examples

1. Continuity of f alone is not sufficient for uniqueness. Consider the ODE $\dot{y}(t) = |y(t)|^{1/2}$ on $t \in [0,1]$ with $y(0) = 0$. $y(t) \equiv 0$ and $y(t) = \frac{t^2}{4}$ are both solutions.
2. The Lipschitz condition on f is not necessary (example here).
3. An ODE can have EXACTLY

2 Properties/extension of solutions

2.1 Definitions

2.2 Useful Theorems

Theorem 1. Continuation of solutions
Suppose $y(t)$ is a solution to the ODE $[\dot{y} = f(t,y)$ with $y(\tau) = \xi]$ on $t \in (a,b)$. Let $D = I \times U$ be the domain of f. If f is bounded on D, then:

1. $\lim_{t \to b^-} y(t)$ and $\lim_{t \to a^+} y(t)$ exist
2. $(a, \lim_{t \to a^+} y(t)) \in D$ and/or $(b, \lim_{t \to b^-} y(t)) \in D \Rightarrow y(t)$ can be continued to the right and/or left.

Depends on: completeness of $\mathbb{R}$ i.e. cauchy criterion

Proof idea: Consider two points $t_1 < t_2 \in (a,b)$ approaching a^+. $|y(t_1) - y(t_2)| \leq \int_{t_1}^{t_2} |f(s,y(s))| ds \leq M|t_2 - t_1| \to 0$ as they both approach a (Likewise for $t_1, t_2 \to b^-$). By cauchy criterion, they $y(t)$ has a finite limit y_a at a. If $(a, y_a) \in D$, one can verify that the function $\tilde{y}(t)$ extending $y(t)$ to $[a,b)$ also satisfies the integral form of the ODE. Reapplying Cauchy-Peano at the point (a, y_a) gives the extension. □

Remark: The only case where continuation fails s if the finite limit is not in $Domain(f)$ or if f is unbounded (i.e. finite-time blowup). See example below.

Theorem 2. (Continuity w.r.t. initial conditions/parameters)
Suppose $f : D = (I \times U) \subseteq (\mathbb{R} \times \mathbb{R}^d) \to \mathbb{R}$ is continuous and Lipschitz in (t,y). Suppose $\psi(t)$ is a solution to the ODE $[\dot{y} = f(t,y)]$. Then $\exists \delta > 0$ s.t. $\forall (\tau, \xi) \in V_\delta \equiv \{(\tau', \xi') : \tau' \in (a,b), |\xi' - \psi(\tau')| < \delta\}$, $\exists$ a unique soln $\varphi(t, \tau, \xi)$

on (a,b) with φ cnts in (t,τ,ξ) on $(a,b)\times V_\delta$.

Depends on: Gronwall, continuation of solution, successive approximation, uniform limit of cnts fns is cnts

Proof idea: 3 steps:

1. Choose $\delta' > 0$ s.t. $V_\delta \subseteq D$ (one must exist). Choose $\delta < \delta' e^{-K(b-a)}$ where K is the Lipschitz constant of f. We know that $\forall(\tau,\xi)\in V_\delta$, $\exists!$ a soln $\varphi(t,\tau,\xi)$ on some small interval containing τ.

2. We can show that $|\varphi(t)-\psi(t)| \le |\xi-\psi(\tau)| + K\displaystyle\int_\tau^t |\varphi(s)-\psi(s)|ds$ $\forall t$ where φ is defined (just expand φ and ψ in their recursive integral form). Apply Gronwall's lemma to show that $|\varphi-\psi| \le \delta' \Rightarrow \varphi(t,\tau,\xi)$ can be continued throughout (a,b) (since $V_{\delta'}\subseteq D$).

3. Show $\varphi(t,\tau,\xi)$ is cnts by showing it is the uniform limit of the sequence of cnts fns defined by:
   $\varphi_0(t,\tau,\xi) = \psi(t)-\psi(\tau)+\xi$, $\varphi_{n+1}(t,\tau,\xi) = \xi + \displaystyle\int_\tau^t f(s,\varphi_n(s,\tau,\xi))ds.$
   Method is similar to before -

   (a) Show by induction that $\forall n\ge 0$, $|\varphi_{n+1}(t)-\varphi_n(t)| \le \frac{K^{n+1}|t-\tau|^{n+1}|\xi-\psi(\tau)|}{(n+1)!}$. This shows uniform convergence.

   (b) Also NTS that $|\varphi_n(t)-\psi|\le\delta'$ so that $\varphi_n(t)$ are all defined on (a,b) (do this by expanding $\varphi_n(t)$ as a telescoping sum). Thus the uniform limit $\varphi(t)$ satisfies desired ODE.

□

Remark: This extends the case where $f = f(t,y,\mu)$ depends on some parameter μ using $V_\delta = \{(\tau',\xi',\mu') : \tau'\in(a,b), |\xi'-\psi(\tau')| + |\mu'-\mu_0'| < \delta\}$.

Theorem 3. (Differentiability w.r.t. initial conditions/parameters)
Let $f, D, \psi, \varphi(t,\tau,\xi)$ be as in previous theorem and suppose also that $J(t,y) = [\frac{df_i(t,y)}{dx_j}]_{i,j}$ exists and is cnts on D. Then $\varphi(t,\tau,\xi)$ is C^1 and det $[\frac{d\varphi_i(t,\tau,\xi)}{d\xi_j}]_{i,j} = exp(\displaystyle\int_\tau^t tr J(s,\varphi(s))ds)$.
Depends on:

Proof idea: Proof is complicated. Refer to Thm 7.2 in Coddington. Very briefly:

1. To show $\frac{d\varphi_i(t,\tau,\xi)}{d\xi_j}$ exists, we note that it is the solution to the linear (matrix-valued) ODE $[\dot{y}(t) = J(t, \varphi(t, \tau, \xi))y]$ (Plug in solution, switch order of differentiation, and apply chain rule).

2. The det part follows from next theorem

$\square$

Theorem 4. (Properties of matrix-valued solutions to ODEs)
Suppose $A(t)$ is a cnts $n \times n$ matrix and that $\Phi(t)$ is a matrix-valued function satisfying the matrix ODE $[\Phi'(t) = A(t)\Phi(t)\ \forall t \in [a, b]$. Then $[\det \Phi(t)]' = (trA(t)) \det \Phi(t)$. As a consequence, $\det \Phi(t) = \det \Phi(\tau) exp(\int_\tau^t trA(s)ds)$.

Depends on:

Proof idea: Pure computation with some tricks. Use the permutation expansion of $\det \Phi(t)$ and then take $\frac{d}{dt}$ by repeated applying product rule to get that $[\det \Phi(t)]' = \sum_{j=1}^{n} \det \Phi_j(t)$ where $\Phi_j(t)$ is obtained by replacing the j^{th} row of $\Phi(t)$, $[\Phi_{j*}(t)]$ with $[\Phi'_{j*}(t)] = [\sum_{k=1}^{n} a_{*k}(t)\Phi_{kj}(t)]$. Perform row operations to normalize j^{th} row to $[a_{jj}(t)\Phi_{j*}(t)] \Rightarrow \det \Phi_j(t) = a_{jj}(t) \det \Phi(t)$, giving the result. $\square$

2.3 Important examples

1. Finite-time blow up (where continuation fails): consider the ODE $[\dot{y} = y^2]$ with $y(1) = -1$. The solution on $(0, \infty)$ is $y(t) = -t^{-1}$. But clearly $f(t, y) = y^2$ is not bounded on $(t, y) \in [0, 1] \times \mathbb{R}$, so the theorem cannot apply, which makes sense since the solutin blows up at $t = 0$.

3 Linear/Autonomous ODE

3.1 Definitions

autonomous ODE, linear ODE, critical point, positive/negative attractor, node, saddle, focus, center, stable/unstable manifold, first integral, fundamental matrix, linearly independent solutions

3.2 Critical points in 2D linear systems

Consider the ODE $\dot{x} = Ax$ where $A \in mathbbR_{2\times 2}$ is a nonsingular matrix and $x = (x_1, x_2) : \mathbb{R} \to \mathbb{R}^2$. Let λ_1, λ_2 be the eigenvalues of A and let $A = SJS^{-1}$ be the Jordan decomposition of A. Consider the following cases at the critical point at $x = 0$:

1. **Node:** λ_1, λ_2 are real and have the same sign.

   (a) A **diagonalizable:** look at $z = S^{-1}x$. Then $\dot{z} = S^{-1}Ax = JS^{-1}x = Jz \Rightarrow z_j = c_j e^{\lambda_j t}$ for $j = 1, 2$. If the eigenvalues are negative, 0 is a positive attractor since $z(t) \to 0 \Rightarrow x(t) = Sz(t) \to 0$. Otherwise it is a negative attractor. Notice also that we have $z_2 = C|z_1|^\alpha$ with $\alpha > \lambda_2/\lambda_1 > 0$

   (b) A **not diagonalizable:** $\lambda = \lambda_1 = \lambda_2$ and solving for z gives $z_2 = c_2 e^{\lambda t}$ and $z_1 = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$. If $\lambda < / > 0$, 0 is again a positive/negative attractor.

2. **Saddle:** λ_1, λ_2 are real and $\lambda_2 < 0 < \lambda_1$ (WLOG). Solving for $z(t)$ again gives $z_j(t) = c_j e^{\lambda_j t}$ for $j = 1, 2$. So $z_2 = c z_1^\beta$ with $\beta < 0$. So $z_1 \to \pm\infty$ and $z_2 \to 0$ as $t \to \infty$. There is a stable subspace $E_S = y - axis$ and unstable subspace $E_U = x - axis$ with $\mathbb{R} = E_S \oplus E_U$.

3. **Focus:** $\lambda_1, \lambda_2 = \mu \pm i\omega$ with $\mu, \omega \neq 0$. We can use the same transformation $z = S^{-1}x$ to get complex solutions $z_{1,2}(t) = c_{1,2} e^{\mu t} e^{\pm i\omega t}$. So $x = Sz_1$ is also a (possibly complex) solution. Noting that the real/imaginary part of a solution is again a solution (since A is real), we get solutions $\Re S z_1$ and $\Im S z_1$. For $\mu < / > 0$ these correspond to a spiral going radially inward/outward.

4. **Centre:** $\lambda_{1,2} = \pm i\omega$, $\omega \neq 0$. Following the same process above, we have solutions that are oscillatory. Since $z_{1,2}(t)$ are 2π-periodic, it follows that so are the solutions $x_{1,2}(t)$.

3.3 Useful Theorems

Theorem 1. Solutions to autonomous ODE's are phase-invariant
Suppose $x(t)$ solves the autonomous ODE $\dot{y} = f(y)$. Then $\forall t_0 \in \mathbb{R}$, $\hat{x}(t) = x(t - t_0)$ is also a solution.

Depends on: chain rule

Proof idea: follows directly from chain rule. □

Theorem 2. Let V be the set of solutions to the ODE $[\dot{x} = A(t)x]$ where $t \in I$, $x \in \mathbb{R}$, and $A(t) : \mathbb{R} \to \mathbb{R}_{n\times n}$ is continuous. Then V is vector space of dimension n.

Depends on: uniqueness of solutions, linear independence in regular vector spaces

Proof idea: Clearly the basic properties of a vector space holds in V (linear combinations of solutions are also solutions). In addition we can construct a basis for V by taking a basis $\{\xi_j\}$ for the range of the solution (e.g., the canonical basis of $\mathbb{R}^n$). Fixing $\tau \in I$, we have n unique solutions $\varphi_j(t)$to the ODE solving $\varphi_j(\tau) = \xi_j$. Then we can use the fact that $\{\xi_j\}$ is a basis to show that $V = span\{\varphi_j\}$ and that $\{\varphi_j\}$ are linearly independent (simply evaluate at τ). □

Theorem 3. (Necessary and sufficient conditions for a fundamental matrix) Suppose $M(t)$ solves the matrix-valued ODE $[\dot{M} = AM]$. Then M is a fundamental matrix $\Leftrightarrow \det M(\tau) \neq 0$ for some $\tau \in I$.

Depends on: uniqueness, linear algebra, property of the determinant of the solution of matrix-valued ODE's

Proof idea: Let $\varphi_j(t)$ be the n columns of M.

1. $\Rightarrow$: Suppose M is a f.m. Then $\sum c_j\varphi_j(t) = 0\ \forall t \Rightarrow c_j = 0,\ \forall j$. Fix $\tau \in I$ and suppose $\sum c_j\varphi_j(\tau) = 0$. By uniqueness (of the zero solution), we have that $\sum c_j\varphi_j(t) = 0\ \forall t \in I \Rightarrow c_j = 0\ \forall j$. So $\varphi_j(\tau)$ are l.i. and so $detM(\tau) \neq 0$.

2. $\Leftarrow$: If $\det M(\tau) \neq 0$ for some $\tau \in I$. By theorem 4 above we have $\det M(t) \neq 0\ \forall t \in I$. Thus the $\varphi_j(t)$'s must be l.i. and so M is a f.m.

□

Theorem 4. (Multiplication by nonsingular matrices preserves f.m's) Suppose $\Phi(t)$ is a f.m. for the ODE $\dot{y} = A(t)y$ and C is a nonsingular constant matrix. Then $\Phi(t)C$ is a f.m. and every f.m. is of the form $\Phi(t)\tilde{C}$ for some nonsingular $\tilde{C}$.

Depends on: linear algebra and matrix calculus, previous theorem

Proof idea:

1. $\Rightarrow$: $(\Phi(t)C)' = A(\Phi(t)C)$ and $(\Phi(t)C)$ is nonsingular since its determinant is the product of 2 nonzero determinants.

2. $\Leftarrow$: Take 2 f.m.'s $\Phi_1(t)$ and $\Phi_2(t)$. Let $\Psi(t) = \Phi_1^{-1}\Phi_2$ so that $\Phi_2 = \Phi_1\Psi$. Differentiating both sides wrt t (using product rule for RHS) and using the fact that Φ_1 and Φ_2 are solutions, we get $\Psi'(t) \equiv 0$ so Ψ is constant and is nonsingular since Φ_1 and Φ_2 are.

□

Theorem 5. (Adjoint systems)
Let $\Phi(t)$ be a f.m. for the ODE $[\dot{y} = A(t)y]$. Then $\Psi(t)$ is a f.m. for the ODE $[\dot{y} = -A(t)^*y] \Leftrightarrow \Psi^*\Phi = C$ for some constant nonsingular matrix C.

Depends on: previous theorem

Proof idea: Use the fact that $\Phi(t)\Phi(t)^{-1} = I \Rightarrow (\Phi(t)^{-1})' = -\Phi(t)^{-1}\Phi'(t)\Phi(t)^{-1} = -\Phi(t)^{-1}A(t) \Rightarrow \Phi^{-*}$ solves the adjoint system and apply the previous theorem (for both directions). □

Remark: If A is antisymmetric ($A = -A^*$) then we have $\Phi^*\Phi = C \Rightarrow$ all solutions have constant norm.

Theorem 6. (Solution for constant coefficient linear ODE)
Suppose $A_{n\times n}$ is a matrix and $\tau \in I \subseteq \mathbb{R}$, $\xi \in \mathbb{R}^n$. Then $\varphi(t) = e^{(t-\tau)A}\xi$ is a solution to the ODE $\dot{y} = Ay$ with $y(\tau) = \xi$.

Depends on: properties of the matrix exponential

Proof idea: Write out the definition of $e^{(t-\tau)A}$ as infinite series, and differentiate the sum term-by-term. □

Remark: e^A is typically computed by writing $A = SJS^{-1} \Rightarrow e^A = Se^JS^{-1}$ where J is the jordan form. If J is a Jordan block (with λ on the diagonal and 1's on the superdiagonal) then e^{tJ} has the same term $t^re^{\lambda t}/r!$ on the r'th superdiagonal.

Theorem 7. (Reduction of order for linear ODE's)
Let $A(t) : \mathbb{R} \to \mathbb{R}_{n\times n}$ and suppose $\{\varphi_j\}_{j=1}^m$ are m linearly independent solutions to the ODE $\dot{y} = Ay$ with $m < n$. Then we can reduce the problem of finding n linearly independent solutions the ODE by solving another ODE of dimension $n - m$.

Depends on: manipulations

Proof idea: 4 steps:

1. Construct the $n \times n$ matrix M to have $\{\varphi_j\}_{j=1}^m$ as the first m columns and the remaining columns as $\{e_j\}_{j=m+1}^n$. WLOG we can assume that the upper-left $m \times m$ submatrix has nonzero determinant on some interval $\tilde{I} \subseteq I$.

2. Make the change of variables $x = My$ to get the new ODE $U'y + Uy' = AUy$. Expand this out separately for the first m rows and the latter $n - m$ rows. Use the fact that φ_j's are solutions to simplify the equations.

3. The first system will allow you to solve $\{y_j'\}_{j=1}^m$ in terms of $\{\varphi_{ij}, a_{ik}, y_k\}_{k=m+1}^n$.

4. Plugging these into the second system results in a linear ODE system of $n - m$ variables $\{y_j\}_{j=m+1}^n$.

$\square$

Theorem 8. (Solution for nonhomogeneous linear ODEs)
Let $A(t) : \mathbb{R} \to \mathbb{R}_{n\times n}$ and $b(t) : \mathbb{R} \to \mathbb{R}^n$. Then if $\Phi(t)$ is a f.m. for the linear ODE $[\dot{y} = A(t)y]$ then $\varphi(t) = \Phi(t)[\xi + \int_\tau^t \Phi^{-1}(s)b(s)ds]$ is a solution to the the ODE $[\dot{y} = A(t)x + b(t),\, y(\tau) = \xi]$.

Depends on:

Proof idea: Guess a solution of the nohomogeneous ODE of the form $\varphi(t) = \Phi(t)\gamma(t)$ where $\gamma : \mathbb{R} \to \mathbb{R}^n$. Plug in to get the constraint $\gamma'(t) = \Phi^{-1}(t)b(t)$ and integrate to get the desired form of $\gamma(t)$.

$\square$

3.4 Important examples

1. A time-dependent matrix $A(t)$ can have linearly independent columns but $\det A(t) \equiv 0\ \forall t$. Consider $\varphi_1(t) = (t, 0)^T$ and $\varphi_2(t) = (t^2, 0)^T$. The point of the above theorem is that such matrices are not fundamental matrices for ODE's.

2.

3.

4 Linear systems with periodic coefficients

4.1 Definitions:

characteristic/Floquet multipliers + exponents, 'log' of a matrix

4.2 Useful theorems:

Theorem 1. (Factorization of f.m.'s for periodic systems)
Suppose $A(t)$ is T-periodic matrix. Then $\exists$ a T-periodic matrix $P(t)$ and a constant matrix R s.t. $\Phi(t) = P(t)e^{tR}$ is a f.m. for the ODE $[\dot{y} = A(t)y]$.

Depends on: Nonsingular conjugation of f.m.'s, invariance to phase-shift for periodic systems

Proof idea: $\Phi(t)$ is an f.m. $\Rightarrow \Phi(t+T)$ is an f.m. Therefore $\Phi(t+T) = \Phi(t)C$ for some nonsingular constant matrix C. Find a 'log' of C, R s.t. $C = e^{tR}$, define $P(t) = \Phi(t)e^{-tR}$. One can check that $P(t) = P(t+T)$ and clearly $P(t)e^{tR} = \Phi(t)$.

□

Remark:

1. One can derive the log of a nonsingular matrix by separately taking the log of each Jordan block. Since the eigenvalues are nonzero, we can always define the complex logarithm $log z = ln|z| + i \arg z$. For diagonal matrices the log is obvious, and for blocks with 1's on the superdiagonal and λ's on the diagonal, write $J = (\lambda I)(I + D)$ where D has zeros everywhere except λ^{-1} along the superdiagonal. Then use $log(1+z) = z - z^2/2 + z^3/3 - ...$ to compute $\log(I + D)$.

2. If we take another f.m. $\hat{\Phi}(t)$ s.t. $\hat{\Phi}(t)C = \Phi(t)$ then applying the same procedure in the proof gives $\hat{Phi}(t+T) = \hat{\Phi}(t)[Ce^{TR}C^{-1}]$ Thus all f.m.'s result in a family of similar matrices (e^{tR}) with unique nonzero eigenvalues called characteristic multipliers.

Theorem 2. (Properties of Floquet exponents/multipliers)
Suppose $A : \mathbb{R} \to \mathbb{R}_{n\times n}$ is T-periodic and let $\{\rho_j\}_{j=1}^n$, $\{\lambda_j\}_{j=1}^n$ be the Floquet multipliers and Floquet exponents of the ODE $[\dot{y} = A(t)y]$, respectively. Then $\displaystyle\prod_{j=1}^n \rho_j = e^{\int_0^T tr(A(s))ds}$ and $\displaystyle\sum_{j=1}^n \lambda_j = \frac{1}{T}\int_0^T tr(A(s))ds$ (mod $\frac{2\pi i}{T}$).

Depends on: Properties of the determinant of an f.m.

Proof idea: Let $\Phi(t)$ be an f.m. of the above ODE with $\Phi(0) = I$. From a previous theorem we have that $\det \Phi(t) = exp(\int_0^t tr(A(s))ds)$. Evaluating at $t = T$ and noting that $\det \Phi(T) = \det(P(T)) \det(e^{TR}) = \prod_{j=1}^{n} \rho_j$ (since $P(T) = P(0) = I$) gives the first result.

$\square$

Theorem 3. (Linearization of autonomous ODE about a periodic solution)

Suppose $\varphi(t)$ is a T-periodic solution to the autonomous ODE $[\dot{y} = f(y)]$. Then 1 is a floquet multiplier of the ODE $[\dot{y} = \frac{df}{dx}|_{\varphi(t)} y]$ obtained via linearization around $\varphi(t)$.

Depends on: Previous theorem

Proof idea: Noting that $\dot{\varphi}(t)$ solves the linearized ODE (by the chain rule), and that the linearized ODE has $A(t) = \frac{df}{dx}(\varphi(t))$ T-periodic, we can see that since we can complete an FM $\Phi(t)$ whose first column is $\dot{\varphi}(t)$ and use the fact that $\Phi(t + T) = \Phi(t)C \Rightarrow 1$ is an eigenalue of $C = e^{TR} \Rightarrow 1$ is a floquet exponent. $\square$

Theorem 4. (Wronskian and linear independence of solutions for special systems)
Suppose $\{\varphi_j\}_{j=1}^n$ are are n solutions to the ODE $[L_n y = 0]$ where $L_n = \sum_{j=0}^{n} a_j \frac{d^{n-j}}{dt^{n-j}}$ is the linear differential operator. Then $\{\varphi_j\}$ are linearly independent $\Leftrightarrow W(\varphi_1, ..., \varphi_n)(t) \neq 0 \ \forall t \in I$.

Depends on: properties of fundamental matrices

Proof idea: First note that the matrix which the Wronskian is a determinant of is a fundamental matrix iff the solutions are LI. By the linearity of the differential operator, clearly the functions $\{\varphi_j(t)\}_{j=1}^n$ are LD $\Leftrightarrow$ the vector-valued functions $\{\hat{\varphi}_j(t)\}_{j=1}^n$, with the kth component being the kth derivative, are LD $\Leftrightarrow W(\varphi_1, ..., \varphi_n)(\tau) \neq 0$ for some $\tau \in I \Leftrightarrow W \neq 0 \ \forall t \in I$.
$\square$

Theorem 5. (Wronskian gives the equation for a given solution)
Let $\{\varphi_j\}_{j=1}^n$ be C^n functions on I with $W(\varphi_1, ..., \varphi_n)(t) \neq 0$ on I.Then there exists a unique homogeneous differential equation of order n for which the

matrix $\Phi(t)$ formed by taking the jth column as $(\varphi_j \varphi_j' ... \varphi_j^{(n-1)})^T$ is a fundamental matrix.The ODE is:
$(-1)^n \frac{W(x,\varphi_1,...,\varphi_n)}{W(\varphi_1,...,\varphi_n)} = 0.$

Depends on: previous thoerem

Proof idea: Linear independence of the φ_j's comes from the previous theorem and the assumption that $W(\varphi_1, .., \varphi_n) \neq 0$ on I. Clearly each φ_j solves the equation. Also by an expansion of $W(x, \varphi_1, ..., \varphi_n)$ in the first column shows that is an nth order ODE of the desired type, i.e. $\sum a_j x^{(j)} = 0$ with $a_n = 1$. Uniqueness comes from the fact that the φ's are a basis for the solution space, and so the coefficient matrix is determined uniquely $(\dot{P}hi(t) = A(t)\Phi(t) \Rightarrow A(t) = \dot{P}hi(t)\Phi^{-1}(t))$.

□

Theorem 6. (Nonhomogeneous solutions with Wronskians)
Suppose $\{\varphi_j(t)\}_{j=1}^n$ are n LI solutions to the system $L_n y = 0$. Then a solution to the nonhomogeneous system $[L_n y = b(t), y(\tau) = \xi]$ is $\psi(t) = \psi_h(t) + \sum_{k=1}^{n} \varphi_k(t) \int_\tau^t \frac{W_K(\varphi_1, ..., \varphi_n)(s)}{W(\varphi_1, ..., \varphi_n)(s)} b(s) ds$ where ψ_h is the unique solution to the homogeneous system with the same initial condition and W_k is the same as the W except the kth column is replaced by $[0, ..., 0, 1]^T$.
Depends on: previous theorem on nonhomogeneous linear ODE, properties of Wronskian

Proof idea: This is just reinterpreting $L_n y = b(t)$ as a system of ODE's in n dimensions, applying the theorem for nonhomogeneous linear ODE, and using the Cramer's representation of the matrix inverse to write the coefficients using the Wronskian □

4.3 Important examples:

1.

2.

3.

5 Stability

Theorem 1. (Stability of solutions to constant coeff linear systems)
Consider the 0 solution to the ODE $[\dot{y} = Ay]$ for some matrix $A \in \mathbb{R}^{n \times n}$ with

eigenvalues $\{\lambda_j\}_{j=1}^n$.Then:

1. $\Re\lambda_k < 0\ \forall k \Rightarrow$ the 0 solution is asymptotically stable
2. $\Re\lambda_k \leq 0\ \forall k$ and λ_k with 0 real part are nondefective $\Rightarrow$ the 0 solution is stable
3. $\exists k$ s.t. $\Re\lambda_k > 0$ or a defective λ_k with 0 real part $\Rightarrow$ the 0 solution is unstable

Depends on: Jordan form and solution to constant coeff linear systems

Proof idea: This follows directly from the fact that the solution is of the form $\varphi(t) = \xi S e^{tJ} S^{-1}$ where $A = SJS^{-1}$. Clearly in the first 2 cases, $\|e^{tJ}\| \to 0$ whereas in the last case, it diverges. □

Theorem 2. (Stability of solutions to perturbed constant coeff linear systems)
Consider the 0 solution to the ODE $[\dot{y} = Ay + B(t)y + f(t,y)]$ for some matrix $A \in \mathbb{R}^{n\times n}$ with eigenvalues $\{\lambda_j\}_{j=1}^n$. Suppose the following hold:

1. $\|B(t)\| \to 0$ as $t \to \infty$
2. $\frac{|f(t,y)|}{|y|} \to 0$ as $|y| \to 0$ uniformly in t

Then if $\Re\lambda_k < 0\ \forall k$, the 0 solution is asymptotically stable. Furthermore, convergence is exponential, i.e. $\exists C, \delta, \mu$ s.t. $|y(t_0)| < \delta \Rightarrow |y(t)| \leq Ce^{-\mu(t-t_0)} \forall t \geq t_0$.

Depends on: Previous theorem,Duhummel-type formula, Gronwall's Lemma

Proof idea: By the assumptions we can show that $\|e^{tA}\| \leq Ce^{-\mu t}$ where $\mu = \max\{\lambda_k\} < 0$. We can write the solution as:
$\varphi(t) = e^{(t-\tau)A}\varphi(\tau) + \displaystyle\int_\tau^t e^{(t-s)A}[B(s) + f(s,\varphi(s))]ds.$
Choose T s.t. $t \geq T \Rightarrow \|B(t)\| \leq \epsilon$ and choose η s.t. $|x| < \eta \Rightarrow |f(t,x)| \leq \epsilon|x|$. Take absolute value of above expression and bound the part of the integrand in brackets by $2\epsilon|\varphi(s)|$. Then apply Gronwall's lemma to the function $g(t) = e^{\mu(t-\tau)}|\varphi(t)|$ to get that: $|\varphi(t)| \leq C|\varphi(\tau)|e^{(2\epsilon C-\mu)(t-\tau)}$.

□

Remark: Any solution of the ODE thus goes to 0 exponentiall, i.e. $\displaystyle\limsup_{t\to\infty} \frac{\log|\varphi(t)|}{t} \leq -\mu$.

Theorem 3. (Stability of solutions to periodic coeff linear systems)
Consider the 0 solution to the ODE [$\dot{y} = A(t)y$] for some periodic matrix $A(t) \in \mathbb{R}^{n \times n}$. The same exact stability results hold above if we substitue 'eigenvalues of the coeff matrix' with the Floquet exponents.

Depends on: Previous theorem,change of variable

Proof idea: We reduce the ODE to a linear constant coeff. ODE by guessing a solution of the form $y(t) = P(t)x(t)$ where $\Phi(t) = P(t)e^{tR}$ is the decomposition of the f.m. for the original ODE. The result is the ODE [$\dot{x} = Rx$], and we apply the above theorem. □

Theorem 4. (Stability of solutions to perturbed periodic coeff linear systems)
Consider the 0 solution to the ODE [$\dot{y} = A(t)y + B(t)y + f(t, y)$] for some matrix $A(t) \in \mathbb{R}^{n \times n}$ is T-periodic and the same assumptions on $B(t)$ and $f(t, y)$ hold. Then if the characteristic exponents of the system all have negative real part, the 0 solution is asymptotically stable.

Depends on: Previous theorem, decomposition of the fundamental matrix for periodic systems, Change of variables

Proof idea: Let $\Phi(t) = P(t)e^{tB}$ be the decomposed fundamental matrix for the linear system. Guess a solution $x(t) = P(t)y(t)$ for the nonlinear system. Substituting it in and manipulating (product rule etc.) gives an ODE for $y(t)$ of the form: $\dot{y} = By + P^{-1}F(t, P(t)y(t))$. Apply the previous theorem. □

Theorem 5. (Instability of solutions to perturbed constant coeff linear systems)
Consider the 0 solution to the ODE [$\dot{y} = Ay + B(t)y + f(t, y)$] for some matrix $A \in \mathbb{R}^{n \times n}$ with eigenvalues $\{\lambda_j\}_{j=1}^n$ with the same restrictions on B, f. Then if $\exists k$ s.t. $\Re\lambda_k > 0$, the 0 solution is unstable.

Depends on: dependencies...

Proof idea: Factor $A = SJS^{-1}$ and take $y = S^{-1}x$. Then $\dot{y} = Jy + S^{-1}F(t, Sy)$. Let R^2 and ρ^2 be the sum of squared-norms of the components of y corresesopnding to positive-real evals and negative-real-part evals, respectively. Show that $\dot{R} \geq \frac{\sigma}{2}R - \epsilon R$ and $\dot{\rho} \leq \epsilon(R + \rho) \Rightarrow \frac{d}{dt}(R - \rho) \geq \frac{\sigma}{4}(R - \rho)$, which implies exponential growth and thus instability. □

Theorem 6. (Relationship between stability of periodic solution and linearization about the solution)
Consider the ODE $[\dot{x} = F(t,x)]$ with F being T-periodic. Suppose $\varphi(t)$ is a T-periodic solution. Now consider the linearization about $\varphi(t)$, i.e. $[\dot{y} = \frac{dF}{dx}(t,\varphi(t))y + g(t,y)]$ (periodic coefficient linear system with $o(y^2)$ nonlinear term). Then:

1. If the n characteristic exponents of the linear system have negative real part, then $\varphi(t)$ is Lyapunov and asymptotically stable.

2. If $F = F(x)$ is independent of t (autonomous system) and $n-1$ char exponents have negative real part, then $\varphi(t)$ is orbitally and asymptotically stable.

Depends on: First variation and application of previous theorem on asuymptotic stability

Proof idea:

1. We see this by applying the previous theorem to the linearization (first variation) about $\varphi(t)$, which gives that 0 is an asymptotically stable solution to the first variation system. In other words, the difference of another solution to $\varphi(t)$ evolves according to the first variation and thus goes to 0 asymptotically, i.e. $\varphi(t)$ is asymptotically Lyapunov stable.

2. Since the system is autonomous, we know that $\dot{\varphi}(t)$ is a periodic solution to the first variation. If we denote $\Phi(t)$ as a fundamental matrix for the first variation system. We know that since $\Phi(t+T)$ is also a fundamental matrix it is related to $\Phi(t)$ by a nonsingular matrix with first column $e_1 = [100...0]^T$. Thus 1 is an eigenvalue of this matrix, i.e. 0 is a characteristic exponent. Clearly we cannot get Lyapunov asymptotic stability since $\varphi(t+\delta)$ is a solution which does not converge to $\varphi(t)$. However, we can get *asymptotic orbital stability*, even a stronger result:
$\exists C$ s.t. $|\varphi(t) - x(t+C)| \to 0$ as $t \to 0$.
The actual proof is really long and complicated - see Coddington p323-327.

□

Theorem 7. (Lyapunov functions and stability)
Consider the ODE $[\dot{y} = f(t,y)]$ with $f(t,0) = 0$. Then:

1. If we can find a positive definite function $V(t,x)$ (i.e. $\exists W(x)$ cnts s.t. $V(t,x) \geq W(x) > 0$ $\forall x \in D\backslash\{0\}$ and $t \geq t_0$) s.t. $\exists$ a neighborhood

of U around 0 on which $(L_t V)(t,x) \equiv \frac{\delta V}{\delta t} + f(t,x)\frac{\delta V}{\delta x} \leq 0$, then 0 is a stable solution.

2. If strict inequality holds ($L_t V < 0$ on U), then 0 is asymptotically stable.

3. If $L_t V$ is positive definite on U, then 0 is unstable. ***Depends on:*** dependencies...

Proof idea: 1. By assumption $V(t,x) \geq m > 0$ on some annulus $B(0,R)\backslash B(0,r)$ centered at 0 and there is a δ s.t. $0 \leq V(t,x) \leq m/2$ on $B(0,\delta)$. If a solution $x(t)$ starts in the δ-ball at $t = t_0$, then $V(t,x(t)) - V(t,x(t_0)) \leq 0$ by integrating the assumption on $L_t V \Rightarrow V(t,x(t)) < m/2 \Rightarrow x(t) \in B(0,r)$. But r is arbitrarily small, so 0 is stable.

2. Suppose there is a trajectory $x(t)$ starting in $B(0,a)$ at time t_0. Then we know that there is a δ s.t. if $x(t)$ enters $B(0,\delta)$ it can never come out (by previous step), so $|x(t)| > \delta \ \forall t \geq t_0$. By integrating the assumption, $V(t,x(t)) - V(t_0,x(t_0)) < -\mu(t-t_0)$ where $L_t V < -\mu < 0$, which contradicts the positive definiteness of V.

3. Similar argument to previous step. Take an annulus around 0 (with arbitrarily small outer ring) where $V(t,x)$ is bounded below and above by positive values. Take a trajectory starting in the annulus and integrate out the assumption to get that $V(t,x(t)) \to +\infty$.

□

Theorem 9. (Poincare-Bendixson theorem)
Consider the autonomous 2D system $[\dot{x} = f(x)]$ with $f : \mathbb{R}^2 \to \mathbb{R}^2$ C^1 on an open set $M \subseteq \mathbb{R}^2$. The asymptotic behavior of the trajectory starting from any point $x \in M$ must be one of 3 cases - it either tends to a fixed point, is a periodic orbit, or tends to a periodic orbit.

More formally, let $\Phi(x,t) : M \to \mathbb{R}^2$ be the location of the trajectory starting at x after time t. Let $\omega(x) = \{y \in M : \exists (t_n) \to \infty \text{ s.t. } \Phi(x,t_n) \to y\}$ (i.e. the set of points 'tended' to by the trajectory). Suppose we are given $x \in M$ with $\omega(x)$ compact and nonempty. Then if $\omega(x)$ does not contain fixed points, it is exactly a periodic orbit in M.
Depends on: lots of things to do with planar geometry

Proof idea: The general approach is as follows:

1. For a given $x \in M$, let T be the trajectory starting from x. Then $\omega(x)$ can intersect any transversal of T not more than once.

2. Any ω-limit-point of an ω-limit point lies on a periodic orbit

3. If $\omega(x)$ contains a nondegenerate periodic orbit P, then $\omega(x) = P$.

□

5.1 Important examples

1. Perturbed constant coeff linear system is given by $x'' + x + \mu x' + x^2 = 0$ where μ is a 'damping term'. Then the nonlinear term is just $[x^2 0]^T$ and the matrix in the linear term has eigenvalues $\frac{-\mu \pm \sqrt{\mu^2 - 4}}{2}$ which is ± 1 for $\mu = 0$.

2. Consider the orbital stability theorem on the 2D ODE system defined by the equation $x'' + f(x)x' + g(x) = 0$, assumoing there is some T-periodic solution $\varphi(t)$. The system is autonomous and we know that the sum of the char. exponenents $\lambda_1 + \lambda_2 = -\frac{1}{T}\int_0^T f(\varphi(s))ds$ and $\lambda_1 = 0$. It follows that if $\int_0^T f(\varphi(s))ds > 0$, then $\varphi(t)$ is asymptotically orbitally stable.

6 Perturbed systems

Theorem 1. (How different is a solution of a 'perturbed' system?)
Consider the perturbed system $[\dot{x} = f_0(t,x) + \epsilon f_1(t,x) + ... + \epsilon^m f_m(t,x) + \epsilon^{m+1} R(t,x)]$ with $x(t_0) = \eta$. Suppose that f_i is cnts in t and C^{m+1-i} in x for $1 \leq i \leq m$, and that R is cnts in both arguments. Then:
$|x - (x_0(t) + ... + \epsilon^m x_m(t))| \leq C\epsilon^{m+1}$ for $t \in [t_0, t_0 + h]$ (where C may depend on h) where $x = x_0 + \epsilon x_1 + ...$

Depends on: Gronwall's lemma, Duhummel's formula

Proof idea: $\dot{x}_i$ can be derived by substituting the ϵ-expansion of $x(t)$ in the original ODE and equating powers of ϵ. This gives a Duhummel formula for each x_i. To apply Gronwall's lemma to the expression to be bounded, subtract the sum of these Duhummel expressions from that of $x(t)$. Bound the integral using the smoothness assumption and apply Gronwall.

□

Theorem 2. (How different is a solution of a 'perturbed' nonautonomous system with periodic solution?)
Consider the perturbed system $[\dot{x} = g(t,x) + \epsilon h(t,x) = f(t,x,\epsilon)]$ with g, h T-periodic and f cnts in all arguments, Lipschitz in x. Suppose that the nonperturbed system has a T-periodic solution $p(t)$. If the first variation of the non-perturbed system has no T-periodic solution then the perturbed solution has a T-periodic solution for ϵ small.

Depends on:

Proof idea: □

Theorem 3. (How different is a solution of a 'perturbed' autonomous system with periodic solution?)
Consider the perturbed system $[\dot{x} = g(x) + \epsilon h(x,\epsilon) = f(x,\epsilon)]$ with f cnts in (x,ϵ) and C^1 in x. Suppose that the nonperturbed system has a T-periodic solution $p(t)$. If 1 is a simple floquet multiplier of the first variation of the nonperturbed system, there exists a periodic solution of the perturbed system with period $T(\epsilon)$.

Depends on:

Proof idea: □