

# ODE Oral Exam Abridged Notes 2008

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## 1 Existence and uniqueness

### 1. Gronwall's Lemma

- (a) **Statement:** If a nonnegative cnts fn  $g$  satisfies the condition that at any time  $t$ , it is bounded above by some constant plus the integral up to time  $t$  of itself weighted by another nonneg. cnts fn  $\beta$ , then its growth is subexponential with rate being the integral of  $\beta$  up to time  $t$  (in other words,  $\log g$  is bounded by the integral of  $\beta$ ).
- (b) **Depends on:** Manipulations and pure computation - no real intuition here.
- (c) **Used for:** This is a very technical lemma that is useful for bounding nonnneg. fns (usually the norm of the difference between 2 fns). Places where it is used include:
  - i. Uniqueness of Picard-Lindelof solution - applied to the norm of the difference between 2 candidate solutions
  - ii. Continuity wrt data - applied the norm of 2 different solutions for 2 different initial conditions
  - iii. Asymptotic stability for perturbed linear systems - applied to the norm of a solution to show that it decays exponentially

### 2. Picard-Lindelof - existence/uniqueness

- (a) **Statement:** If a  $f(t, x)$  is Lipschitz in  $x$  and cnts in  $(t, y)$  on some open time/space domain, then for every point  $(\tau, \xi)$  in the domain, there exists a *unique* solution satisfying this initial condition. The solution is guaranteed to exist at least in a small open time interval around  $\tau$ .
- (b) **Depends on:** Choosing the interval to be small enough to be making sense (i.e. so that the solution cannot escape the domain). Using the Duhummel formula to formulate an iteration scheme whose fixed point is necessarily a solution. Showing that the limit of the iteration exists and is indeed such a fixed point. Gronwall's lemma gives uniqueness.

**Used for:** duh

### 3. Cauchy-Peano - existence

- (a) **Statement:** If  $f(t, x)$  cnts in  $(t, y)$  on some open time/space domain, then for every point  $(\tau, \xi)$  in the domain, there exists a solution satisfying this initial condition. The solution is guaranteed to exist at least in a small open time interval around  $\tau$ . Note here that Lipschitz condition is not assumed, but that uniqueness is not guaranteed.
- (b) **Depends on:** Defining a family of recursively defined functions which are obtained by integration of oneself up to  $\epsilon$  back in time. Applying Arzela-Ascoli to this family to get a convergent subsequence (need uniform bound and equicontinuity - in fact their derivatives are uniformly bnded). Show that the limit fn satisfies the Duhummel formula.
- (c) **Used for:** duh

## 2 Properties of solutions and relationship with data

### 1. Continuation of solutions

- (a) **Statement:** If  $f$  is bounded on its domain, then we can always continue solutions to the endpoints of the interval on which a solution exists (Then we can reapply existence theorems at the endpoints to continue the solution further!).
- (b) **Depends on:** The cauchy criterion - just show that as we approach the endpoint of the interval the solution is converging to some point, plugging in the limit value does not change the fact that it satisfies the ODE.
- (c) **Used for:** getting global solutions (continuing throughout the real line for instance - use a contradiction argument to show that this can always be done as long as  $f$  is bounded). Finite-time blowup ( $f$  unbounded) is an important way in which continuation fails. The only other way it fails is if the limit at the endpoints is outside the domain.

### 2. Continuity and differentiability wrt initial data or parameters

- (a) **Statement:** If we have the continuity and Lipschitz conditions on  $f$  and a solution on an open interval satisfying some initial

condition, we can draw a ' $\delta$ -cylinder' around the solution trajectory (in the domain)  $D$  for which we can define a function that gives the value at time  $t$  of a solution satisfying a initial condition in  $D$ . This function is cnts wrt the initial condition  $(\tau, \xi)$ . This also generalizes to the case where  $f$  depends on a parameter as well. In addition, the degree of smoothness of this function will always match that of  $f$  with respect to its arguments.

- (b) **Depends on:** Choosing first a  $\delta'$ -band that fits in the domain, and cleverly choosing the real  $\delta$  so that the difference between two solutions with inital conditions in the  $\delta$ -band is always less than  $\delta'$  (this uses Gronwall's Lemma), which ensures that we can continue all solutions in this band throughout the original interval. Get continuity by iterating a sequence of (unif. cvgnt) cnts fns derived by applying the Duhummel formula. As the first iterate use the original solution translated (in space) to satisfy the given initial condition.
- (c) **Used for:** This basically says 'solutions starting close to each other stay close together on any *finite* time interval.' It does NOT say anything about *asymptotic* relationships between 2 solutions starting close by. Those issues are addressed by other theorems later.

### 3 Properties of fundamental matrices of linear systems

#### 1. The determinant of a fundamental matrix

- (a) **Statement:** For a linear system, the determinant of the fundamental matrix is the exponentated intergral of the trace of the coefficent matrix.
- (b) **Depends on:** Showing that the determinant satisfies a linear ODE where the coefficient is the trace. Do this by using the permutation expansion of the determinant.
- (c) **Used for:** Since the product of the eigenvalues is the determinant, this gives a relation on the product of the Floquet multipliers (or equivalently, the sum of the Floquet exponents) for periodic coefficient systems. In particular, if a 2D periodic linear system admits a periodic solution, then we know one exponent is 0, so the other is fully determined.

## 2. Basic properties of a fundamental matrices

- (a) **Statement:** A matrix is a fundamental matrix for an ODE on an interval iff it is nonsingular at *some* time  $\tau$  in the interval and satisfies the ODE in question. We can get any f.m. from another f.m. via right-multiplication by a nonsingular matrix. Finally, a f.m. for the adjoint system (where the coefficient matrix is negated and conjugate-transposed) is the inverse-adjoint of the f.m. for the original system (so we can get any f.m. for the adjoint system by right-multiplying nonsingular matrices).
- (b) **Depends on:** The first uses hinges on properties of linear independence from regular linear algebra. The second involves showing that the inverse of an f.m. times another f.m. has time-derivative 0. The third involves taking the derivative of the inverse of the f.m. of the original system.
- (c) **Used for:** The first fact is useful in seeing how f.m.'s are different from any matrix-valued function with L.I. columns. The right-multiplication by nonsingular matrices is useful in decomposing f.m.'s of periodic systems (since the f.m. is related to itself at a later time by a nonsingular matrix). Note that if the coefficient matrix is antisymmetric, then the adjoint and original systems are the same, and so the f.m. times its adjoint is a constant matrix, i.e. all solutions have constant norm (and pairwise-conjugate inner products).

## 4 Linear ODEs

### 1. Constant coeff. linear ODE

- (a) **Statement:** For a linear system with constant coefficient matrix, the f.m. is just the exponential of  $(t - \tau)$  times the coefficient matrix.
- (b) **Depends on:** properties of the matrix exponential - easy to verify.
- (c) **Used for:** This is very useful especially since we can use the Jordan form of the coefficient matrix and the exponential involves a Jordan-block matrix, which we can exactly characterize (each super diagonal has the same element). This is exactly why the

eigenvalues of the coefficient matrix are so important in determining stability of solutions (in particular, the 0 solution) of the ODE.

## 2. Reduction of order for linear ODE

- (a) **Statement:** If we know  $k$  linear indep. solutions to an  $n$ -dimensional ODE, we can reduce the problem of solving the remaining solution to the problem of solving a  $n - k$ -dimensional ODE.
- (b) **Depends on:** Any matrix of rank  $r$  has an  $r \times r$  nonsingular submatrix. So we can apply this principle to a 'filled out' f.m. with the first  $k$  columns our given solutions. Then make a change of variables with respect to this matrix to get an ODE for which we can solve the first  $k$  coordinates of the solution in terms of the others.
- (c) **Used for:** Not sure if this is practical - in 2D its obviously very useful.

## 3. Nonhomogeneous linear ODE

- (a) **Statement:** If we add a nonhomogeneous term  $b(t)$  to a homogeneous linear ODE, then given an initial condition and an f.m. for the homog. ODE, we can get a solution to the nonhomog. ODE by adding the homogeneous solution with the f.m. times the integral of its inverse times  $b$ .
- (b) **Depends on:** Variation of constants - easy to verify.
- (c) **Used for:** Obviously useful.

## 4. Periodic coefficient linear ODE

- (a) **Statement:** The f.m. of a periodic coefficient linear ODE can always be decomposed into a periodic matrix times the exponential of  $t$  times a constant matrix.
- (b) **Depends on:** The time-shifted f.m. is also an f.m., and all f.m.'s are related via nonsingular matrices. Any nonsingular matrix has a 'log' matrix.
- (c) **Used for:** This relates behavior of a periodic ODE to that of a constant coeff. ODE and motivates the definition of Floquet multipliers/exponents. The floquet multipliers are the eigenvalues of the exponential matrix in the decomposition, evaluated at  $t = T$ . The exponential factors at  $t = T$  for different f.m.'s are all

similar to each other. So we can define Floquet multiplier as the unique nonzero eigenvalues of these. One application is that we can characterize the stability of a periodic coeff. ODE just by looking at the characteristic exponents/multipliers.

## 5. Autonomous linear ODE with periodic solution

- (a) **Statement:** For any ODE and a solution we can define another ODE that is the first variation about this solution. If the solution is periodic, the first variation is a periodic system. If the original system is autonomous, then 1 is a Floquet multiplier of the first variation.
- (b) **Depends on:** Follows from the fact that if the system is autonomous, then by the chain rule, the derivative of the periodic solution is a solution to the first variation system. Writing the f.m. of the first variation as a matrix times itself one period later gives the relation.
- (c) **Used for:** This will be used along with the fact (proved later) that stability of a periodic solution is 'roughly equivalent' to stability of the 0 solution to the first variation around that fixed point. If the system is autonomous we only need all but one Floquet exponent to have negative real part.

# 5 Wronskians and n'th order linear ODE's

## 1. Wronskian and linear independence

- (a) **Statement:** Consider the special type of  $(n + 1)$ -dim ODE in which the  $n$ 'th order derivative is a linear combination of the  $n$  previous derivatives. Given  $n$  solutions, they are linearly independent iff their Wronskian is nonzero on the whole time interval. Conversely, given any  $n$  linearly indep. solutions, we can use the Wronskian to define the unique ODE of this special type for which the given functions are solutions.
- (b) **Depends on:** The first is just a restatement that solutions are linearly independent iff. the determinant of the matrix in which they are columns, is nonzero. For the second, take the ratio of the Wronskian with 'x' as the first column with the Wronskian of the given  $n$  functions.

- (c) **Used for:** This will be used to show that stability of a periodic solution is 'roughly equivalent' to stability of the 0 solution to the first variation around that fixed point. This is highly intuitive - if 0 is stable for the first variation, it means that for any solution starting close to the periodic solution, the different to the periodic solution should stay small, in some sense.

## 2. Wronskian and nonhomogeneous ODE

- (a) **Statement:** Given the abovementioned special type ODE and a fundamental matrix for it, we can derive a solution to a non-homogenous version of the ODE by simply adding the homogeneous solution and a weighted linear combination of the homogeneous solutions where the  $j$ 'th weight is the integral of the non-homogeneous term times the ratio of the Wronskian with  $j$ 'th solution removed to the regular Wronskian.
- (b) **Depends on:** This is basically a restatement of the general form of solution for nonhomogeneous linear ODE, but taking advantage of the special form. Use the 'Cramer' expansion for determinants.
- (c) **Used for:** no idea.

# 6 Stability

## 1. Stability of constant coeff. linear ODE

- (a) **Statement:** The stability of the 0 solution is completely determined by the Jordan form of the coefficient matrix. If the eigenvalues have all negative real part, then 0 is asymptotically stable. If all the eigenvalues have nonpositive real part and the ones with 0 real part do not have Jordan blocks, then 0 is stable. If any eigenvalue has positive real part, or if all eigenvalues are nonnegative and the ones with 0 real part are defective, then 0 is unstable.
- (b) **Depends on:** The form of the exponential of a Jordan matrix.
- (c) **Used for:** obvious. There is also an extension to perturbed const. coeff. linear systems below.