Real Analysis Oral Exam Notes 2008

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1 Basic properties of $\mathbb{R}$

1.1 Definitions

sup/inf, open/closed sets, G_δ and F_σ sets, borel sets, Lindelof property, finite intersection property, open cover, subcover.

1.2 Useful Theorems

Theorem 1. ($\mathbb{R}$ satisfies the Lindelof property)
$\bigcup_\alpha G_\alpha \supseteq A \Rightarrow \exists\{\alpha_j\}$ s.t. $\bigcup_j G_{\alpha_j} \supseteq A$

Depends on: axiom of choice, countability of rationals

Proof idea: For each $x \in A$, take an interval I_x with rational endpoints that contains x and is in some G_α from the open cover. Use countability. □

Theorem 2. (Heine Borel Theorem)
$C \subseteq \mathbb{R}$ closed and bounded $\Rightarrow$ every open covering $\{G_\alpha\}$ of C contains a finite subcover.

Depends on: existence of sup/inf

Proof idea: Do first for $C = [a, b]$. Let $A = \{x \in C : (a, x]$ has a finite subcover$\}$. Show that $\sup A = b$. Extend to arbitrary sets by adding C^c to $\{G_\alpha\}$, and applying to some interval $I \supseteq C$. □

Theorem 3. ($\mathbb{R}$ satisfies finite intersection property.)
Let $\{C_\alpha\}$ be a collection of closed sets in $\mathbb{R}$ containing at least one bounded set C. Then $\bigcap_{j=1}^{n} C_j \neq \phi\ \forall n \Rightarrow \bigcap_\alpha C_\alpha \neq \phi$

Depends on: Heine Borel (#2)

Proof idea: Since no finite union from $\{C_\alpha^c\}$ can cover C, $\{C_\alpha^c\}$ cannot cover C by Heine-Borel

□

Theorem 4. Every open set $O \subseteq \mathbb{R}$ is a disjoint countable union of open intervals $\bigcup_j I_j$

Depends on: def. of open set, sup/inf, countability of rationals

Proof idea: $\forall x \in O$, let I_x be the biggest open interval containing x s.t. $I_x \subseteq O$. Then any two intervals in $\{I_x\}$ are disjoint or the same, and $O = \bigcup_x I_x$. $\{I_x\}$ is countable since each I_x contains a distinct rational.

□

1.3 Important examples

1. Something not Lindelof
2. Spaces in which compactness is not equivalent to closed and bounded

2 Measure theory

2.1 Definitions

field, sigma field, semi-ring, (finite/sigma-finite) measure, countable additivity, outer measure, subadditivity, monotone class

2.2 Useful Theorems

Theorem 1. (Equivalent definitions of countable additivity)

1. $\forall (A_j)$ disjoint, $\mu(\bigcup_j A_j) = \sum_j \mu(A_j) \Leftrightarrow$
2. $\forall (A_j)$ increasing/decreasing, $\mu(\bigcup_j A_j / \bigcap_j A_j) = \lim_{j \to \infty} \mu(A_j) \Leftrightarrow$
3. $A_j \downarrow \phi \Rightarrow \lim_{j \to \infty} \mu(A_j) = 0$

Depends on: set theory, limits of sums

Proof idea: Reinterpret increasing/decreasing sequences of sets as countable disjoint unions and vice versa. □

Theorem 2. (Caratheodory extension theorem)
If $\mathcal{F}$ is a field of subsets of X and μ c.a. on $(X, \mathcal{F}) \Rightarrow \exists$ an extension μ^* of μ that is c.a. on $(X, \sigma(\mathcal{F}))$.

Depends on: properties of countable additity, definition of sigma algebra, monotone class theorem (for uniqueness only)

Proof idea: 5 steps:

1. Introduce outer measure μ^* (defined on 2^X) and show it is nonneg, monotonic, agrees with μ on $\mathcal{F}$ (easy) and is subadditive (use $2^-j\epsilon$ trick and def. of outer measure).

2. Define $S = \{E \in 2^X : \forall A \in 2^X,\ \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)\}$ to be the class of measurable sets. Show S is a σ-field (show finite unions which also implies finite additivity of μ on S, then take limit and use subadditivity to show countable unions).

3. Show μ^* is countably additive on S (take limit with finite additivity, and then use subadditivity for the reverse inequality)

4. Show that $S \supseteq \mathcal{F}$ (use the definition of outer measure and apply countable additivity of μ on $\mathcal{F}$).

5. Use monotone class theorem to show uniqueness (show that the class of subsets on which 2 extensions agree is a monotone class and therefore contains $\sigma(\mathcal{F})$).

$\square$

Theorem 3. A monotone class $\mathcal{M}$ that is a field is a σ-algebra.

Depends on: definitions

Proof idea: Take $A_n \in \mathcal{M}$ disjoint. Then since each partial union is in $\mathcal{M}$, and since monotone classes are closed under increasing limits, the result fullows. $\square$

Theorem 4. Monotone class theorem
Let $\mathcal{F}$ be a field and let $\mathcal{M}$ be the smallest monotone class containing $\mathcal{F}$. Then $\mathcal{F} \supseteq \sigma(\mathcal{F})$.

Depends on: Previous theorem

Proof idea: STS that $\mathcal{M}$ is an algebra. Define for each $A \in \mathcal{F}$ the set $\mathcal{M}_A = \{B \in \mathcal{M} : A \cap B \in \mathcal{M}, A \cap B^c \in \mathcal{M}, A^c \cap B \in \mathcal{M}\}$. Then one can show that $\mathcal{M}_A$ is a monotone class and is thus equal to $\mathcal{M}$. But this means that $\forall B \in \mathcal{M}$, $\mathcal{M}_B = \mathcal{M} \Rightarrow \mathcal{M}$ is an algebra. □

Theorem 5. Existence of Lebesgue measure
$\exists$ a c.a. measure λ on $(\mathbb{R}, \mathcal{B})$ that is translation-invariant and agrees with the length of intervals.

Depends on: Caratheodory extension, Heine Borel

Proof idea: Define $\lambda((a,b]) = b - a$ for all intervals $(a, b]$ and then show that the extension theorem applies. STS λ defined on the field $\mathcal{F}$ generated by these intervals is countably additive. We do this in 2 steps:

1. Show on the semi-ring of semi-closed intervals. Suppose $I = (a,b] = \bigcap_{j=1}^{\infty} I_j$. Clearly, $\lambda(I) \geq \sum \lambda(I_j)$. To show the reverse, make the I_j's $2^{-j}(\epsilon/2)$ bigger and open, and make I $\epsilon/2$ smaller and closed. Then the stretched I_j's are an open cover for the shrunken I, and then we can use Heine Borel to conclude that $\sum \lambda(I_j) \geq \lambda(I) - \epsilon$, $\forall \epsilon > 0$.

2. Show on the field. Suppose $A = \bigcup_{j=1}^{N} I_j = \bigcup_{j=1}^{\infty} A_j$ for some $A, A_j \in \mathcal{F}$ with A_j disjoint. Then show that $\lambda(A) = \sum_{j=1}^{\infty} \lambda(A_j)$ by commuting sums and applying the previous step (just manipulations).

□

2.3 Important examples

1. item1

2. item2

3 Measurable sets and functions

3.1 Definitions

measurable set (Caratheodory definition), measurable function, a.e. convergence, uniform convergence

3.2 Useful Theorems

Theorem 1. (Measurable sets are like open/closed sets)

1. E measurable $\Leftrightarrow$
2. $\forall \epsilon > 0\ \exists O \supseteq E$ open s.t. $\mu(O \cap E^c) < \epsilon \Leftrightarrow$
3. $\forall \epsilon > 0\ \exists C \subseteq E$ closed s.t. $\mu(E \cap C^c) < \epsilon \Leftrightarrow$
4. $\forall \epsilon > 0\ \exists$ a G_δ set $G \supseteq E$ s.t. $\mu(G \cap E^c) < \epsilon \Leftrightarrow$
5. $\forall \epsilon > 0\ \exists$ a F_σ set $F \subseteq E$ s.t. $\mu(E \cap F^c) < \epsilon$

Depends on: def. of outer measure, set theory

Proof idea: Finish this proof!

1. (1) $\rightarrow$ (2): use def. of outer measure to find a set of open intervals (tightly) containing E, use def. of E measurable to bound $\mu^*(O \cap E^c)$
2. (2) $\rightarrow$ (4): Take $\epsilon_n = 1/n$ in (2) to get open set O_n. Take $G = \bigcap_n O_n$
3. (4) $\rightarrow$ (???)

$\square$

Theorem 2. Alternative definitions of real-valued measurable functions:
$f : (X, \Omega) \rightarrow (\mathbb{R}, \mathcal{B})$ measurable $\Leftrightarrow$
$\forall a \in \mathbb{R}$, $\{x \in X : f(x) <, >, \leq, \geq, = a\} \in \Omega$

Depends on: commuting of preimage and countable unions, def. of sigma algebra

Proof idea: $\Rightarrow$ is clear.
$\Leftarrow$: NTS $\forall A \in \mathcal{B}$, $f^{-1}(A) \in \Omega$
$f^{-1}(I_j) \in \Omega \Rightarrow f^{-1}(\bigcup_j I_j) \in \Omega$ since preimage and countable unions commute. $\geq$ is the complement of $<$, and $\leq$ is the countable union of $<$. $\square$

Remark: It is easily shown that measurable functions are closed under $\{+, *, \sup, \inf, \liminf, \limsup, \lim\}$

Theorem 3. (Measurable fns on finite intervals are almost simple/step/cnts)
Let $f : [a, b] \to \mathbb{R}$ be measurable s.t. $f = \pm\infty$ on a set of measure 0. Then:

1. $\forall \epsilon > 0$, $\exists$ a step fn $g : [a, b] \to \mathbb{R}$ and a set $A \subseteq [a, b]$ s.t. $\mu([a, b] \cap A^c) < \epsilon$ and $|f(x) - g(x)| < \epsilon$ on A
2. $\forall \epsilon > 0$, $\exists$ a cnts fn $h : [a, b] \to \mathbb{R}$ and a set $B \subseteq [a, b]$ s.t. $\mu([a, b] \cap B^c) < \epsilon$ and $|f(x) - h(x)| < \epsilon$ on B

Depends on: finite measure set, def. of measurable function/set, simple functions

Proof idea: This is an $\epsilon/3$ argument:

1. Find M s.t. $|f| \leq M$ except on a set S_1 of measure $< \epsilon/3$
2. Approximate f on ${S_1}^c$ by a simple function ϕ to ϵ-accuracy by binning the range $[-M, M]$.
3. Approximate ϕ by a step function g by approximating the support sets A_j of ϕ by disjoint intervals I_j s.t. $g = \phi$ except on a set S_2 of measure $< \epsilon/3$
4. Approximate g on $[a, b]$ by a cnts function h except on a set S_3 of measure $< \epsilon/3$ (connect the steps)

$\square$

Remark: This is combined Egoroff's theorem to show that f is the uniform limit of cnts functions (see Lusin's Theorem)

Theorem 4. ([Nonnegative] measurable fns are [monotone] pointwise limit of simple functions)
Let $f : X \to \mathbb{R}$ be measurable. Then $\exists \phi_n$ simple s.t. $\forall x \in X$, $\phi_n(x) \to f(x)$
If $f \geq 0$ then we can construct (ϕ_n) to be a monotone sequence.

Depends on: definition of measurable fn

Proof idea: Construct ϕ_n by binning the range $[-n, n]$ (or $[0, n]$ if $f \geq 0$) into finitely many bins and let ϕ_n equal the lower bound on the preimage of each bin. Set $\phi_n(x) = n$ on $f^{-1}([-n, n]^c)$ (or $f^{-1}([0, n]^c)$ if $f \geq 0$). $\square$

Remark: If f is bounded, then convergence is uniform.

Theorem 5. Egoroff's Thm (A.e. cvgnce on finite-measure sets is "almost" uniform) Let $f_n \to f$ a.e. on $E \subseteq X$ with $\mu(E) < \infty$. Then $\forall \epsilon > 0, \delta > 0, \exists A_{\epsilon,\delta} \subseteq E, N$ s.t. $\mu(E \cap A^c) < \delta$ and $n > N \Rightarrow |f_n - f| < \epsilon$ on A. Furthermore, $\exists C_\delta \subseteq E$ s.t. $\mu(E \cap C^c) < \delta$ and $f_n \to f$ uniformly on C

Depends on: finite measure of E

Proof idea: Let $G_n = \{x \in E : \exists k \geq n \text{ s.t. } |f_k(x) - f(x)| \geq \epsilon\}$ Then $G_n \downarrow \phi \Rightarrow \exists N$ s.t. $\mu(G_N) < \delta$. Let $A_{\epsilon,\delta} = E \cap {G_N}^c$
We can get C_δ by repeatedly applying above argument to get $A_{1/n, 2^{-n}}$ and taking $\bigcap_n (A_{1/n,2^{-n}})^c$.

$\square$

Theorem 6. Lusin's Thm (Measurable fns on intervals are cnts except on small sets)
Let $f : [a, b] \to \mathbb{R}$ be measurable. Then$\forall \delta > 0$, $\exists g$ cnts on $[a, b]$ s.t. $\mu(\{x \in [a, b] : f \neq g\}) < \delta$

Depends on: Approx. of measurable fns by cnts fns, Egoroff's Theorem, the uniform limit of continuous functions is continuous

Proof idea: $\epsilon/3$ argument:

1. Find a set $S_1 \subseteq [a, b]$ of measure $< \delta/3$, outside of which $|f| \leq M$.

2. Construct cnts functions $h_n \to f$ outside a set $S_2 \subseteq {S_1}^c$ of measure $< \delta/3$.

3. Apply Egoroff's Thm to get a set $S_3 \subseteq ({S_1}^c \cap {S_2}^c)$ of measure $< \delta/3$ s.t. $h_n \to f$ uniformly $\Longrightarrow f$ cnts outside the set $(S_1 \cup S_2 \cup S_3)$ of measure $< \delta$.

$\square$

3.3 Important examples

1. A nonmeasurable set:

   (a) Take $[0,1]/\mathbb{Q}$ i.e. $x \sim y \Leftrightarrow (x-y) \in \mathbb{Q}$ (equivalence relation).

   (b) Use axiom of choice to construct a set P containing exactly one element from each equivalence class.

   (c) If P is measurable, then $P_i = P + q_i$ is also measurable and has same measure (by translation invariance of Leb. measure), $\forall q_i \in \mathbb{Q} \cap [0,1]$.

   (d) But we know $P_i \cap P_j = \phi$ for $i \neq j$ (use the definition of P), and also that $[0,1) \subseteq \bigcup_i P_i$ (every $x \in [0,1)$ is in some P_i).

   (e) But then $\mu([0,1)) = \sum_i \mu(P_i) = \sum \mu(P) = 0$ or $\infty \Rightarrow$ contradiction (we know the measure is less than 3 but not 0).

2. Nonmeasurable functions: characteristic function of a nonmeasurable set

3. The approximation of measurable fns by cnts fns fails for infinite measures: consider $f(x) = x^2$ on $\mathbb{R}$

4. Egoroff's Thm fails for infinite measures: consider $f_n(x) = 1_{[-n,n]}$, $f = 1$

4 Integration and convergence

4.1 Definitions

integral of nonnegative simple fn, integral of nonnegative measurable fn, integral of measurable fn, a.e. convergence, convergence in measure

4.2 Useful Theorems

Theorem 1. (Bounded convergence theorem)
Suppose $f_n \to f$ a.e. on E with $\mu(E) < \infty$ and $\forall n\ |f_n| \leq M$. Then $\int_E f_n d\mu \to \int_E f d\mu$

Depends on: finite measure space, Egoroff's Theorem

Proof idea: By Egoroff take $A \subseteq E$ s.t.$f_n \to f$ uniformly on $E \cap A^c$ with $\mu(A)$ arbitrarily small. Then $|\int_E (f_n - f)d\mu| \leq \int_A |f_n - f|d\mu + \int_{E\cap A^c} |f_n - f|d\mu$. Both integrals can be bounded by $\epsilon/2$ for n sufficiently large. □

Theorem 2. (Fatou's lemma)
Suppose $f_n \to f$ a.e. on E with $f_n \geq 0$. Then $\int_E f d\mu \leq \liminf \int_E f_n d\mu$
Depends on: definition of integral of nonnegative measurable function

Proof idea: STS $\forall$ simple $\phi \leq f$, $\int_E \phi d\mu \leq \int_E f_n d\mu$ for suff. large n. If $\int_E \phi d\mu = \infty$ show that eventually $\int_E f_n d\mu = \infty$ (there is a support set of infinite measure on which $\phi = a > 0$).
Otherwise, restrict E to where $\phi > 0$ and look at the sets $A_n = \{x \in E : \forall k \geq n,\ f_k(x) > (1-\epsilon)\phi(x)\}$. Then $A_n^c \downarrow \Phi$ so $\mu(B_n) < \epsilon$ for n suff. large. So for n suff. large, we can show that $\int_E f_n d\mu \geq \int_E \phi d\mu - C\epsilon$ by restricting the integral to A_n, comparing to ϕ, and using the fact that ϕ is bounded and A_n is small. □

Theorem 3. (Monotone convergence theorem)
Suppose $f_n \to f$ a.e., monotonically from below on E with $f_n \geq 0$. Then $\int_E f_n d\mu \to \int_E f d\mu$

Depends on: Fatou's lemma

Proof idea: By monotonicity, $\limsup \int_E f_n d\mu \leq \int_E f d\mu$. By Fatou, $\liminf \int_E f_n d\mu \geq \int_E f d\mu$ □

Remark: We can prove Fatou from MCT and vice versa

Theorem 4. (Dominated convergence theorem)
Suppose $f_n \to f$ a.e. on E, and $\exists g$ integrable s.t. $\forall n$, $f_n \leq g$ on E. Then $\int_E f_n d\mu \to \int_E f d\mu$

Depends on: Fatou's lemma

Proof idea: Apply Fatou's lemma to $(f + g_n)$ and $(f - g_n)$ to show that $\int_E f d\mu \leq \liminf \int_E f_n d\mu$ and $\int_E f d\mu \geq \limsup \int_E f_n d\mu$, respectively. □

Theorem 5. (Generalized DCT)
Suppose $f_n \to f$ a.e. on E, and $\exists g_n \to g$ a.e. on E s.t. $\forall n$, $f_n \le g_n$ on E, (g_n) and g are integrable, and $\displaystyle\int_E g_n d\mu \to \int_E g d\mu$. Then $\displaystyle\int_E f_n d\mu \to \int_E f d\mu$

Depends on: Regular DCT

Proof idea: Proof follows same format as DCT with more manipulation. □

4.3 Important examples

1. The BCT fails for infinite measure spaces: consider $f_n(x) = 1/n \to 0$ on $\mathbb{R}$.

2. The MCT and Fatou's lemma can fail if the f_n's are not nonnegative: consider $f_n(x) = \frac{-1}{n} \to 0$ pointwise but $\liminf$ of the integrals is $-\infty$.

5 L_p spaces

5.1 Definitions

$L_p(X, \Omega, \mu)$ - make sure to specify how to deal with functions that at are equal a.e., $\|.\|_p$, completeness, vector space, banach space, ess sup, normed space, L_p convergence, Cauchy-L_p convergence, Cauchy-measure convergence

5.2 Useful Theorems

Theorem 1. (Minkowski's Inequality, i.e. triangle inequality for L_p)
$\forall p \ge 1$, $f, g \in L_p \Rightarrow \|f + g\|_p \le \|f\|_p + \|g\|_p$
Depends on: convexity of $|x|^p$

Proof idea: For $p = \infty$, use properties of ess sup
For $p < \infty$, $|f+g|^p \le (|f|+|g|)^p = (\|f\|_p+\|g\|_p)^p(\frac{|f|}{\|f\|_p}\frac{\|f\|_p}{\|f\|_p+\|g\|_p}+\frac{|g|}{\|g\|_p}\frac{\|g\|_p}{\|f\|_p+\|g\|_p})^p \le (\|f\|_p + \|g\|_p)^p((\frac{|f|}{\|f\|_p})^p\frac{\|f\|_p}{\|f\|_p+\|g\|_p} + (\frac{|g|}{\|g\|_p})^p\frac{\|g\|_p}{\|f\|_p+\|g\|_p})$
Integrating both sides and take $\frac{1}{p}$-power. □

Theorem 2. (Holder's inequality)
Suppose $1 \le p, q \le \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then $\forall f \in L_p$, $\forall g \in L_q$, $fg \in L_1$ and $\|fg\|_1 \le \|f\|_p\|g\|_q$

Depends on: concavity of log

Proof idea: The case $p = 1$, $q = \infty$ or if $\|f\|_p = 0$ or $\|g\|_q = 0$ is easy. Otherwise, first show Young's inequality: $\forall A, B \geq 0$, $AB \leq \frac{A^p}{p} + \frac{B^q}{q}$. To this by taking log of both sides and using concavity of log. Then plug in $A = f/\|f\|_p$ and $B = g/\|g\|_q$ and integrate both sides. $\square$

Theorem 3. Riesz-Fischer Theorem (completeness of L_p)
(f_n) cauchy in L_p sense $\Rightarrow (f_n) \to f \in L_p$ in L_p sense.

Depends on: absolute convergence of series a.e. implies regular convergence a.e., Fatou's Lemma, Minkowski Inequality, Dominated Convergence Theorem

Proof idea: For $p = \infty$, use the completness of $\mathbb{R}$ and properties of ess sup (basically use uniform covergence). For $p < \infty$, the basic strategy is to construct a subsequence g_n that converges a.e. and in L_p to a limit function $f \in L_p$ and then use the cauchy property to show that $f_n \to f$ in L_p.

1. Extract a subsequence g_n s.t. $\|g_{n+1} - g_n\|_p < 2^{-n}$
2. Let $g(x) = |g_1(x)| + \sum_k |g_{k+1}(x) - g_k(x)|$. Then use Fatou's lemma and Minkowski inequality to show that $g \in L_p$. Conclude that the series defining g converges a.e.
3. Since the series defining g is an absolute series, this means that the series without absolute values must converge a.e. Let $f(x)$ be this limit, i.e. $f(x) = g_1(x) + \sum_k (g_k(x) - g_{k+1}(x))$. Then $g_k \to f$ a.e.
4. Since $|f - g_n|^p \leq |2g|^p$ integrable, by DCT we have $f \in L_p$ and $g_k \to f$ in the L_p sense.
5. Use Cauchy-L_p property and Fatou's Lemma to show that $f_n \to f$ in L_p sense (by $\epsilon/2$ argument using g_n's) .

$\square$

Theorem 4. (Dominated convergence theorem for L_p)
Suppose $f_n \to f$ a.e. and $|f_n| \leq g \in L_p$. Then $f \in L_p$ and $f_n \to f$ in L_p sense.

Depends on: Regular DCT

Proof idea: Use $|f_n - f|^p \leq |2g|^p$ and apply DCT $\square$

Theorem 5. (L_p fns are almost bnded/cnts/step fns on finite-measure sets)
Suppose $f \in L_p(X, \Omega, \mu)$ with $\mu(X) < \infty$ and $1 < p \leq \infty$. Then $\forall \epsilon > 0$, $\exists$ a bounded fn $\overline{f}$, a step fn g and cnts fn h s.t. $\|f - \overline{f}\|_p < \epsilon$, $\|f - g\|_p < \epsilon$, and $\|f - h\|_p < \epsilon$.

Depends on: finite measure, approximation of measurable fns on finite-measure sets, L_p-DCT, Minkowski's inequality

Proof idea: 4 steps:

1. Approximate f in L_p by a bounded function $\overline{f}$ simply by using a cutoff value N to define f_N and applying L_p-DCT to $\|f - f_N\|_p$.
2. Apply approximation of bnded measurable fns on finite-measure sets to get a step function g s.t. $|g - \overline{f}|$ is small outside a set S_1 of small measure. Bound $\|\overline{f} - g\|_p{}^p$ by splitting the integral into integrals over S_1 and $X \cap S_1^c$
3. Since $\|f - \overline{f}\|_p$ small and $\|\overline{f} - g\|_p$ small, use Minkowski's and a standard $\epsilon/2$ argument.
4. To get a cnts fn, just approxmate $g(x)$ except on a set of arbitrarily small measure.

$\square$

Theorem 6. (L_∞ norm as the limit of L_p norm for finite measures)
Let (X, Ω, μ) be a measurable space with $\mu(X) < \infty$. Then $\forall f : X \to \mathbb{R}$ measurable, $\lim\limits_{p \to \infty} \|f\|_p = \|f\|_\infty$.

Depends on: finite measure, properties of ess sup, $\lim\limits_{p \to \infty} x^{1/p} = 1$

Proof idea: 2 parts:

1. $\|f\|_\infty \geq \limsup \|f\|_p$: $\forall B$ s.t. $|f| \leq B$ a.e., $\|f\|_p{}^p = \int |f|^p d\mu \leq B^p \mu(X)$. Take $1/p$ power of both sides and take limit.
2. $\|f\|_\infty \leq \liminf \|f\|_p$: Let $B = essup(f)$. Then $B - \epsilon$ is not $\Rightarrow |f| > B - \epsilon$ on a set of measure $\delta > 0 \Rightarrow \int |f|^p \geq (B - \epsilon)^p \delta$. Take $1/p$ power of both sides.

$\square$

Theorem 7. (Inclusion of L_p spaces for finite measures)
If $\mu(X) < \infty$. Then $p \leq q \Rightarrow L_q \subseteq L_p$.
Depends on: finite measure

Proof idea: For $q = \infty$ it is obvious. Otherwise, take $f \in L_q$ and let $E = \{x \in X : |f(x)| < 1\}$. Then $\|f\|_p = \int_X |f|^q d\mu = \int_E |f|^p d\mu + \int_{E^c} |f|^p d\mu \leq \mu(X) + \|f\|_q < \infty$. $\square$

5.3 Important examples

1. For infinite measures, $p \leq q \nRightarrow L_q \subseteq L_p$: consider $f(x) = 1/x$. One can show by the integral test that $f \in L_2([1, \infty), \Omega, l)$ but $f \notin L_1([1, \infty), \Omega, l)$. For any measure, $p \leq q \nRightarrow L_p \subseteq L_q$. Consider $f(x) = x^{-1/2}$. $f \in L_1([0, 1], \Omega, l)$ but $f \notin L_2([0, 1], \Omega, 1)$.

6 Modes of convergence

6.1 Definitions

uniform integrability, types of convergence (a.e., a.u., pointwise, uniform, measure, Lp, and cauchy versions of all of these)

6.2 Summary of relationships between modes of convergence

1. General measures:

   (a) measure $\leftrightarrow$ cauchy-measure (see below)

   (b) $L_p \leftrightarrow$ cauchy-L_p (Riesz-Fischer Thm)

   (c) pointwise/a.e./uniform $\leftrightarrow$ cauchy pointwise/a.e./uniform, resp. (completeness of $\mathbb{R}$)

   (d) a.u. $\leftrightarrow$ cauchy-a.u. (see below)

   (e) uniform $\rightarrow$ pointwise $\rightarrow$ a.e. (obvious)

   (f) a.u. $\rightarrow$ a.e. (see below)

   (g) $L_p \rightarrow$ measure $\rightarrow$ subsequence a.e. and subsequence a.u. (see below)

   (h) a.e. $\rightarrow$ subsequence in measure (see below)

   (i) (measure + unif. integrable + "pseudo-finite-measure") $\rightarrow L_p$

2. Finite measures (in addition to the those for general measures)
   (a) a.e. $\rightarrow$ measure and a.u. (Egoroff's Thm)
   (b) (measure + unif. integrable) $\rightarrow L_p$
   (c) (uniformly bounded p-norm, $p > 1$) $\rightarrow$ unif. integrable.
3. Dominated convergence (in addition to all those above)
   (a) measure (and therefore also a.e. and a.u.) $\rightarrow L_p$

6.3 Useful theorems

Theorem 1. (Cauchy-measure convergence is equivalent to measure convergence)
Let (f_n) be measurable fns on (X, Ω, μ). Then (f_n) are cauchy in measure on $E \subseteq X \Leftrightarrow \exists f$ measurable s.t. $f_n \rightarrow f$ in measure on E.

Depends on: $\limsup$ of a set, completeness of $\mathbb{R}$

Proof idea: $\Leftarrow$ is clear. For $\Rightarrow$:

1. Construct subsequence g_k s.t. $\mu(\{x : |g_{k+1} - g_k| \geq 2^{-k}\}) < 2^{-k}$
2. Define: $E_k = \{x \in E : |g_{k+1} - g_k| \geq 2^{-k}\}$, $F_k = \bigcup_{n \geq k} E_n$

   $G = \limsup E_k = \mu(\bigcap_n F_n)$
3. Show that $\forall k$, $\mu(F_k) < 2^{-k+1}$ and so $\mu(G) = 0$. Show that $(g_n(x))$ is a *uniformly* cauchy sequence and therefore converges uniformly to a limit $f(x)$ outside G.

   Details: $m > n \geq k \Rightarrow \forall x \notin F_k$, $|g_m(x) - g_n(x)| \leq |g_m(x) - g_{m-1}(x)| + ... + |g_{n+1}(x) - g_n(x)| \leq 2^{-n+1} \Rightarrow \forall x \notin F_k$, $(g_n(x))$ is uniformly cauchy. Let $f(x)$ be the limit fn.
   By letting $m \rightarrow \infty$, $\forall n \geq k$, $x \notin F_k$, $|f(x) - g_n(x)| \leq 2^{-n+1}$. *It follows that the subsequence $g_k \rightarrow f$ a.e. (i.e., outside G) and a.u. (i.e., outside F_k).*
4. Show $g_k \rightarrow f$ in measure: take $\epsilon, \delta > 0$ and let N be large enough s.t. $\mu(F_N) < \min(\epsilon, \delta)$. Since we know that $\forall n \geq N$, $x \notin F_N \Rightarrow |f(x) - g_n(x)| \leq 2^{-k+1}$, and that $\mu(F_N) < 2^{-k+1}$, the result follows easily.

5. Show $f_n \to f$ in measure: $|f - f_n| \leq |f - g_k| + |g_k - f_n| \Rightarrow \{x \in X : |f - f_n| \geq \epsilon\} \subseteq (\{x \in X : |f - g_k| \geq \epsilon/2\} \cup \{x \in X : |g_k - f_n| \geq \epsilon/2\})$, from which the result follows.

$\square$

Remark: the subsequence (g_k) converges a.e. and a.u., thus showing that cauchy-measure convergence implies a subsequence converging a.e. and a.u. One can show that $f_n \to f$ in measure directly implies such a subsequence by choosing n_j to be s.t. $n > n_j \Rightarrow \mu(\{x \in E : |f_n(x) - f(x)| \geq 2^{-j}\}) < 2^{-j}$.

Theorem 2. (Lp cvgnce implies cvgnce in measure)
Suppose f_n,$f \in L_p(X, \Omega, \mu)$ and $\|f_n - f\|_p \to 0$. Then $f_n \to f$ in measure.

Depends on: generalized chebyshev's inequality

Proof idea: $\mu(\{x \in X : |f - f_n| \geq \epsilon\}) \leq \frac{1}{\epsilon^p} \int |f - f_n|^p d\mu \to 0$ $\square$

Theorem 3. (cauchy-a.u. convergence is equivalent to a.u. convergence)
Let (f_n) be measurable fns on (X, Ω, μ). Suppose $\forall \delta > 0 \exists A_\delta \subseteq E$ with $\mu(A_\delta) < \delta$ outside of which (f_n) converges uniformly (to some function that may depend on δ). Then $\exists f$ s.t. $f_n \to f$ a.u. on E.
Depends on: the same $\limsup$ construction from Theorem 1

Proof idea: Similar to proof of Theorem 1:

1. Let E_k be the set of measure $< 2^{-k}$ outside which (f_n) converges unif. to some limit. Define F_k and G as before s.t. $\mu(F_k) < 2^{-k+1}$, $\mu(G) = 0$.

2. Define $g_k(x) = \lim f_n(x) 1_{F_k^c}$. Since $g_n(x)$ does not change outside F_k for $n \geq k$, the function $f(x) = g_k(x)$ on F_k^c is well-defined and $f_n \to f$ outside G, (i.e. a.e).

3. By def. of f, for any δ we take N s.t. $2^{-N+1} < \delta$ and show that $f_n \to g_N \equiv f$ outside F_N, implying that $f_n \to f$ a.u. on E.

$\square$

Remark: The subsequence (g_k) converges a.e. thus showing that cauchy-a.u. convergence implies a.e convergence. One can show that $f_n \to f$ a.u. directly implies a.e. convergence by showing that $f_n \to f$ outside of $\limsup E_k$

Theorem 4. (a.u. cvgnce implies cnvgnce in measure)
Let (f_n) be measurable fns on (X, Ω, μ) s.t.$f_n \to f$ a.u. on $E \subseteq X$. Then $f_n \to f$ in measure on E.

Depends on:

Proof idea: Follows from definitions. $\forall \epsilon, \delta > 0$, $\exists A \subseteq E$ with $\mu(A) < \delta$ outside of which $f_n \to f$ uniformly $\Rightarrow \{x \in E : |f_n(x) - f(x)| \geq \epsilon\} \subseteq A$ for suff. large n. □

Theorem 5. Vitali cvgnce thm (necessary/suff. conditions for L_p cvgnce) Let (f_n) be measurable fns on (X, Ω, μ). Then $f_n \to f$ in $L_p \Leftrightarrow$:

1. $f_n \to f$ in measures
2. f_n are uniformly integrable
3. $\forall \epsilon > 0$, $\exists E \subseteq X$ s.t. $\mu(E) < \infty$ and $\forall F \in \Omega$ disjoint from E, $\int_F |f_n|^p d\mu < \epsilon^p \ \forall n$.

Depends on:

Proof idea: □

Theorem 6. Suppose (f_n) are a sequence of measurable fns on (X, Ω, μ) and $p > 1$. Suppose that $\|f_n\|_p \leq C \ \forall n$. Then (f_n) are uniformly integrable.
Depends on:

Proof idea: For a given set A and number $l > 0$, notice that
$\int_A |f_n| d\mu \leq \frac{1}{l^{p-1}} \int_{A \cap \{|f_n(x)| \geq l\}} |f_n(x)|^p d\mu + \int_{A \cap \{|f_n(x)| < l\}} |f_n| d\mu \leq \frac{M}{l^{p-1}} + l\mu A$. Choose l large s.t. the first term is less than $\epsilon/2$ and then choose A s.t. $\mu(A) < \epsilon/(2l)$. □

6.4 Important examples

1. Cvgnce a.e. but not in measure: $f_n(x) = min(n, |x|) \to |x|$ pointwise but not in measure (need finite measure)
2. Cvgnce a.e./a.u./measure but not L_p: $f_n(x) = n1_{[1/n, 2/n} \to 0$ pointwise on $[0, 1]$ but integral is constantly 1 $\forall n$. (even with a finite measure, still need a dominating integrable function)
3. Cvgnce in measure/L_p but not a.e./a.u. : $f_n(x) = 1_{I_n}$ where $\{I_n\} = \{[0, 1], [0, 1/2], [1/2, 1], [0, 1/3], [1/3, 2/3], ...\}$ (cycling but shrinking intervals).
4. Cvgnce a.e. but not a.u.: $f_n(x) = 1_{[n, n+1]} \to 0$ pointwise (need finite measure).

7 Functional analysis and L_p

7.1 Definitions

(bounded) linear functional, absolutely continuous measures, weak convergence, Hilbert space, dual space, signed measure, totally positive/negative/null set, measure decomposition

7.2 Useful theorems

Theorem 1. The dual space of a banach space is a banach space
Let $(B, \|.\|_B)$ be a banach space. Let $B^* = \{\Lambda : B \to \mathbb{R} : \Lambda$ linear and $C_\Lambda = \sup\limits_{\|x\|_B=1} |\Lambda x| < \infty\}$. Then $(B^*, \|.\|_{B^*})$ is a banach space with norm $\|\Lambda\|_{B^*} = C_\Lambda$.

Depends on: completeness of $\mathbb{R}$, definitions

Proof idea: Easy to check that B^* is a vector space and that C_Λ is a valid norm. To show it is complete, suppose $\|\Lambda_m - \Lambda_n\| \to 0$.

1. We know that $\forall x \in B$ s.t. $\|x\|_B = 1$, $|\Lambda_m(x) - \Lambda_n(x)| \to 0$ and so by completeness of $\mathbb{R}$, $\Lambda_n(x) \to \Lambda(x)$

2. $\Lambda(x)$ is linear (easy) and bounded (choose N large s.t. $\forall x$ s.t. $\|x\|_B = 1$, $|\Lambda_n(x) - \Lambda_N(x)| < \epsilon \Rightarrow \forall n \geq N$, $|\Lambda_n(x)| < |\Lambda_N(x)| + \epsilon \Rightarrow \|\Lambda\| < \|\Lambda_N\| + \epsilon < \infty$ by taking the limit).

3. Show that $\|\Lambda_n - \Lambda\|_{B^*} \to 0$ using the fact that $|\Lambda_n(x) - \Lambda(x)| = \lim\limits_{m\to\infty} |\Lambda_n(x) - \Lambda_m(x)| \leq \limsup \|\Lambda_n - \Lambda_m\|_{B^*} \|x\|_B \to 0$ uniformly on $\|x\|_B = 1$.

□

Theorem 2. (Properties of Hilbert spaces)
Let H be Hilbert space with inner product $\langle *, * \rangle$. The following properties hold:

1. $\forall x, y \in H$, $\langle x, y \rangle \leq \|x\|\|y\|$ (Cauchy-Schwarz)

2. $\forall x, y \in H$, $2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$ (Parallelogram law)

3. $\forall H_0 \subseteq H$ (i.e. H_0 is a closed,linear subspace), $H = H_0 \oplus H_0^\perp$

4. $\forall H_0 \subseteq H$, $H_0^\perp \subseteq H$ and $(H_0^\perp)^\perp = H_0$

Depends on:

Proof idea: Proofs use a variety of tricks:

1. Look at the quadratic nonnegative form $f(\lambda) = \langle x + \lambda y, x + \lambda y \rangle$ and set the discriminant ≤ 0

2. Prove using geometry on a parallelogram whose sides are x, y and whose diagonals are $x + y, x - y$.

3. Take $x \in H$. Let $C = \inf_{y \in H_0} \|x - y\|$. If $C = 0 \Rightarrow \exists y_n$ s.t. $\|x - y_n\| \rightarrow 0 \Rightarrow x \in H_0$ (by closure of H_0. Else, show that the minimizing seqience (y_n) is cauchy using the Parallelogram law with $x' = x - \frac{y_n + y_m}{2}, y' = \frac{y_n - y_m}{2}$. Thus $y_n \rightarrow y \in H_0$. By optimality of y we also have that $\frac{d}{d\lambda}\|x - y + \lambda z\|^2 = 0$ at $\lambda = 0 \forall z \in H_0 \Rightarrow \langle x - y, z \rangle = 0 \forall z \in H_0 \Rightarrow x - y \in H_0^{\perp}$.

4. Clearly $H_0 \subseteq (H_0^{\perp})^{\perp}$. For every $y \in (H_0^{\perp})^{\perp}$, decompose $y = y_1 + y_2$ with $y_1 \in H_0$ and $y_2 \in H_0^{\perp}$. But then $0 = \langle y, y_2 \rangle = \langle y_2, y_2 \rangle \Rightarrow y_2 = 0 \Rightarrow y \in H_0$.

□

Theorem 3. (bounded linear functionals are exactly continuous linear functionals)
Let B be a banach space. A linear functional $\Lambda : B \rightarrow \mathcal{B}$ is bounded $\Leftrightarrow$ it is continuous (at one point).

Depends on: definitions of continuity, boundedness, linearity

Proof idea: 2 steps:

1. $\Rightarrow$: $\forall \epsilon > 0$, choose $\delta < \epsilon / \|\Lambda\|_{B^*}$ to show (uniform) continuity.

2. $\Leftarrow$: Suppose Λ cnts at x_0. Use continuity to obtain δ corresponding to $\epsilon = 1$ and let $\eta = \delta/2$. For any $y \in B$ s.t. $\|y\|_B = 1$, $\eta\Lambda(y) = \Lambda(\eta y + x_0) - \Lambda(x_0)$ which has norm < 1 by definition of $\eta, y \Rightarrow |\Lambda(y)| \leq \eta^{-1}$

□

Theorem 4. Riesz representation theorem
Let H be a Hilbert space with inner product $\langle *, * \rangle$ and let $\Lambda : H \rightarrow \mathbb{R}$ be a bounded linear functional. Then $\exists y \in H$ s.t. $\forall x \in H$, $\Lambda(x) = \langle y, x \rangle$.

Depends on:

Proof idea: If $\Lambda \equiv 0$ then choose $y = 0$. Otherwise, let $H_0 = \ker \Lambda$ which is a closed subspace by the continuity of Λ (preimage of a point is closed). Therefore, $H = H_0 \oplus H_0^\perp$.Choose $e \in H_0^\perp$ with $\|e\| = 1$. Then since $\forall x \in H$, $\Lambda(x)e - \Lambda(e)x \in H_0$, we have $\langle \Lambda(x)e - \Lambda(e)x, e\rangle = 0 \Rightarrow \Lambda(x) = \langle y, x\rangle$ where $y = \Lambda(e)e$. ☐

Theorem 5. Hahn-Banach Thm: (Extension of BLF's)
Let V be a vector space, and let $W \subseteq V$ be a subspace. Suppose $\Lambda : W \to \mathbb{R}$ be a linear functional, and let $p : V \to \mathbb{R}$ s.t.

1. $p(x) \geq 0 \ \forall x \in V$
2. $p(\lambda x) = |\lambda| p(x) \ \forall \lambda \in \mathbb{R}, x \in V$
3. $p(x + y) \leq p(x) + p(y) \ \forall x, y \in V$

Then there exists a linear functional $\overline{\Lambda} : V \to \mathbb{R}$ s.t. $\overline{\Lambda} = \Lambda$ on W and $\overline{\Lambda} \leq p$ on V.
Depends on: axiom of choice (or Zorn's lemma?)

Proof idea: Trivial if $W = V$. Otherwise:

1. Choose $y \in V \cap W^c$. We want to extend Λ to include $\{x + cy : x \in W, c \in \mathbb{R}\}$.
2. In order to choose $\Lambda(y)$, use the constraint of linearity of Λ and bound $p(x)$ to show that $\Lambda(y) \leq \inf_{x \in W, c > 0} \dfrac{p(x + cy) - \Lambda(x)}{c}$ and $\Lambda(y) \geq \sup_{x \in W, c < 0} \dfrac{\Lambda(x) - p(x - cy)}{c}$
3. Show that this is in fact possible, i.e. that $\forall x_1, x_2 \in W$ and $\forall c_1, c_2 > 0$, $\frac{p(x_1 + c_1 y) - \Lambda(x_1)}{c_1} \geq \frac{\Lambda(x_2) - p(x_2 - c_2 y)}{c_2}$.
4. Argue that you can iterate this procedure for all $y \in W^c$.

☐

Theorem 6. Hahn decomposition theorem
Suppose μ is a countable additive signed measure on (X, Σ). Then $\exists P, N \in \Sigma$ disjoint s.t. $X = P \cup N$, and $\forall E \in \Sigma$, $\mu(E \cap P) \geq 0, \mu(E \cap N) \leq 0$.

Depends on: dependencies...

Proof idea: 2 steps:

1. First show that $\mu(A) \leq 0 \Rightarrow \exists D \subseteq A$ totally negative s.t. $\mu(D) \leq \mu(A)$. The general idea is obtain D by iteratively removing subsets of A with positive measure:

   (a) Let $A_0 = A$ and define $t_n = \sup\{\mu(B) : B \in \Sigma,\ B \subseteq A_n\}$.

   (b) Let B_n be the set with $\mu(B_n) \geq \min\{1, t_n/2\}$. Let $A_{n+1} = A_n \cap B_n^c$ and keep going.

   (c) Let $D = A \cap (\cup_j B_j)^c$ and use c.a. to show that $\mu(D) \leq \mu(A)$ and also that D is totally negative since μ cannot take the value $-\infty$.

2. Start with $N_0 = X$ and let $s_n = \inf\{\mu(B) : B \in \Sigma$ and $B \subseteq X \cap N_n^c\}$. Let C_n be the corresponding sets with $\mu(C_n) \leq \max\{s_n/2, -1\}$ and use step 1 to get totallly negative subsets $D_n \subseteq C_n$. Then define $N = \cup_j D_j$ and show that it is totally negative and its complement is totally positive.

$\square$

Remark: One can show uniqueness that if (P', N') is another Hahn decomposition thatn the symmetric differences $P\Delta P'$ and $N\Delta N'$ have only subsets of measure 0 by using the fact that the intersection of a totally positive and a totally negative set have this property.
The Hahn decomposition also yields the Jordan decomposition theorem which says that any signed measure μ is the difference of 2 positive measures $\mu_+ - \mu_-$ with $\mu_+(E) = \mu(E \cap P)$ and $\mu_- = \mu(E \cap N)$.

Theorem 7. Radon-Nikodym theorem
Let μ and λ be 2 σ-finite measures on (X, Σ) with $\lambda << \mu$. Then $\exists f : X \to \mathbb{R}$ measurable and nonnegative s.t. $\forall A \in \Sigma$, $\lambda(A) = \displaystyle\int_A f(x)d\mu$. f is unique almost everywhere w.r.t. μ.

Depends on: Hahn decomposition OR Riesz Representation

Proof idea: 1. Suppose we have the result for finite measures. Then if λ, μ are σ-finite, we have $A_j \uparrow X$ with $\lambda(A_j), \mu(A_j) < \infty$. Let h_n be the Radon-Nikodym derivative on A_n and vanishing everywhere else. Define $f_n(x) = \sup\{h_j(x) : j = 1, ..., n\}$ and let f be the monotone limit of f_n. Then by Monotone convergence thereom, $\lambda(A) = \displaystyle\int_A f d\mu$.

2. Assume μ, λ are finite. The general strategy is to construct f by defining its level sets to be the increments in an increasing sequence of sets obtained via Hahn decomposition:

(a) Define $\mu_k = \lambda - k\mu$ for $k \in \mathbb{Q}^+$ and let (P_k, N_k) be the Hahn decomposition of X for μ_k. Then, for any A satisfying $A \in P_k \forall k$,$\lambda(A) - k\mu(A) \geq 0\ \forall k \Rightarrow \mu(A) = \lambda(A) = 0$. Putting all such null sets in the negative sets, we can conclude that $P_k \downarrow \phi$ and $N_k \uparrow X$ as $k \to \infty$.

(b) Define $f(x) = \inf\{k \in \mathbb{Q}^+ : x \notin P_k\}$ (level sets). Partition X into $\{P_{j\epsilon} \backslash P_{(j+1)\epsilon} : j \geq 0\}$. The key is that: $\forall x \in P_{l\epsilon}$, $l\epsilon\mu \leq \lambda \leq (l+1)\epsilon\mu$
Define f_ϵ^- and f_ϵ^+ to be the lower/upper bounds on the levels sets $P_{l\epsilon}$, and define the measures λ_ϵ^- and λ_ϵ^+ to be their respective integrals w.r.t. μ.

(c) For any $A = \cup(A \cap P_{l\epsilon}) = \cup A_l$, we can show that $\lambda(A) = \int_A f d\mu$ by showing that $\forall l \geq 0$, $\lambda_\epsilon^-(A_l) \leq \lambda(A_l) \leq \lambda_\epsilon^+(A_l)$. Show that $\lambda_\epsilon^+, \lambda_\epsilon^-$ converge to λ monotonically as $\epsilon \to 0$. Take $f = \lim_{\epsilon \to 0} f_\epsilon^{+/-}$.

$\square$

Alternate proof (Von Neumann): Assume finite measure and let $\nu = \lambda + \mu$ and consider $L_2(X, \Sigma, \nu)$. Then $\Lambda(f) = \int f d\lambda$ is a bounded linear functional $\Rightarrow \exists g \in L_2(X, \Sigma, \nu)$ s.t. $\int f d\lambda = \int f g d\nu$ by Riesz representation theorem. Let $h = g/(1-g)$. Need to show:

1. $0 \leq g < 1$ a.e.: $\forall A \in \Sigma$, $\displaystyle\int_A g d(\lambda + \mu) = \lambda(A) \leq (\lambda + \mu)(A) \Rightarrow 0 \leq g \leq 1$. Also, $\lambda(\{x : g = 1\}) = (\lambda + \mu)(\{x : g = 1\}) \Rightarrow \mu(A) = \lambda(A) = 0$.
2. $h \in L_2(X, \Sigma, \mu)$: follows from the fact that $\displaystyle\int f d\lambda = \int f g d(\lambda + \mu)$
3. $\displaystyle\int_A h d(\lambda + \mu) = \langle g, \frac{1}{1-g} \rangle = \int_A \frac{1}{1-g} d\lambda \Rightarrow \int_A h d\mu = \int_A 1 d\lambda = \lambda(A)$

$\square$

Theorem 8. The dual of L_p is L_q
Let $1 \leq p < \infty$ and suppose μ is σ-finite on (X, Σ). Then $L_p(X, \Sigma, \mu)^* \cong L_q(X, \Sigma, \mu)$ where $1/p + 1/q = 1$.

Depends on: Radon-Nikodym theorem, MCT/DCT, density of simple functions

Proof idea: Clearly each $g \in L_q$ defines a bounded linear functional $\int f g d\mu$ (use Holder's inequality to show boundedness)r, so $L_q \subseteq L_p^*$.

1. Assume finite measure and take any $\Lambda \in L_p^*$ and define $\lambda(A) = \Lambda(\chi_A)$. Let $\varphi = \frac{d\lambda}{d\mu}$. Since simple functions are dense in L_p, we have $\Lambda(f) = \int f\varphi d\mu \ \forall f \in L_p$.

2. Show that $\varphi \in L_q$:

   (a) Define a sequence of nonneg. simple fns $\psi_n \uparrow |\varphi|^q$ and define $\phi_n = (\psi_n)^{1/p} sgn(\varphi)$ so that $\phi_n \varphi \geq \psi_n$, $\forall n$.

   (b) Use the fact that $\int \varphi g d\mu = \Lambda(g) \leq \|\Lambda\| \|g\|_p$ for all simple g to show that $\int \psi_n d\mu \leq M[\int \psi_n d\mu]^{1/p}$.

   (c) Multiply both sides by $[\int \psi_n d\mu]^{-1/p}$ to get that $\int \psi_n d\mu \leq M^q$ and apply MCT.

   (d) For σ-finite μ with $X = \cup A_n$, apply argument to each A_n to get g_n and take g as the combination of the g_n's. Use MCT to show $g \in L_q$, and DCT to show that $\Lambda(f) = \int fg d\mu$.

□

7.3 Important Examples

1. Radon-Nikodym fails if the measure is not σ-finite. Consider the counting measure μ that assigns finite sets their cardinality and ∞ otherwise. Then the lebesgue measure $\lambda << \mu$. But if there were a Radon-Nikodym derivative f then we have $0 = \lambda(\{a\}) = \int_{\{a\}} f d\mu = f(a)$, $\forall a \in \mathbb{R}$, which is a contradiction since then that means $\lambda \equiv 0$.

2. $L_\infty^* \supset L_1$. FILLIN.

3.

8 Real line integration/differentiation

8.1 Definitions:

monotone function, bounded variation, absolutely continuous, differentiable, Vitali cover

8.2 Useful theorems

Theorem 1. Vitali covering lemma
Let $A \subseteq \mathbb{R}$ have finite measure and suppose Γ is collection of intervals covering A in the sense of Vitali. Then $\forall \epsilon > 0$, $\exists I_1, ..., I_N \in \Gamma$ s.t. $\mu(A \backslash \cup_{j=1}^{N} I_j) < \epsilon$.

Depends on: properties of sup

Proof idea: WLOG assume that intervals in Γ are closed and restricted to an open set $O \supseteq A$.

1. Let $I_1 \in \Gamma$ be any interval and define inductively:
   (a) $k_n = \sup\{\mu(I) : I \in \Gamma \text{ and } \forall 1 \leq j \leq n,\ I \cap I_j = \phi\}$
   (b) $I_{n+1} \in \Gamma$ is the interval disjoint from $I_1, ..., I_n$ s.t. $\mu(I_{n+1}) > k_n/2$
2. Then $\mu(I_n) \to 0$ since $\cup_{j=1}^{\infty} I_j \subseteq O$ which has finite measure. Choose N s.t. $\displaystyle\sum_{j=N+1}^{\infty} \mu(I_j)$ is small.
3. Show that $\mu(B = A \backslash \cup_{j=N+1}^{\infty} I_j) < \epsilon$ as follows:
   (a) Take any $x \in B$ and let $I \in \Gamma$ be a small interval containing x disjoint from $I_1, ..., I_N$. Then I has to intersect I_n for some minimal $n > N$ since we have $\mu(I) < k_n \to 0$. It follows that $x \in J_n$ where J_n has the same midpoint as I_n but 5 times the length since $\mu(I) \leq k_{n-1} < 2\mu(I_n)$.
   (b) Then $B \subseteq \cup_{j=N+1}^{\infty} J_j$, which has measure less than 5 times the tail $\displaystyle\sum_{j=N+1}^{\infty} \mu(I_j)$, which is arbitrarily small.

$\square$

Theorem 2. (Monotone functions are everywhere differentiable on closed intervals)
Suppose $F : \mathbb{R} \to \mathbb{R}$ is monotone nondecreasing on $[a, b]$. Then F is differentiable a.e. on $[a, b]$. Further more, $\displaystyle\int_a^b F'(x)dx \leq F(b) - F(a)$.
Depends on: Vitali covering lemma, Fatou's lemma

Proof idea: It suffices to show that $A_{q1,q2} \equiv \{x \in [a,b] : \limsup_{h\downarrow 0} \frac{F(x+h)-F(x)}{h} > q_2 > q_1 > \liminf_{h\downarrow 0} \frac{F(x+h)-F(x)}{h}\}$ has measure 0 $\forall q_1, q_2 \in \mathbb{Q}$ by countable additivity and definition of derivatives.

1. Apply Vitali covering lemma to A_{q_1,q_2} with $\Gamma = \{[x, x-h] : h > 0$ and $F(x) - F(x-h) > q_1 h\}$ to get intervals $\{[x_j, x_j - h_j]\}_{j=1}^N$ that almost cover A. Notice that F jumps by at most $q_1 \sum_{j=1}^{N} \mu(I_j)$ on these intervals.

2. For each $1 \le j \le N$, apply Vitali covering lemma to $A_{q_1,q_2} \cup I_j$ with $\Gamma = \{[x, x+h] : h > 0$ and $F(x+h)-F(x) > q_2 h\}$ to get intervals $\{J_{j,k}\}_{k=1}^{N_j}$. Notice that F jumps by at least $q_2 \sum_{j=1}^{N} \sum_{k=1}^{M_j} \mu(J_{j,k}) \to q_2 \sum_{j=1}^{N} \mu(I_j)$. This gives a contradiction since $q_1 < q_2$.

3. Therefore $F'(x)$ exists a.e. in $[a,b]$ and it is measurable and nonnegative since it can be expressed as the limit as $h_n = 1/n \to 0$ of some nonnegative fn involving F and h_n.

4. WLOG assume $F(x) = F(b)$ for $x \ge b$ and $F(x) = F(a)$ for $x \le a$. By Fatou's lemma, $\int_a^b F'(x) d\mu \le \liminf [\frac{1}{n} \int_b^{b+1/n} F(x) d\mu - \frac{1}{n} \int_a^{a+1/n} F(x) d\mu] \le F(b) - F(a)$ by monotonicity.

$\square$

Theorem 3. (BV functions are the differences of monotone functions) Suppose $f : [a,b] \to \mathbb{R}$. Then f is of BV $\Leftrightarrow \exists$ monotone functions g, h on $[a,b]$ s.t. $f = g - h$.

Depends on: definition of BV

Proof idea: Let P_x, N_x, T_x be the positive/negative/total variations of f on $[a,x]$.

1. $\Rightarrow$:

   (a) Show that if f is BV then $\forall x \in [a,b]$, $f(x) - f(a) = P_x - N_x$ and $T_x = P_x + N_x$. Do this by fixing a partition of $[a,x]$ and using the fact that $p = n + f(b) - f(a)$ and then taking suprema over partitions.

(b) Let $g(x), h(x)$ be the positive/negative variations of f on $[a, x]$,respectively. Then $g, h - f(a)$ are nondecreasing fns and $f = g - (h - f(a))$.

2. $\Leftarrow$: Bound the total variation by the sum of individual variations of the monotone functions, which reduce to the difference at the endpoints a, b.

$\square$

Theorem 4. (Properties of BV/monotone functions)
Let $f : [a, b] \to \mathbb{R}$ be a BV function. Then:

1. $\forall c \in (a, b)$, $\lim\limits_{x \to c^-} f(x)$ and $\lim\limits_{x \to c+} f(x)$ exist.

2. The set of points $\{x \in [a, b] : f \text{ discnts at } x\}$ is countable

Depends on: countable additvity, countability of rationals

Proof idea:

1. If at any point c the lefthand limit doesn't exist, the lim sup must be strictly greater than the lim inf. If either of them is infinite then we can always find a sequence of points $(x_i) \to c^-$ such that if we partitioned $[a, b]$ using these points, the variation would go to infinity. If they are both finite, we can still do this by constructing a sequence that alternates between points where f comes close to the lim sup and then close to the lim inf.

2. STS that a monotone fn has countable discontinuities. $f monotone \Rightarrow f$ BV $\Rightarrow \forall c \in (a, b)$, the LH and RH limits exist. Let A be the set of discontinuities. Then for each $x \in A$, there is a corresponding interval $[\alpha_x, \beta_x]$ corresponding to the limits approached from each side. But each of these intervals is disjoint by monotonicity, and so each contains a distinct rational. Thus A is countable.

$\square$

Theorem 5. (Continuity/differentiability of indefinite integrals)
Let $f : [a, b] \to \mathbb{R}$ be integrable and let $F(x) = \displaystyle\int_a^x f(t)dt$. Then:

1. F is continuous

2. F is of BV

3. F is absolutely continuous

4. $F \equiv 0$ on $[a, b] \Rightarrow f \equiv 0$ on $[a, b]$

5. $\|\frac{F(x+h)-F(x)}{h} - f(x)\|_1 \to 0$ as $h \to 0$ where $\|.\|_1$ is the L_1 norm on $[a,b]$.

6. $F'(x) = f(x)$ a.e. on $[a,b]$.

Depends on: Properties of BV functions

Proof idea: Show each separately:

1. Continuity of F follows from the fact that for f integrable, $\displaystyle\int_A f d\mu$ is small whenever $\mu(A)$ is small. To see this, WLOG assume $f \geq 0$ (express as difference of 2 nonneg fns) and construct the sequence $f_n \uparrow f$ but cutoff at n. Choose N such that $\int f_N$ is close to $\int f$ (by MCT). Then choose $\mu(A)$ to be small s.t. $\int_A f_N$ is small.

2. To show F is of BV, take any partition, and express the variation as a sum of absolute values of integrals. Upper bound by bringing the absolute value inside and bound by $\displaystyle\int_a^b |f(x)|dx$.

3. To show absolute continuity, we know that for given $\epsilon > 0$, we can choose $\delta > 0$ s.t. $\mu(A) < \delta \Rightarrow \displaystyle\int_A f d\mu < \epsilon$, then choose A to be any finite union of intervals.

4. Suppose $f > 0$ on a set A of positive measure. Find a closed set $F \subseteq A$ of positive measure and since $F(b) = 0$ we have $\displaystyle\int_{[a,b]\backslash F} f dx = -\int_F f dx \neq 0$. Express $[a,b]\backslash F$ as a countable union of intervals and conclude that the integral of f is nonzero on some interval $[\alpha, \beta] \subseteq [a,b]$. Then we can show that either $F(\alpha)$ or $F(\beta)$ is nonzero, which is a contradiction.

5. (a) First show that for any fn $W(x) = \displaystyle\int_a^x w(x)dx$ with $w \in L_1$, we can bound $\displaystyle\int_a^b |W(x+h) - W(x)|dx$ by $h\|w\|_1$ by just writing it out manipulating integral limits.

   (b) Since $f \in L_1$ we can write $f = g + w$ where g is a nice cnts fn with compact support and $\|w\|_1$ is small. Then STS the result separately for G, W, but this is easy since G is nice and $\|w\|_1$ is small, using previous step.

6. (a) Show first for $|f|$ bounded by M. We know that $F'(x)$ exists a.e. Then $|\frac{F(x+h)-F(x)}{h}| \leq M$ so by BCT we have $\int_a^c F'(x)dx = \lim \frac{1}{h}\int_a^c (F(x+h_n)-F(x))dx = \lim[\frac{1}{h}\int_c^{c+h} Fdx - \frac{1}{h}\int_a^{a+h} Fdx] = F(c)-F(a) = \int_a^c f(x)dx$ by continuity of F. Then $\int_a^c (F'-f)dx = 0\ \forall c \in [a,b] \Rightarrow F' = f$ a.e. on $[a,b]$ by step (3).

   (b) For general f, assume WLOG that $f \geq 0$. Approximate f by n-cutoff functions (f_n).Then since $\int_a^x (f-f_n)(t)dt$ is monotone nondec., it has an a.e. nonnegative derivative. It follows that $F' \geq f_n$ a.e. $\forall n \Rightarrow F' \geq f$ a.e. $\Rightarrow \int_a^b F'dx = F(b)-F(a) \Rightarrow \int_a^b (F'-f)dx = 0 \Rightarrow F' = f$ a.e.

$\square$

Theorem 6. (Properties of absolutely continuous fns)
Let $F : [a,b] \to \mathbb{R}$ be absolutely continuous. Then:

1. F is of BV (and therefore has a.e. derivative)
2. $F' = 0$ a.e.on $[a,b] \Rightarrow F$ is constant
3. $F(x) = \int_a^x F'(t)dt$ a.e. on $[a,b]$. (Absolutely cnts fns are exactly indefinite integrals)

Depends on: Vitali covering lemma,

Proof idea: 1. This is easy. Just take δ corresp. to $\epsilon = 1$. Show that the total variation for any partition is bounded by ceiling of $(b-a)/\delta$.

2. Take any $c \in [a,b]$ and show that $|f(c)-f(a)|$ is arbitrarily small by applying the Vitali Covering lemma to $[a,b]$ with $\Gamma = \{[x, x+h] \in [a,c] : F(x+h)-F(x) < \eta h\}$ (need $F' = 0$ a.e.) to get intervals $I_1, ..., I_N$ almost covering $[a,b]$ except a set of measure δ corresponding to the ϵ via abs. continuity. Then we can bound $|f(c)-f(a)|$ by $C_1\epsilon + C_2\eta$ by using a telescoping sum and bounding separately, those corresp. to I_j's and the leftover intervals.

3. Apply the previous step to the function $f = F - G$ where $G = \int_a^x F'(t)dt$. Show first that G is integrable by writing F as a difference of monotone fns, bounding the derivative by the sum of the derivatives and then bounding the integral by using the property of integrals of the derivatives of monotone fns.

$\square$

Remark: Other properties:

1. Closed under addition, multiplication, nonzero quotient, composition
2. If f is absolutely continuous then: f is Lipschitz $\Leftrightarrow$ $|f'|$ is bounded

8.3 Important examples

1. Continuous but not absolutely continuous function: consider $f(x) = sin(1/x)x$ for $x \neq 0$ and $f(0) = 0$ on $[-1, 1]$.
2. Monotone continuous but not absolutely continuous: the cantor ternary functiom (maps the counter set to $[0, 1]$)

9 Product measures

9.1 Definitions

measurable rectangle, product measure space, product measure, cross-section set

9.2 Useful theorems

Theorem 1. (Existence of product measures)
Let (X, Σ_1, μ) and (Y, Σ_2, ν) be measurable spaces. Then $\exists$ a countably additive measure $(\mu \times \nu)$ on $(X \times Y, \Sigma)$ where Σ is the σ-field generated by measurable rectangles $\{A \times B : A \in \Sigma_1, B \in \Sigma_2\}$

Depends on: BCT, Caratheodory extension

Proof idea: We define $Q(A \times B) = \mu(A)\nu(B)$ for measurable rectangles $(A \times B)$. It STS that if a measurable rectangle $(A \times B) = \cup_{j=1}^{\infty}(A_j \times B_j)$ then $Q(A \times B) = \sum Q(A_j \times B_j)$, and therefore this easily extends to finite disjoint unions of rectangles, and to countable unions of rectangles by the

Extension thm. The key to showing this is to notice that for each $x \in A$, we can partition B into the sets $\{B_j : x \in A_j\}$ (draw a rectangle and partition it a bunch of random ways and test it). Thus, $\nu(B)1_{x\in A} = \sum \nu(B_j)1_{x\in A_j}$. Integrating both sides wrt μ and exchanging integral and sum by MCT gives $\nu(B)\mu(A) = \sum \nu(B_j)\mu(A_j)$. □

Theorem 2. (Fubini's theorem)

Let $f(x, y)$ be a measurable function on $(X \times Y, \Sigma)$ (product space as described above). Then:

1. The functions $g_x(y) = h_y(x) = f(x, y)$ are measurable for a.e. x and a.e. y respectively.
2. If f is integrable, then:
   (a) $g_x(y)$ is integrable (for a.e. x) and $h_y(x)$ is integrable (for a.e. y)
   (b) $G(x) = \displaystyle\int_Y g_x(y)d\nu$ and $H(y) = \displaystyle\int_X h_y(x)d\mu$ are measurable, finite a.e. and integrable wrt mu, ν, respectively
   (c) $\displaystyle\int_{X\times Y} fd(\mu \times \nu) = \int_X G(x)d\mu = \int_Y H(y)d\nu$
3. Conversely, if $f \geq 0$ is measurable and if either G and H is integrable, so is the other and f is also integrable with coinciding integral value.

Depends on: dependencies...

Proof idea: 1. We first establish that for any $E \in \Sigma$, then $\mu(E_y)$ and $\nu(E_x)$ are measurable and $P(E) = \displaystyle\int_Y \mu(E_y)d\nu = \int_X \nu(E_x)d\mu$. We do this in the following steps:
   (a) If $E = A \times B$ for some $A \in \Sigma_1$, $B \in \Sigma_2$, then it is clear since $E_y = A1_{y\in B}\ \forall y$ and $E_x = B1_{x\in A}\ \forall x$.
   (b) For $E \in \mathcal{F}$ (finite disjoint union of rectangles) then the sums add up.
   (c) The class of sets for which the claim holds is a monotone class since we can exchange sums and integral by the MCT. Thus it holds for all sets in the σ-field Σ.
2. We note that if $f = 1_E$ for some $E \in \Sigma$, then the above step proves the theorem.

3. It holds for simple functions (by linearity) $\to$ bounded measurable functions (pass the uniform limit) $\to$ nonneg. measurable fns (monotone limits) $\to$ all integrable measurable fn.

$\square$

9.3 Important examples

1. Product measures: give decimal/binary/ternary expansions
2. It is possible that a set A is not measurable wrt to the product σ-field, but all its cross-sections A_x, A_y are measurable, and $\int_Y \mu(A_y)d\nu \neq \int_X \nu(A_x)d\mu$
3. Nonnegativity is important in the converse statement of Fubini's thm: GIVE EXAMPLE

10 Topological aspects

10.1 Definitions

topological space, separable, conditionally/sequentially compact, metric space, dense, totally bounded, lindelof, complete, countable basis, hausdorff, T1, normal, lattice, modulus of continuity, tight measure, regular measure.

10.2 Useful theorems

Theorem 1. (Equivalent notions of separable space)
Let X be a metric space. Then the following are equivalent:

1. X has countable dense subset
2. X has a countable basis (any open set is the union of sets in a basis)
3. Given any $x \in U \subseteq X$, $\exists G_\alpha$ s.t. $x \in G_\alpha \subseteq U$.
4. X has the lindelof property

Depends on: set theory, definitions

Proof idea: 1. (2) $\Leftrightarrow$ (3): If $x \in U = \cup G_\alpha$, then clearly $\exists$ such a G_α containing x. Conversely if we take any open $U \subseteq X$, and each $x \in U$ is contained by some $G_\alpha \subseteq U$, then one can show that the union of these G_α's is exactly U.

2. (1) $\Rightarrow$ (3): Will show that $\{B(x_i, 1/n) : i, n \geq 1\}$ is a basis, where $\{x_i\}$ is the countable dense set. Take any $x \in U \subseteq X$. Then there is $B(x, 1/n)$ contained in U. Find x_i s.t. $d(x, x_i) < 1/(8n)$. Then we clearly $x \in B(x_i, 1/(4n)) \subseteq U$.

3. (2) $\Rightarrow$ (4): Take a countable basis $\{H_i\}$. This is an axiom of choice/countability argument - if $\{G_\alpha\}$ is a cover for A, then each $x \in A$ is contained in some G_α which in turn is equal to some $\cup H_{\alpha_j}$. Take the H's that show up and these are countable.

4. (4) $\Rightarrow$ (1): Construct a countable dense set as follows - repeatedly apply the Lindelof property to the countable sequence of open covers $\{\{B(x, 1/j) : x \in X\} : j \geq 1\}$ of X to get a countable cover of X consisting of balls of the form $B(x_i, 1/j)$. Then for any x, ϵ, $x_i^j \in B(x, \epsilon)$ for some i, j with j arbitrarily large.

$\square$

Theorem 2. The countable product of separable spaces is separable wrt a special product metric
Depends on: product metric definition, countability

Proof idea: We define the 'product metric' on the countable product of metric spaces $(X_1 \times X_2 \times ...)$ as $d(\{x_i\}, \{y_i\}) = \sum_{i=1}^{\infty} 2^{-i} \dfrac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)}$ where $d_i(.,.)$ is the metric corresponding to X_i. Fix some sequence $\{z_i\}$ in X and for each X_i extract a countable dense set E_i. Define $\mathcal{D}_k = E_1 \times ... \times E_k \times z_{k+1} \times ...$ and let $\mathcal{D} = \cup \mathcal{D}_i$. Since the tail of $\sum s^{-i} \to 0$, the $\mathcal{D}$ is a countable dense set for the product space. $\square$

Theorem 3. Any metric space (X, d) can be made complete, i.e. $\exists$ a complete metric space (Y, D) with $(X, d) \cong (U, D) \subseteq (Y, D)$ and U dense in Y.
Depends on: Diagonalization of sorts, def. of cauchy sequences

Proof idea: We take Y to be the set of cauchy sequences of elements in X with $D(\{x_i\}, \{y_i\}) = \lim_{n \to \infty} d(x_i, y_i)$ (can show this limit exists using cauchy property of the individual sequences,triangle inequality, and completeness of $\mathbb{R}$) . Then:

1. X is isomorphic to the set $U = \{(x, x, x...) : x \in X\} \subseteq Y$ and clearly $D(\{x\}, \{y\}) = d(x, y)$ $\forall x, y, \in X$.

2. U is dense in (Y, D) since for any cauchy sequence $\{x_j\} \in Y$ and ϵ we can choose N large s.t. $n \geq N \Rightarrow d(x_n, x_N) < \epsilon$. Then the sequence $D(\{x_N, x_N, ...\}, \{x_j\}) < \epsilon$.

3. (Y, D) is complete - take a cauchy sequence of cauchy sequences $\{(x_j)_{j\geq 1}^{(n)} : n \geq 1\}$. Then construct a sequence $(y_j)_{j\geq 1}$ where y_j is chosen from the jth sequence at the point after where the tail of the sequence oscillates by less than $1/j$. Can verify that this is itself a cauchy sequence and is in fact the limit sequence.

□

Theorem 4. (Equivalent notions of compactness)
Let X be some set. The following notions are equivalent:

1. Every sequence in X has a convergent subsequence. (This also implies separability - see proof).

2. Every collection of closed sets with the FIP has nonempty intersection

3. Every open covering of X has a finite subcovering

4. X complete and totally bounded

5. If X is a metric space then the above are also equivalent to: X closed and bounded.

Depends on: dependencies...

Proof idea:

1. (1) $\Rightarrow$ (3): Since there is no sequence having no convergent subsequence, $\forall \epsilon > 0$ there is a finite covering $\{B(x_j, \epsilon) : 1 \leq j \leq N_\epsilon\}$ of X. Take any open cover of X - a weird argument shows that there is an ϵ s.t. $B(x_j, \epsilon)$ is contained in some G_α of the cover, $\forall j$. The result follows easily.

2. (2) $\Leftrightarrow$ (3): simple complementation argument

3. (2) $\Rightarrow$ (1): Take any sequence (x_j) and look at the sets $B_n = \{x_n, x_{n+1}, ...\}$. Then $(\overline{B_n})_{n\geq 1}$ has the FIP so therefore it has nonempty intersection, i.e. there is a cluster point of the sequence (x_j), and thus a convergent subsequence.

4. (1,2,3) $\Rightarrow$ (4): If (x_j) is a cauchy sequence, it has a convergent subsequence by (1) and so it converges to a limit. Thus X is complete. It is totally bounded by applying (3) to the cover $\{B(x, \epsilon) : x \in X\}$.

5. (4) $\Rightarrow$ (1): Pigeonhole principle - For each n take a finite set of balls of radius $1/n$ covering X. Given a sequence (x_j) take the ball B_1 of radius 1 with infinitely many terms. Then there must exist a ball B_2 of radius $1/2$ intersecting B_1 containing infinitely many terms. Continuing this way, we get a sequence of balls B_k each with radius $1/k$ and each containing infinitely many terms. We can extract x_{n_k} from B_k and this must be a cauchy sequence, which converges if we assume complteness.

6. (5) $\Leftrightarrow$ (3): $\Leftarrow$ is the Heine-Borel theorem. The other way uses the properties of continuous functions on compact sets (they achieve max/min and are bounded). Use the distance function as the particular continuous function.

$\square$

Theorem 5. (Properties of cnts fns on compact sets) Let (X, d) and (Y, d') be a compact metric spaces and suppose $f : X \to Y$ is cnts. Then:

1. f is bounded and achieves its sup/inf on X
2. f is uniformly cnts.
3. $K \subseteq X$ comapact $\Rightarrow f(K) \subseteq Y$ is compact.

Depends on: various notions of compactness

Proof idea: 1. Use the convergent subsequence property. We have a sequence (x_j) with $f(x_j) \uparrow \sup\limits_{x \in X} f(x)$. But this has a convergent subsequence $(x_{n_j}) \to x_0 \in X$. Consequently, $f(x_0) = \lim f(x_{n_j}) = \sup f(x)$.

2. Use the finite subcover property. For each $x \in X$ there is a ball B_x of radius δ_x in which f oscillates by less than $\epsilon/2$. These B_x's with radii $\delta_x/2$ are an open cover of X and so has a finite subcover. Let δ be less than half of the minimum radius of the balls in the finite subcover. Take $x, y \in X$ with $d(x, y) < \delta$. Use triangle inequality to bound both $d'(f(x), f(y))$ and $d(x, z)$ where z is chosen from the subcover with $y \in B_z$.

3. Use convergence subsequence property. Take a sequence (y_j) in the image $f(K)$. Thus there is a sequence (x_j) with $f(x_j) = y_j$. Extract a convergent subsequence (x_{n_j}) which yields a convergent subsequence $(y_{n_j} = f(x_{n_j}))$ in the image domain.

$\square$

Theorem 6. The countable product of compact space is compact wrt the product metric $d(\{x_i\}, \{y_i\}) = \sum_{i=1}^{\infty} 2^{-i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)}$.

Depends on: product metric definition, diagonalization

Proof idea: Use the convergent subsequence property of compactness. First show that convergence wrt product metric is equivalent to convergence in each individual coordinate (this is easy to show using just definitions). Then just take a convergent subsequence in first coordinate, then take a subsequence of this subsequence in the second coordinate, and so on to get a subsequence converging in all coordinates and thus in the product metric.

$\square$

Theorem 7. (Sufficient conditions for normal space)

1. X is compact and hausdorff space $\Rightarrow X$ is regular, which is equivalent to being normal.

2. X is any metric space $\Rightarrow X$ is normal.

Depends on: finite subcovers,cnts fns

Proof idea:

1. (a) To show that it is regular, we need disjoint open sets separating a point x from a closed set C. Use the hausdorff property to get a pair (O_y, O_{yx}) of disjoint open sets separating each $y \in C$ from x. Then the O_y are an open cover for C and so there is a finite subcover O_{y_j}. Then the desired open sets are $\cup O_{y_j}$ and $\cap O_{xy_j}$.

   (b) To show the equivalence between regular and normal spaces, take C_1, C_2 disjoint and use regularity to get pairs (O_{Cy}, O_y) separating the set C_1 from each $y \in C_2$. Then O_y is an open cover of C_2 and so has a finite subcover O_{y_j}. The union of this finite subcover along with the intersection $\cap O_{Cy_j}$ is the desired pair.

2. Take C_1, C_2 disjoint. Define $d(x, C) = \inf_{y \in C} d(x, y)$ and let $f(x) = \frac{d(x,C_1)}{d(x,C_1)+d(x,C_2)}$. Show that f is cnts using continuity of d. The result follows by properties of continuity.

$\square$

Theorem 8. (Urysohn's lemma)
Let X be a normal space. Suppose C_1, C_2 are disjoint subsets of X. Then $\exists f : X \to \mathbb{R}$ continuous with $0 \leq f \leq 1$ and $f(x) = 0$ on C_1 and $f(x) = 1$ on C_2.

Depends on: interpolation

Proof idea: Use the normality inductively. Find open sets (U_0, V_0) separating C_1 from C_2. Since $\overline{U_1} \subseteq C_2^c$, we can again find a set $U_{1/2}$ s.t. $U_0 \subseteq \overline{U_0} \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq C_2^c$. Continuing in this way by interpolation over the diadic rationals in $[0, 1)$, and setting $U_1 = X$, we have a family $\{U_r\}$ of open sets containing C_1 and contained in C_2^c satisfying $r < s \Rightarrow \overline{U_r} \subseteq U_s$. Construct $f(x)$ using U_r's as level sets, i.e. $f(x) = \inf\{r : x \in \overline{U_r}\}$. Then $f = 0$ on C_1 and $f = 1$ on C_2 and f is cnts. $\square$

Theorem 9. (Baire category theorem)
Let X be a complete metric space. Then $X \neq \cup_{j=1}^{\infty} C_j$ where C_j are closed and C_j^c is dense $\forall j$.

Depends on: Cauchy criterion, def. of dense

Proof idea: Taking complements and partial unions, it STS that it cannot be the case that $\cap G_n = \phi$ where (G_n) is a decreasing sequence of dense sets. We show this by constructing a Cauchy sequence whose limit lies in $\cap G_n$. We fix $x_1 \in G_1$ and know by density of G_2 that $\exists x_2$. s.t. $B(x_2, \epsilon_2) \subseteq B(x_1, 1) \cap G_2$. Thus $\exists x_3$ s.t. $B(x_3, \epsilon_3) \subseteq B(x_2, \epsilon_2) \cap G_3$, and so on. (x_n) is a cauchy sequence and by completeness has a limit $x \in \cap G_n$. $\square$

Remark: In fact, we can show that $H = \cap G_n$ is dense! If it weren't then this violates Baire category on the complete metric space $(U, \tilde{d})$ where $U = \overline{H}^c$ and $\tilde{d}(x, y) = d(x, y) + |\frac{1}{d(x,U^c)} - \frac{1}{d(y,U^c)}|$.
Remark: This motivates classification of sets into 1st category (union of sets with no interior) and 2nd category. Baire category says that complete spaces cannot be 1st category. It can be shown that 1st category sets are closed under countable union, while 2nd category sets are closed under countable intersection.

Theorem 10. (Dinis's theorem - convergence is uniform on compact sets)
Let (f_n) be a sequence of upper-semicontinuous functions (i.e.$\forall n$, $\limsup_{x \to x_0} f_n(x) \leq f_n(x)$) with $f_n \downarrow 0$. Then on any compact space $EsubseteqX$, $f_n(x) \to 0$ uniformly.

Depends on: finite subcover notion of compactness, continuity neighborhoods

Proof idea: For each $x \in E$ we know that $\exists n_0(x)$ s.t. $n \geq n_0(x) \Rightarrow f_n(x) < \epsilon/2$. Now since for each fixed x, $f_{n_0(x)}(.)$ is upper-smicnts, so there is a neighborhood N_x of x s.t. $f_{n_0(x)}(y) \leq f_{n_0(x)}(x) \leq \epsilon/2$ for $y \in N_x$. $\{N_x\}$ is an open cover of E and so there is a finite subcover $\{N_{x_j}\}$. Take $n_0 \geq \max\{n_0(x_j)\}$. Then for any x, it is in some N_{x_j}, so for $n \geq n_0$, $f_n(x) \leq f_{n_0(x_j)}(x) < \epsilon$.

□

Theorem 11. (Stone-Weierstrass theorem - suff. conditions for density in $C(X)$)
Let X be a compact Hausdorff space and let $\mathcal{A} = \{f_\alpha(x)\}$ be a collection of continuous functions s.t.:

1. $\mathcal{A}$ contains all constant functions
2. $\forall x, y \in X$ and $a \neq b \in \mathbb{R}$, $\exists f \in \mathcal{A}$ s.t. $f(x) = a, f(y) = b$
3. $\mathcal{A}$ is a field/algebra

Then $\overline{\mathcal{A}}$ is dense in $\mathcal{C}(X)$.

Depends on: Lattices, finite subcovers, continuity neighborhoods

Proof idea: Basic steps:

1. Show that $\overline{\mathcal{A}}$ is closed under finite max and min operations. Do this by expressing $\max(f, g) = \frac{|f+g|+|f-g|}{2}$ and proving that any $f \in \{$, $|f|$ is in $\overline{\mathcal{A}}$ since it is a uniform limit of polynomial functions ($|f| = \sqrt{1 + f^2 - 1} = \sum a_n (f^2 - 1)^n$).
2. Take any $g \in C(X)$ and define f_{x_1,x_2} to be the function in $\mathcal{A}$ which agrees with g at points x_1, x_2. By continuity $|f_{x_1,x_2} - g| < \epsilon$ for x in some neighborhood N_{x_1} of x_1.
3. The N_{x_1} form an open covering so it has a finite subcovering $\{N_{x^j}\}$. Define $f_{*,x_2} = \max\{f_{x^j,x_2}\}$ so that $f_{*,x_2}(x) \geq g(x) - \epsilon \ \forall x_1, x_2$.
4. Now use continuity again to define neighborhoods N_{x_2} where $|f_{*,x_2} - g| < \epsilon$. Extract a finite subcover $\{N_{x^k}\}$. Define $f_{*,*} = \min f_{*,x^k}$ so that $f_{*,*}(x) \leq g(x) + \epsilon$. Thus $|f_{*,*} - g| < \epsilon$ on X.

□

Theorem 12. (Arzela-Ascoli theorem - necessary/suff. conditions for conditional compactness in $C(X)$)
Let X be a compact Hausdorff space. Consider a subset $\mathcal{A}$ of the space $\mathcal{C}(X)$ of real-valued continuous functions on X. Then $\overline{\mathcal{A}}$ is compact $\Leftrightarrow$

1. $\sup\limits_{f\in\mathcal{A}}\sup\limits_{x\in X}|f(x)| < \infty$ (Uniform boundedness)

2. $\lim\limits_{\delta\to 0}\sup\limits_{x,y:d(x,y)<\delta} f(x) - f(y) = 0$ (Equicontinuity)

Depends on: dependencies...

Proof idea: Proof here $\square$

Theorem 13. (Necessary/sufficient conditions for separability of $\mathcal{C}(X)$)
Let X be a normal space. Then $\mathcal{C}(X)$ is separable $\Leftrightarrow$ X is a compact metric space. ***Depends on:*** Continuity, Dini's theorem, separability of X

Proof idea: Long. Here is the outline:

1. $Rightarrow$: Suppose $\mathcal{A}$ is compact.

   (a) Define $\Phi(f) = \sup\limits_x |f(x)|$ and since it $|\Phi(f_1)-\Phi(f_2)| \leq \sup\limits_x |f_1(x)-f_2(x)|$, it follows that Φ is cnts and so is bounded on compact $\mathcal{A}$.

   (b) Define $\omega_f(\delta) = \sup\limits_{x,y:d(x,y)<\delta} |f(x)-f(y)|$. Show that $|\omega_f(\delta)-\omega_g(\delta)| \leq 2\sup|f-g|$ so it is cnts. and goes to 0 as $\delta \to 0$. Apply Dini's theorem.

2. $Leftarrow$: We will construct a convergent subsequence from any sequence f_n in $\mathcal{A}$. Since X is separable take a countable dense set $\{x_i\}_{i\geq 1}$. At each point x_i use the uniform bound on f_n and compactness of $[-M, M]$ to extract a subsequence $(f_{n_k}(x_i))_{n\geq 1}$.

3. Diagonalize to get a sequence f_{n_k} in $\mathcal{A}$ converging to some a_i at each x_i $\forall i \geq 1$.

4. Show that $\sup\limits_x f_{n_j}(x) \to 0$. Take δ s.t. $\omega_f(\delta) < \epsilon/3$, $\forall f \in \mathcal{A}$. Use the triangle inequality (take an $x_i \in B(x,\delta)$ and use cauchy property of the sequence $f_{n_j}(x_i)$.

$\square$

Theorem 14. (Tight measures)
Any countably additive measure μ on a complete separable metric space X is a tight measure.

Depends on: dependencies...

Proof idea: Proof here $\square$

Theorem 15. (Riesz representation on metric spaces)
Let X be a compact metric space along with the Banach space of continuous $\mathbb{R}$-valued functions on X, $\mathcal{C}(X)$ (with sup norm). Suppose $\Lambda(f)$ is a linear functional on $\mathcal{C}(X)$ s.t. $f \geq 0 \Rightarrow \Lambda(f) \geq 0$ and $\Lambda(1) = 1$. Then $\exists$ a measure μ on $(X, \mathcal{B})$ s.t. $\Lambda(f) = \displaystyle\int_X f(x) d\mu$.

Depends on: dependencies...

Proof idea: Proof here $\square$

10.3 Important examples

1. The countable product of separable spaces need not be countable wrt other metrics. Take $\mathbb{R}^\infty$ along with the sup metric $d(\{x_i\}, \{y_i\}) = \sup\limits_i \dfrac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)}$. WHY NOT?

2. An example of applying Baire Category: $\mathbb{Q}$ cannot be complete under any metric. Suppose it was then. Then $\mathbb{Q} = \cup\{r_i\}$, each of which has empty interior, which violates Baire category.

3. Another application of Baire Cateogory: there exists a continuous nowhere-differentiable function on the unit interval. Take $X = C([0, 1])$ under the sup norm. Let G_n be the the set of functions for which the LH and RH derivative limits are order n. One can show that each G_n is open and dense and so $\cap G_n$ is nonempty.