Real Analysis Abridged Notes 2008

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1 Basic properties of $\mathbb{R}$

1. **Lindelof property**
   (a) **Statement:** all open covers have a countable subcover
   (b) **Depends on:** axiom of choice, countability of rationals
   (c) **Used for:** alternative definition of separable space
2. **Heine Borel**
   (a) **Statement:** a closed and bounded subset is compact
   (b) **Depends on:** sup/inf construction
   (c) **Used for:** showing countable additivity of Lebesgue measure on the semiring of intervals.
3. **Finite intersection property**
   (a) **Statement:** any collection of closed sets that has nonempty finite intersections has nonempty total intersection
   (b) **Depends on:** (equivalent to Heine-Borel by complementation)
   (c) **Used for:** same things as Heine Borel, more directly used in probability theory to show equivalence of distribution functions and prob. measures and also for Kolmogorov's consistency theorem.
4. **Characterization of open sets**
   (a) **Statement:** any open set of $\mathbb{R}$ is a disjoint countable union of open intervals
   (b) **Depends on:** countability of rationals, sup/inf construction
   (c) **Used for:** Used to show that if the indefinite integral $F(x) = \int_a^x f d\mu$ is identically 0 then so is $f(x)$.

2 Measure Theory

1. **Characterizations of countable additivity**

   (a) **Statement:** For finite measures, countable additivity can be shown either by showing that the measure of a disjoint union is the sum, or the measure of a limit of an increasing/decreasing sequence of sets is the limit of measures. The most common application is to show that if a sequence of sets decreases to the null set than the measure goes to 0.

   (b) **Depends on:** (set theory and reinterpretations of unions of sets)

   (c) **Used for:** anytime we want to show that a measure is countably additive on some class of sets. Countable additivity is often applied to a sequence of sets $(A_n) \downarrow \phi$ to get N s.t. $\mu(A_n) < \epsilon$ for $n \geq N$.

2. **Caratheodory extension theorem**

   (a) **Statement:** any countably additive measure on a field extends uniquely to its generated sigma-field

   (b) **Depends on:** Countable additivity, outer measure construction, definition of measurable sets, monotone class theorem (for uniqueness)

   (c) **Used for:** Constructing the Lebesgue measure (apply to field generated by intervals), constructing an infinite product measure (apply to field of measurable rectangles)

3. **Monotone class theorem**

   (a) **Statement:** the smallest monotone class containing a field also contains its fenerated sigma-field.

   (b) **Depends on:** definition of monotone class, relationship between fields and monotone classes (monotone classes that are fields are sigma-fields)

   (c) **Used for:** Any time we need to show that a property holds on a sigma-field, we can show that the class of sets it holds is a monotone class. For example, uniqueness of Caratheodory extension measures, and showing that cross-sections of measurable rectangles are measurable/integrable for Fubini's theorem.

4. **Existence of Lebesgue measure**

(a) **Statement:** there is a c.a. measure on $\mathbb{R}$ that agrees with the length function on intervals and is translation invariant.

(b) **Depends on:** Caratheodory extension theorem, Heine borel (to show on semiring of intervals) or equiv. FIP.

(c) **Used for:** same things as Heine Borel, more directly used in probability theory to show equivalence of distribution functions and prob. measures and also for Kolmogorov's consistency theorem.

3 Measurable sets and functions

1. **Characterization of measurable sets**

   (a) **Statement:** a measurable set can be over-approximated by an open set, under-approximated by a closed set to arbitrary ϵ-measure error.

   (b) **Depends on:** definition of measurable set, outer measure.

   (c) **Used for:** Approximating functions by step fns (which in turn are approx. by cnts fns). For instance, the support sets of a simple fn can be approx. by intervals.

2. **Approximation of measurable functions**

   (a) **Statement:** any measurable function (taking finite values a.e.) on a finite interval can be ϵ-approximated by a simple/step/cnts function outside a δ-measure set, $\forall \epsilon, \delta$.

   (b) **Depends on:** measurable fn –¿ bnded fn –¿ simple fn –¿ step fn (approx. of measurable sets by intervals) –¿ cnts fns

   (c) **Used for:** Any time we want to show that a $\mathbb{R}$-valued function is measurable.

3. **Approximation of measurable functions by limits**

   (a) **Statement:** all (nonnegative) [bounded] measurable fns are (monotone) [uniform] limits of simple fns

   (b) **Depends on:** truncation, approxmation of bnded fns by simple fns.

   (c) **Used for:** Any time we want to show verify a property of a measurable fn, we can verify it for indicator fns, then simple fns, and pass monotone limits to verify for measurable fns - e.g.:

  i. definition of the Lebesgue integral for nonneg. measurable fns
  ii. showing that the dual of L_p is L_q
  iii. Fubini's theorem

4. **Egoroff's theorem**

   (a) **Statement:** a.e. convergence on a finite-measure set implies uniform convergence outside a δ-measure subset. In particular, for every ϵ, δ we can find a point in the sequence (f_n) after which the limit (f) is approximated up to ϵ-accuracy outside of a δ-measure subset.

   (b) **Depends on:** Construction of 'ϵ-nonconvergence sets' and their relation with a.e. convergence (the intersection goes to Φ), countable additivity. Uniform convergence comes from repeated application of the more general statement with $\epsilon = \delta = 2^-n$.

   (c) **Used for:** Lusin's theorem, bounded convergence theorem

5. **Lusin's theorem**

   (a) **Statement:** measurable fns on finite intervals are continuous except on δ-measure sets.

   (b) **Depends on:** Approximation of measurable fns by cnts fns, repeated application of Egoroff's theorem, closure of cnts fns under uniform limits.

   (c) **Used for:** Nothing yet. Interesting fact, though.

4 Integration and convergence

1. **Bounded convergence theorem**

   (a) **Statement:** For finite measures, integral and limits can be exchanged as long as the integrands are uniformly bounded.

   (b) **Depends on:** Egoroff's theorem allows easy interchange of integral and (uniform) limit except on a small set, whose integral can be made arbitrarily small.

   (c) **Used for:** interchange of limit and integral for uniformly bounded sequences of functions - finite measure only!

2. **Fatou's Lemma**

(a) **Statement:** if a nonnegative sequence of functions f_n converges in measure to a limit f, then the integral of f is bounded above by the lim inf of the integrals of the f_n's

(b) **Depends on:** Approximation of nonneg. measurable fns by simple functions (monotone limits), construction of the '$\phi_{1-\epsilon}$-exceeding' set (whose complement goes to Φ), countable additivity.

(c) **Used for:** Interchanging limit with integral for general measures, proving dominated convergence theorems, Riesz-Fischer theorem, differentiability of monotone functions

3. **Monotone convergence theorem**

(a) **Statement:** if a nonnegative sequence of functions monotically converges to another function from below, we can exchange limit and integral.

(b) **Depends on:** Fatou's lemma - the assumption shows the reverse inequality.

(c) **Used for:** Fubini's theorem (measurability of cross-section), L_p duality theorem, in general can be used in a truncation argument(a function truncated at N for $N = 1, 2, ...$ is a monotone sequence of functions converging to the original) - used e.g., to show continuity of indefinite integral functions.

4. **Dominated convergence theorem**

(a) **Statement:** if a sequence of functions are all dominated by an integrable function, then we can exchange limit and integral.

(b) **Depends on:** using Fatou's lemma on $(f_n + g)$ and $(f_n - g)$ to characterize both the $\liminf$ and $\limsup$ of the integrals. In fact we can generalize this to the case where f_n dominated by integrable g_n which converge pointwise and in L_1 to some $g \in L_1$.

(c) **Used for:** Interchanging limit with integral for general measures

5. **Uniform integrability and convergence**

(a) **Statement:** Assume finite measure and lots of small sets. If a sequence of functions converges in L_1 to an L_1 function, then the functions are uniformly integrable. Conversely, if the sequence is uniformly integrable and converges to some function in measure, then it converges in L_1 to that function.

(b) **Depends on:** The forward is an $\epsilon/3$ argument. Take a set small enough so that the integral of the L_1 limit is small. Take N large s.t. L_1 difference is small. Then take δ_i's s.t. the f_n integrals are small. Take the min of the first N of these δ's. For the reverse, split the L_1 integral by a cutoff l. Use BCT for the bounded part, use uniform integrability and Fatou's lemma to show the limit fn is L_1 and there is L_1 cvgnce.

(c) **Used for: Uniform integrability is more like a necessary and suff. condition for when we can interchange limit and integral.** Here is one important application - suppose we have a sequence of *integrable nonneg.* fns cvging a.e. to some fn (Fatou's lemma holds with equality). Then one can show by a 'subtract and add fn' trick that the f_n are uniformly integrable. From this we get L_1 convergence.

5 L_p spaces

1. **Minkowski's inequality**

   (a) **Statement:** the triangle inequality holds for the L_p norm for $1 \leq p \leq \infty$

   (b) **Depends on:** monotonicity and convexity of $|x|^p$, manipulations

   (c) **Used for:** Anything that needs triangle inequalit in the L_p setting, e.g. Riesz-Fishcher theorem

2. **Holder's inequality**

   (a) **Statement:** if $f \in L_p$, $g \in L_q$, then fg is integrable with L_1 norm less than the product of the norms in their respective spaces.

   (b) **Depends on:** Young's inequality and concavity of the logarithm to show $AB \leq \frac{A^p}{p} + \frac{B^q}{q}$. Manipulations with 'normalized' functions.

   (c) **Used for:** Cauchy-schwartz inequality ($p = 2$ case), L_p duality (showing that $\int fgd\mu$ is a bounded linear functional on $f \in L_p$ for $g \in L_q$.

3. **Riesz-fischer theorem**

   (a) **Statement:** L_p spaces are complete

(b) **Depends on:** constructing a subsequence that cvgs a.e. and in L_p to a function in L_p (this employs Fatou's Lemma and Minkowski's inequality and completeness of $\mathbb{R}$). Convergence of absolute series implies convergence of regular series, which allows application of DCT. To extrapolate from the L_p convergent subsequence, use Fatou's lemma and the L_p-cauchy assumption.

(c) **Used for:** This is an important topological result for L_p spaces - now it is enough to show a sequence in cauchy in order to show existence of a limit.

4. **Dominated convergence in L_p**

   (a) **Statement:** If a sequence of fns cnvgs a.e. and are dominated by an L_p function, then their limit is in L_p and the converges is wrt L_p norm as well.

   (b) **Depends on:** Just use regular DCT functions to the p'th power.

   (c) **Used for:** Showing existence of limits in L_p when we have an a.e. limit, e.g., approximating L_p fns with simple functions.

5. **Approximation of L_p functions**

   (a) **Statement:** For finite measure spaces, any L_p function can be approximated to ϵ-accuracy wrt the L_p norm by a simple, step, or cnts function.

   (b) **Depends on:** the usual sequence of approximations (truncate to a bounded, then approximate by a simple function, etc.). The key is to use the L_p-DCT to get L_1 convergence of the truncations to the original function, then use the previous approximating techniques along with Minkowski's inequality.

   (c) **Used for:** This just says that L_p functions are 'nice' functions wrt the L_p norm.

6. **Characterization of the L_∞ norm**

   (a) **Statement:** For finite measure spaces, the L_∞ norm is the limit of the L_p norm.

   (b) **Depends on:** the fact that $x^{1/p} \to 1$ as $p \to \infty$ for any x, properties of the $ess\sup$.

   (c) **Used for:** (???) It indicates that L_∞ fits well into the L_p framework (i.e. the essential supremum norm is not arbitrary, but is a natural limit).

7. **Inclusion of L_p's**

   (a) **Statement:** For finite measure spaces, $L_p \subseteq L_{p'}$ for $1 \leq p' \leq p \leq \infty$.

   (b) **Depends on:** Splitting integral into where f is bigger/smaller than 1.

   (c) **Used for:** To prove L_p convergence for multiple p, we need only prove for the maximal p for which it holds (for finite measures), since it follows that L_p convergence implies $L_{p'}$ converges for $p' \leq p$.

6 Modes of convergence

1. **Cauchy-in-measure cvgnce implies cvgnce in measure**

   (a) **Statement:** If a sequence is cauchy in measure, then it is convergent in measure. In particular, there is a subsequence that converges a.e. and a.u. to this limit.

   (b) **Depends on:** Construction the subsequence along with corresponding 'error sets' (where 2 consecutive functions in the subsequence differ by more than 2^{-k}). Outside the $\limsup$ of the error set, the function values are a uniformly cauchy and thus convergent. The tail unions of the error sets provide sets where a.u. convergence occurs. Showing the subsequence cvgs in measure is easy and use the cauchy property to show that the whole sequence cvgs in measure.

   (c) **Used for:** This proof technique is very useful anytime we want to construct an a.e. cvgt subsequence from a sequence converging in measure. 'Cauchy-in-measue' is also one of the weakest forms of convergence, so this tells us exactly what we can squeeze out of it.

2. L_p **cvgnce implies cvgnce in measure**

   (a) **Statement:** Any sequence cvging in L_p to a fn also cvgs in measure

   (b) **Depends on:** Chebyshev's inequality

   (c) **Used for:** Proving L_p convergence implies $L_{p'}$ convergence for finite measure spaces if $p' < p$.

3. **Vitali convergence theorem - necessary/suff. conditions for L_p cvgnce**

   (a) **Statement:** A sequence convergens to a fn in L_p iff they converge to the fn in measure, the p'th powers are uniformly integrable, and there is a set of finite measure outside of which the p'th power integral of every fn in the sequence is $O(\epsilon^p)$ (always true for finite measures).

   (b) **Depends on:** We can ignore the 'infinite measure' component using Minkowski's inequality (to order ϵ) so that the remaining set has finite measure A We can construct 'error sets' H_{nm} where f_n and f_m differ by more than ϵ/A and show that there is cauchy-L_p-cvgnce on the finite-measure set by splitting the integral into a set H_{nm} of small enough measure (using cvgnce in measure/uniform integrability) and its complement (using Minkowski's).

   (c) **Used for:** Relationship between uniform integrability and modes of convergence (it is exactly the condition we need along with cvgnce in measure to get L_p convergence).

4. **Relationship between uniform integrability and L_p-norm**

   (a) **Statement:** Uniform boundedness in L_p-norm for $p > 1$ implies uniform integrability.

   (b) **Depends on:** Splitting integral over a set A into a part where $f_n \geq l$ and its complement, for arbitrary l and using a 2-step argument (choose l large s.t. *** and then choose $\mu(A)$ small enough s.t. *** satisfied).

   (c) **Used for:** Marks an important difference between L_p norm for $p > 1$ and $p = 1$. Uniform integrability is kind of like a 'domination' condition. Consider the example that is n from 1/n to 2/n.

7 Approximation theorems

In approximating a function with another function, keep in mind the following:

1. What is the accuracy of the approximation? With respect to what norm?

2. If the norm is the sup norm, where is the approximation valid? What is the measure of the set where the approximation is not valid?

3. Even if the approximation is not wrt to a norm, can we define someform of approximation (e.g., pointwise limit a.e.).

7.1 Measurable functions: ϵ-accuracy outside δ-sets

1. **Any measurable function on a set of finite measure taking finite values a.e. can be ϵ-approximated outside a δ-set by a bounded, simple, step, or cnts function.**

2. **Any measurable function is the pointwise limit (a.e.) of a sequence of bounded,simple,step, or cnts fns.** We can make the limit monotone and uniform if the function is bounded (bin the range and take the lower limit at each bin). We can make the limit monotone if the function is nonnegative (just take the cutoff value on the set being ignored).

3. **If the measure is sigma-finite, then we can ϵ-approximate any measurable function outside a δ-set by a step or cnts function.** Just take a step/cnts $2^{-j}\epsilon$-approximation on each A_j except on a $2^{-j}\delta/3$-set, then $2^{-j}\delta/3$-approximate the A_j by intervals, and 'connect them up' if needed (by another $2^{-j}\delta/3$-approximation). Note that this doesnt work with simple functions since they can only assume finitely many values.

7.2 L_p functions - What's dense in L_p?

1. **For finite measures, bounded/simple/step fns are dense in L_p wrt $\|.\|_p$ for $1 \leq p \leq \infty$. Cnts fns are also dense if $p < \infty$.**

   This follows from the DCT for L_p functions to get L_p convergence of the successive (bounded) truncations to the original L_p fn, followed by Minkowski's inequality (for other approximations).

2. **For sigma-finite measures, bounded/simple fns are dense in L_p wrt $\|.\|_p$ for $1 \leq p \leq \infty$. Step/cnts fns are dense for** $1 \leq p < \infty$.

   Just take approximations on a sequence of bounded sets increasing to the whole space. Since the fn is p-integrable, the p-power integral outside of a suff. large set is negligible. If we want a cnts approx, we just need the function to fall of very quickly outside the bounded set. **We cannot do this for $p = \infty$ - e.g., how do we approx the heaviside fn by a cnts fn wrt $\|.\|_\infty$?.**

3. **For finite measures, polynomials (and thus C^∞ and real-analytic fns) are dense in L_p wrt the** sup **norm as well as $\|.\|_p$ for $1 \leq p < \infty$.**

   By the Stone-Weierstrass theorem (see below) are dense in the space of cnts fns wrt sup-norm. Use Minkowski and above results.

4. **For sigma-finite measures, C^∞ fns are dense in L_p for $1 \leq < p \infty$.**

   Take a polynomial approximation on a sequence of bounded sets increasing to the whole space. Weight the approximation by the C^∞ 'bump' function $e^{-1/(1-x^2)}$ so that it falls off very quickly (in a smooth way) outside the each bounded set.

5. **For sigma-finite measures, L_p is separable (wrt $\|.\|_p$) for $1 \leq p < \infty$. L_∞ is not separable wrt $\|.\|_\infty$.**

   For $p < \infty$ and finite measure, we can take polynomials with rational coefficients as the countable dense set by Stone-Weierstrass. For $p < \infty$ and sigma-finite measure, just take these polynomial approximations on a countable sequence of increasing bounded sets. For $p = \infty$, if there was a countable dense set, divide up the domain into a countable partition A_j, and define a function that is different by at least ϵ from the j'th function in the dense set on the set A_j.

7.3 Things we cannot do

1. ϵ-approximate a measurable function on an entire set (even of finite measure). We can, however, use simple functions to uniformly approximate a bounded measurable function for any measure space.

2. Approximate an L_p fn by a cnts fn wrt sup-norm or $\|.\|_\infty$. Just consider the heaviside function.

3. Approximate an L_p fn wrt $\|.\|_p$ on a sigma-infinite measure by a real-analytic function????

8 Functional analysis and L_p

1. **Characterization of the dual space**

   (a) **Statement:** The set of bounded (or equivalently, continuous) linear funcitonals on a Banach space is itself a Banach space under the induced 'sup on unit ball' norm.

   (b) **Depends on:** Manipulations and definitions. Completeness is a bit tricky but basically use the completeness of $\mathbb{R}$ and show that the point-wise limit is in fact the limiting BLF of the cauchy BLF sequence.

   (c) **Used for:** This is important, especially since dual spaces are important in studying L_p spaces.

2. **Properties of Hilbert spaces**

   (a) **Statement:** In Hilbert spaces, we have the Cauchy-Schwartz inequality, the Parallelogram Law, the Decomposition by orthogonal complements.

   (b) **Depends on:** Cauchy-Schwartz can be derived using quadratic forms. Parallelogram Law is manipulation (or geometry). Decomposition theorem - define the projection onto a subspace as the closest point on the subspace to the point. Show that the minimum is actually achieved by showing that the minimzing sequence is Cauchy (use the parallelogram law with corners $x, \frac{y_1+y_2}{2}, y1, (\frac{y_1-y_2}{2}+x))$.

   (c) **Used for:** proving the Riesz Representation theorem for Hilbert spaces, In probability theory, conditional expectations are orthogonal projections for L_2 random variables.

3. **Characterization of BLFs**

   (a) **Statement:** A bounded linear functional on a Banach space is bounded iff it is continuous at one point.

   (b) **Depends on:** Linearity of the functional. manipulation and dual-space norm.

   (c) **Used for:** (???)

4. **Riesz representation theorem for Hilbert spaces**

(a) **Statement:** All bounded linear functionals on Hilbert spaces are equivalent to inner products with a point in the space.

(b) **Depends on:** The decomposition theorem and characterization of the kernel of the linear functional in question. Take a point e in the kernel's orthogonal complement and use the fact that $(\Lambda(e)x - x\Lambda(e))$ is in the kernel and so is orthogonal to e.

(c) **Used for:** Von-Neumann proof of the Radon-Nikodym theorem. Generalization of the linear algebraic theorem for finite-dimensional inner product spaces.

5. **Hahn-banach theorem**

    (a) **Statement:** Any bounded linear functional on a subspace that is dominated by a semi-norm can be extended to a a BLF on the whole space, dominated by the same semi-norm.

    (b) **Depends on:** Axion of choice and iteration. Given a y in the space for which the BLF is not yet defined how can we choose $\Lambda(y)$ to satisfy all the constraints (linearity and domination by the seminorm). Show there always exists a valid value.

    (c) **Used for:** Convex sets (???). Showing the dual of L_∞ contains more than just L_1.

6. **Hahn decomposition theorem**

    (a) **Statement:** Any a measurable space with a signed measure can be partitioned into a 'totally positive' and 'totally negative' set.

    (b) **Depends on:** Iterative process using sups and infs - repeatedly remove sets of negative measure and show that what is left over is totally positive. A key fact is that if the measure of a set is positive (negative) than there is a totally positive (negative) subset of more (less) measure.

    (c) **Used for:** Jordan decomposition of signed measures, proof of the Radon-Nikodym theorem

7. **Radon Nikodym theorem**

    (a) **Statement:** If a nonneg. measure μ is absolutely cnts wrt another nonneg. sigma-finite measure ν, then there is a measurable nonneg. function f so that on any measurable set A we can get $\mu(A)$ by integrating f on A wrt ν.

(b) **Depends on:** A sequence of Hahn decompositions on the signed measure $\mu - c\nu$ for rational c. The totally positive components decrease to null, and so we can define f from its level sets (for what c does x go from the positive to the negative component?).

(c) **Used for:** Lots of things - L_p duality, conditional expectations

8. L_p **duality theorem**

   (a) **Statement:** The dual space of L_p is L_q where $1/p + 1/q = 1$ IF $p \in [1, \infty)$.

   (b) **Depends on:** Define a measure using the linear functional on characteristic functions of sets. Take a Radon-Nikodym theorem and use density of simple functions in L_p to show that this is in fact the function we want but we have to show it is in L_q. This also used the density of simple functions, particularly as monotone limits (do first for finite measures).

   (c) **Used for:** This is huge for understanding topology of L_p spaces. Don't know any applications yet.

9. **Weak * compactness in Banach spaces**

   (a) **Statement:** Suppose $B*$ is th dual of some banach space B. Then any bounded set in $B*$ is compact wrt the weak* topology, i.e. every sequence has a subsequence that converges 'point-wise' (apply to any point $x \in B$). In particular if B is reflexive ($B = B**$), then it every bounded set in B is weakly compact.

   (b) **Depends on:**

   (c) **Used for:** This applies to the L_p spaces for $p > 1$ since these are duals of L_q's (they are reflexive). In other words, every uniformly bounded sequence in L_p,$p > 1$ has a subsequence that converges in the sense of integrating against any function in L_q. Since L_1 is not the dual of L_∞ this does not hold. Consider the sequence $f_n = n1_{1/n,2/n}$ inside the unit ball of L_1. There is no subsequence that could possibly converge when integrated against $g \equiv 1 \in L_\infty$.

9 Calculus on the real line

1. **Vitali covering lemma**

(a) **Statement:** If a set of finite measure is covered by a collection of intervals in the sense of Vitali, then there is a finite number of intervals from the collection that 'almost' covers the set to arbitrary accuracy.

(b) **Depends on:** Axiom of choice and iterative process with a sup construction. Form a maximal disjoint infinite sequence of intervals whose tail must have 0 measure. Show that the measure of the set outside a large enough partial union is small.

(c) **Used for:** proving monotone functions are differentiable.

2. **Monotone functions and differentiability**

   (a) **Statement:** Monotone measurable functions on finite closed intervals are differentiable a.e. Furthermore they satisfy a 'sub' FTOC condition that the difference in function values at the endpoints of an interval is at least the integral of the derivative on that interval.

   (b) **Depends on:** Show that the measure of the set where derivates (limsup/liminf coming from the right/left) are unequal is 0 using a Vitali covering argument. Do this by estimating the amount the function jumps across the interval. Use countability of rationals and countable additivity. To show the FTOC condition, use Fatou's lemma and limit quotient expression for the derivative.

   (c) **Used for:** discovering the class of differentiable function - e.g., it is used to show functions of BV are a.e. differentiable.

3. **Characterization of functions of bounded variation**

   (a) **Statement:** BV functions (on finite intervals) are differences of monotone functions (so they are diff. a.e.).

   (b) **Depends on:** definition of BV, positive, negative, total variation.

   (c) **Used for:** further expanding the sets of diff. fns

4. **Characterization of BV functions**

   (a) **Statement:** Functions of BV have LH and RH limits for all interior points and only a countable number of discontinuities.

   (b) **Depends on:** Use the fact that a limit exists iff the limsup/liminf are the same. To show countable discontinuities, just show it for a monotone function using the countability of rationals.

(c) **Used for:** further understanding what a certain class of a.e. diff. fns looks like.

Characterization of indefinite integrals

(a) **Statement:** An indefinite intetgral on a closed finite interval of an L_1 function is absolutely cnts, of BV, and is identically 0 if the integrand is identically 0. Furthermore, the derivative exists a.e. and equals the integrand a.e. The derivative quotient also converges to the integrand in L_1.

(b) **Depends on:** Abs. continuity follow from the fact that an L_1 function is uniformly integrable (show first for bnded, then use the 'subtract and add truncation' trick). The variation is bounded by the L_1 norm of the integrand. To show the 0 property, assume otherwise and take a closed set where the integrand is positive and express the complement as a union of open intevals to get a contradiction. To show the derivative is the integrand, do bounded case by showin that the integral of the difference between integrand and derivative is 0 a.e. and use the previous property. For the general case, use the 'subtract and add truncation' trick and sub-FTOC property. To show L_1 convergence, approximating the integrand by a 'nice' function + an error function, showing the result for nice functions, and getting uniform control on the error.

(c) **Used for:** Fundamental theorem of calculus

5. **Characterization of absolutely cnts fns**

(a) **Statement:** Absolutely cnts fns are of BV (and so are a.e. differentiable), are constant a.e. if their derivative is 0 a.e., and are the indefinite integral of their derivative, a.e.

(b) **Depends on:** Showing they are BV is easy. If the derivative is 0, we can make a Vitali cover by taking all intervals where the change in the function is less than a constant times the length of the interval. Applying the Lemma, and using absolute continuity for the remaining set gives a bound on $|f(c)-f(a)|$ for all $c \in [a, b]$. To show they are the integral of their derivative, assume monotonicity and the difference between the fn and integral-of-derivative is 0 a.e. and use previous property.

(c) **Used for:** This establishes that absolutely cnts functions are exactly the indefinite integrals and for which the FTOC holds.

10 Product measures

1. **Existence of product measures**

   (a) **Statement:** There exists a product measure on the sigma-field generated by measurable rectangles (i.e. $\mu(A \times B) = \mu_1(A)\mu_2(B)$).

   (b) **Depends on:** Show that the product measure is c.a. on the field generated by measurable rectangles and use the extension theorem. The hardest part is to show that if $A \times B$ is the countable union of rectangles, then the measures add up right. The key is noticing that for any point $x \in A$, we can partition the B into the second components of the each rectangle in the union for which x is in the first component. Integrate both sides and use MCT.

   (c) **Used for:** This is essential in order for multidimensional integrals to make sense. Fubini's theorem would not make sense otherwise.

2. **Cross-section set/function measurability**

   (a) **Statement:** For any set in the product sigma-field, the x and y cross-sections are measurable for a.e. x and y resp. For any function on the product space meas. wrt product measure, the x and y cross-section *functions* are measurable.

   (b) **Depends on:** Show it for simple rectangles and show that class of sets for which the statement holds is closed under subtraction and countable union (in fact, cross-sections and unions 'commute'). To show for functions, just interpret the set for which the function is greater than α as the cross-section of some product-measurable set.

   (c) **Used for:** Now we can define functions that are measures of cross-sections (since we know now they are measurable sets) - see next theorem.

3. **Cross-section *measure* measurability**

   (a) **Statement:** If the 2 measures are sigma-finite, then for any set in the product sigma-field, the individual measures evaluated at the cross-sections ($\mu_1(E_y)$ and $\mu_2(E_x)$) are measurable functions, and their integrals wrt μ_2, μ_1 (resp.) is equal to the product measure of the original set.

   (b) **Depends on:** Show it for simple rectangles (the functions in question are just char fns times measures of sets). Show closure under finite unions and under sequences of increasing/decreasing sets (assume finite measure - easy to generalize for sigma-finite) (use MCT). Then apply the monotone class theorem.

(c) **Used for:** Fubini's theorem

4. **Fubinis/Tonelli's theorem**

   (a) **Statement:** Suppose a function is measurable on the product space. If it is nonnegative an the 2 measures are sigma-finite, then the integrals of the functional cross-sections are measurable and the double integrals in either order are both equal to the integral of the original function (even if they are all infinite!). If there are not necessarily any assumptions about sigma-finiteness or nonnegativity, but the function integrable wrt product measure, then the same conclusion holds.

   (b) **Depends on:** Show for characteristic, then simple (by linearity), nonneg (MCT), and finally measurable fns (nonneg decomposition). The result for char fns is just the previous theorem.

   (c) **Used for:** Switching order of integration.

11 Topological aspects

1. **Characterizations of separability**

   (a) **Statement:** Separability can be defined equivalently as having a countable dense set, a countable basis, a countable number of 'smaller open containers', and satisfying the Lindelof property (countable subcover for every cover). Furthermore, infinite products of separable spaces are separable wrt a special product metric that decays exponentially for higher coordinates.

   (b) **Depends on:** Manipulations - a countable basis and the countable smaller container can be shown to be equivalent. If we have a countable dense set, we can take a countable sequence of smaller and smaller balls around each element and this constitutes a basis. Using an axiom of choice argument we can use a countable basis to extract a countable subcover from an arbitrary open cover. Finally, if we have the Lindelof property, then we can take balls around each point of arbitrarily small size and extract a countable subcover. Concatenating the centers of these balls is a countable dense set. For infinite products, just take a countable dense set in each coordinate space and take a countable union of larger and larger 'cylinders'.

(c) **Used for:** (???)

2. **Completion of spaces**

(a) **Statement:** Any space can be made complete.

(b) **Depends on:** Embedding trick - take the space of all cauchy sequences with metric being the limiting cooridnate-wise distance. Then the original space is isomorphic to those sequences with only one distinct element, and the metrics agree. In fact the embedding is dense in the space of cauchy sequences. One can show using a 'diagonalization'-type trick that a cauchy sequence of cauchy sequences has a limiting cauchy sequence.

(c) **Used for:** (???)

3. **Characterizations of compactness**

(a) **Statement:** A compact set can be defined equivalently as having a finite subcover for every open cover, having a convergent subsequence for every sequence, having the nonempty intersection property for every collection satisfying the FIP, being totally bounded and complete, being closed and bounded (if we are in a metric space). Compact spaces are also closed under countable products (wrt to product measure).

(b) **Depends on:** The FIP and subcovering notions are equivalent by complementation. Closed and bounded implies finite subcover by Heine-Borel. If we have the Bolzano-Weierstrass property, then we can take a finite maximal set of disjoint (arb. small) balls and for every open covering each of these is contained in one of the open sets, so we can get an finite subcover. If we have the FIP property, then we can take any subsequence and take the set of closures of partial tails of the sequence which satisfies FIP and so has nonempty intersection, i.e. the sequence has a cluster point. If we are totally bounded and complete, then for any sequence we can repeatedly apply pigeonhole principle to the finite cover of $1/n$-radius balls to extract a cauchy (and thus convergent) subsequence. If we have finite subcovers in a metric space, use properties of continuous functions on compact spaces applied to the distance function. To show countable products are compact, use diagonlazation, the convergent subsequence property and the fact convergence in the product metric is equiv. to coordinate-wise convergence.

(c) **Used for:** Proving that anything involving compactness (e.g., next point)

4. **Cnts fns on compact spaces**

   (a) **Statement:** Cnts fns on compact sets are bounded, uniformly cnts, achieve their max/min, and take compact sets to compact sets.

   (b) **Depends on:** Boundedness and achieving max/min follow from the Bolzano-Weierstrass property of compactness. Show uniformy continuity by constrcting a special cover of the space using continuity neighborhoods. In the image of a compact set, a sequence corresponds to a sequence in the domain, and extraction of a convergent subsequence in either one is equivalent by continuity.

   (c) **Used for:** Obvious applications in calculus.

5. **Dini's theorem**

   (a) **Statement:** Convergence in a compact set is uniform.

   (b) **Depends on:** Taking a cover of continuity neighborhoods of $f(.) = f_{n_0(x)}(.)$.

   (c) **Used for:** Obvious applications.

6. **Characterization of normal spaces**

   (a) **Statement:** Compact hausdorff spaces regular spaces. Regular spaces are normal. In particular, metric spaces are normal.

   (b) **Depends on:** To show its regular, just apply the Hausdorff property and take a finite subcover of the closed subset. To show it's normal, apply Regularity property and again extract finite subcovers.

   (c) **Used for:** Normal spaces are important since Urysohn's lemma applies to them. So it is good if we know which spaces are normal.

7. **Urysohn's Lemma**

   (a) **Statement:** For any 2 closed subsets in a normal space, there is a $[0, 1]$-valued cnts function that is 0 on one set and 1 on the other.

   (b) **Depends on:** Repeated application of the normality property, interpolation of $[0, 1]$ by diadics, and construction of the function by level sets.
   (c) **Used for:** Showing other properties of normal spaces (i.e. the extension of bnded cnts fns), (??? convex sets)

8. **Tietze's extension theorem**
   (a) **Statement:** In a normal space, if a cnts bnded function is defined on a closed subset, then it can be extended to a cnts bnded function on the whole space.
   (b) **Depends on:** Repeated application of Urysohn's lemma - assume $f \in [0, 1]$ and then apply Urysohn's lemma to the preimages of the first and last 3rds of the unit interval to get a function $g \in [1/3, 2/3]$ that differs from f by no more than $1/3$ on the closed set. Then repeat the process for $|g - f|$ which has range less than $2/3$ and so on to get a sequence g_i. Take the limit of this sequence.
   (c) **Used for:** showing what is special about normal spaces

9. **Baire category theorem**
   (a) **Statement:** A complete metric space cannot be the union of closed sets each having no interior. Equivalently, the intersection of a decreasing sequence of dense sets is nonempty (and is actually dense!)
   (b) **Depends on:** construction of a cauchy sequence using properties of density, and then applying completeness to get a limit point in the intersection.
   (c) **Used for:** Categorizing spaces as '1st' or '2nd' category. Various functional analysis theorems (open mapping, closed graph, uniform boundedness theorems)

10. **Stone-weierstrass theorem**
   (a) **Statement:** In a compact Hausdorff space, a class of cnts functions that contains all constants,has a function that takes any 2 values at any 2 distinct points, and is an algebra has a closure which is dense in the space of cnts functions.
   (b) **Depends on:** This is a finite subcover with continuity neighborhood argument. Show that the class is closed under max/min operations. Then take any cnts function g and index the functions

in the collection by 2 points at which they agree with g. Construct a finite subcover of continuity neighborhoods in the first coordinate and take the max. Then do it for the second coordinate and take the min.

(c) **Used for:** Tells you that polynomials are dense in the space of cnts functions (wrt the sup norm)

11. **Arzela-Ascoli theorem**

(a) **Statement:** In a compact Hausdorff space, a class of cnts functions has a compact closure (in the space of cnts fns) iff it is equicontinuous and uniformly bounded wrt the sup norm.

(b) **Depends on:** To go forwards, show the continuity of 2 'functions on the space of cnts fns', namely the sup function, and the modulus of continuity. Use properties of cnts fns on comapact spaces and Dini's theorem. To go backwards, show that every sequence has a convergent subsequence. First construct a subsequence that converges at every point in the countable set (need uniform boundedness and Heine-Borel here), then use equicontinuity to apply a sort of triangle inequality to get a subsequence converging everywhere.

(c) **Used for:** A ton of things - for instance Cauchy-Peano existence of solutions for ODE, or Montel's theorem in complex analysis. Basically any time we want to extract a convergent subsequence from a sequence of functions, it is enough to show that the functions in the sequence come from a class satisfying the assumptions of Arzela-Ascoli.

12. **When is C(X) separable?**

(a) **Statement:** If a domain space is normal, then the space of cnts functions on it is separable iff the domain space is a compact metric space.

(b) **Depends on:** If X is a compact metric space, then construct a countable dense set in $C(X)$ by taking functions that separate points (by Urysohn's Lemma) and applying Stone-Weierstrass. Conversely, take a countable dense set and define a special metric on the domain space which is a decaying sum so that the distance btw 2 points is small iff each of functions in the set evaluate the points to close values. This is a valid metric. To show compactness, assume there's a sequence with no convergent subsequence.

Each function in the countable dense set has a pointwise limit for the sequence (since it's bounded) and we can use their density and diagonalization to get a subsequence for which the any given cnts function will cvg. But then we can apply Urysohn's lemma to the odd and even elements of this subsequence to get a function that alternates 0 and 1 on the sequence which is a contradiction.

(c) **Used for:** (???)

13. **Measures on metric spaces**

    (a) **Statement:** A c.a. Borel measure on a metric space is a regular measure. If the metric space is complete and separable, the measure is tight.

    (b) **Depends on:** To show regularity, show that the class for which it holds is a sigma-field and contains open sets. To show tightness, just show total boundedness. To do this, use the Lindelof notion from separability to extract countable subcovers consisting of balls of arbitrarily decreasing sizes $(1/k)$. Take the intersection of these subcovers over k.

    (c) **Used for:** Tight measures are used for Komogorov's consistency theorem in probability theory.

14. **Riesz representation for cnts fns**

    (a) **Statement:** Let X be a compact metric space. Then all non-neg. linear functionals with $\Lambda(1) = 1$ on $C(X)$ correspond to integrating the function with respect tto some probability Borel measure.

    (b) **Depends on:** Approximation by closed and open sets. Given Λ define 'inner measure' of a closed set C as the infimum over all functions dominating χ_C. Define 'outer measure' of an arbitrary set as the supremum of the inner measure of all under-approximating closed sets. Show that the class of sets for which the inner and outer measures agree contain both closed/open sets and is a sigma-field. Define the corresponding measure as the common value of these measures and show its countably additive.

    (c) **Used for:** This says that (part of) the 'dual space' of $C(X)$ is essentially measures on X.

12 Fourier analysis on Hilbert spaces

1. **Orthonormal sets**

   (a) **Statement:** Consider a countable orthonormal set in a Hilbert space H. If we take weighted sum of this basis elements with weights coming from an l_2 sequence, the the weighted sum converges to some point in H.

   (b) **Depends on:** The squared-norm of any partial sum is just the sum of squares of the weights. This implies that the tail of the infinte sequence goes to 0 (i.e. it's cauchy) and Hilbert spaces are complete.

   (c) **Used for:**

2. **Bessel's inequality**

   (a) **Statement:** Given an orthonormal set in H and any point in H, we can define a sequence of weights corresponding to the inner-product with the point and each element of the ONS.The squared-norm of the point is always greater than any finite sum-of-squares of these weights. This extends to infinite countable ONS or even arbitrary ONS. However, if H is separable, then any ONS is at most countable.

   (b) **Depends on:** For finite ONS's, just expand out the squared-norm of the difference between points and finite weighted sum of elements from the ONS using inner products.To generalize to arbitrary ONS's, take sup's over finite sums. To show countability, use the fact that the distance between any 2 elements of an ONS is 2 - take ϵ-balls around each element of a countable dense set and use pigeonhole.

   (c) **Used for:** Pareseval's theorem

3. **Parseval's theorem (? - not sure if this is actually called this)**

   (a) **Statement:** Take any ONS S in H. Then S is complete iff every $f \in H$ can be expanded out in its coordinates wrt S iff the squared norm of any f is the sum-of-squares of its coordinates wrt S

   (b) **Depends on:** If S is complete show that the inner product between any element of S with the difference between f and its coordinate expansion is 0. This implies that this difference is 0.

The third easily follows from the second by expanding squared-norm using inner product. If we have the third statement, the first is easy since if f is orthogonal to all elements of S then its squared-norm is 0.

(c) **Used for:** Parseval's identity.

4. **Parseval's identity**

   (a) **Statement:** Take a complete ONS S in H. Then the inner product between any f and g is just the inner product (in the complex sense) of their coordinate expansion sequences (each coordinate expansion is an l_2 sequence).

   (b) **Depends on:** Just write it out. Only the diagonal terms in the inner product survive.

   (c) **Used for:** This shows that the Fourier transform (taking a point in H to a unique l_2 coordinate sequence) preserves inner product (isometry).

5. **Riesz-Fischer for L_2**

   (a) **Statement:** The set $\{\varphi_n(x) = e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a complete ONS for $L_2([0,1), \mathcal{B}, dx)$.

   (b) **Depends on:** Orthonormality is easy (just compute the integrals). To show completeness take several steps. Use Stone-Weierstrass to show that polynomials of these elements is dense in $C([0,1))$ wrt sup-norm which implies density wrt L_2-norm. Then since $C([0,1))$ is dense in $L_2([0,1))$, we can take a polynomial p that is arbitrary close to any L_2 fn f. If f is orthogonal to all elements of the set, one can show (using Bessel's inequality) that both $\|p\|_2$ and $\|f-p\|_2$ are arbitrarily small, giving f is 0.

   (c) **Used for:** Now Parseval's theorem etc. can be applied to this special ONS in $L_2([0,1))$ to show that any L_2 fn can be expanded as a sum of trigonometric functions (i.e. a Fourier series). More precisely the convergence is wrt the norm of the Hilbert space, which in this case is $\|.\|_2$.

   (d) **Remark:** Alternatively, instead of using Stone-Weierstrass we can relate this to approximation of identity. We take the convolution of $f(x)$ with the partial sum $D_N(y) = \displaystyle\sum_{n=-N}^{N} e^{2\pi nx}$, which is called the Dirichlet kernel D_N. As $N \to \infty$, $(f * D_N)(x) \to f(x)$ since D_N is an approximation of identity.

6. L_p **convergence of Fourier expansions**

    (a) **Statement:** For any $f \in L_1([0,1))$ we define f_N to be the sum of coordinates indexed in the range $[-N, N]$. We know that $f_N \to f$ in L_2 if $f \in L_2$. In fact if $f \in L_p([0,1))$ for $1 < p < \infty$, then $f_N \to f$ in L_p. This is *not* always the case for $p = 1$ or $p = \infty$.

    (b) **Depends on:**

    (c) **Used for:** Now Parseval's theorem etc. can be applied to this special ONS in $L_2([0,1))$ to show that any L_2 fn can be expanded as a sum of trigonometric functions (i.e. a Fourier series). More precisely the convergence is wrt the norm of the Hilbert space, which in this case is $\|.\|_2$.

7. **Riemann-Lebesgue Lemma**

    (a) **Statement:** For any integrable function on a σ-finite measure space, The fourier transform tends to 0 as $|\omega| \to \infty$ in the frequency domain.

    (b) **Depends on:** Standard procedure. It is obvious that the integral of $e^{i\omega x}$ on a compact interval I goes to 0 for ω large (so the lemma is true for char. fns of intervals). Thus it holds true for all step $\to$ nonneg. $\to$ L_1 fns with compact support. We can generalize to all L_1 functions using the MCT.

    (c) **Used for:** This theorem implies that the contributions of rapid (high-frequency) oscillations of the function to its integral over the domain is very small. It is used to prove validity of asymptotic approximations of integrals.