Research Statement

BY

Joseph Coffey

Stanford University

I am symplectic geometer, at Stanford University. My thesis, and earlier work, looks at symplectic geometry through the lens of algebraic topology. However, more recently I have worked on perspectives coming from geometric measure theory and geometric analysis. I served an NSF Post-Doctoral fellow at Courant/IHES. Please find the attached letters of support from Helmut Hofer, Dusa McDuff, Dennis Sullivan, Richard Hind, and Yakov Eliashberg.

This statement is divided into two sections. The first gives an introduction to my papers [8, 6, 7, 5]. The actual papers, and all of my application materials, are available on my website: math.nyu.edu/coffey. The second describes a project to understand connections between convex surfaces with a contact manifold and holomorphic curves in its symplectization. There have been two major threads in the explosion of research in contact topology in the last decade. One -- the theory of convex surfaces -- provides a ‘‘cellular decomposition” in the contact category, the other -- contact homology -- provides a homology theory constructed out of holomorphic curves in the symplectization. This two threads have thus far proceeded separately, there is yet no way of computing contact homology from the convex decomposition. The major aim of this project is to fill this gap, and we describe their our progress along this path. This would have far reaching consequences, greatly increasing both our computational power and theoretical understanding.

Table of contents

1 Description of papers

1 Failure of the parametric h-principle for maps with prescribed Jacobian

Let M and N be closed n-dimensional manifolds, and equip N with a volume form σ. Let μ be an exact n-form on M which alternates in sign. Arnold then asked the question: When can one find a map $f: M \rightarrow N$ such that $f^*\sigma = \mu$?

In 1973 Eliashberg and Gromov showed that this problem is, in a deep sense, trivial. It satisfies an h-principle; whenever one can find a bundle map $f_{bdl}: TM \rightarrow TN$ which is degree 0 on the base and such that ${f_{bdl}}^*(\sigma) = \mu$ one can homotop this map to a solution f. That is, if the naive topological conditions are satisfied on can find a solution. There is no further interesting geometry in the problem. In [5] we show the corresponding parametric h-principle fails; if one considers *families* of maps inducing μ from σ, one can find interesting topology in the space $\mathcal{G}_\mu$ of solutions which is not predicted by an h-principle.

Denote the space of bundle maps of degree 0, inducing μ by $\mathcal{B}\mathcal{G}_\mu$. Then there is a natural inclusion:

$$\begin{aligned} i: \mathcal{G}_\mu &\rightarrow \mathcal{B}\mathcal{G}_\mu \\ i(g) &\rightarrow (g, Dg) \end{aligned}$$

Eliashberg and Gromov showed that the induced map on π_0:

$$i_*: \pi_0(\mathcal{G}_\mu) \rightarrow \pi_0(\mathcal{B}\mathcal{G}_\mu)$$

is surjective. In [5] we provide an example showing that it need not be a homotopy equivalence.

The situation is similar to that of Legendrian embeddings. Finding a Legendrian embedding in a given isotopy class is a relatively simple matter (it satisfies an h-principle), however the space of such embeddings is quite interesting (the corresponding inclusion into the space of bundle maps is not a homotopy equivalence). This complicated topology has led to a great deal of interesting mathematics: Thurston-Benniquin invariant, relative contact homology etc... In [5] we show that maps with prescribed Jacobian have a similar character.

1.1 Brief description of example: quantized topology from collapsing maps

The details of [5] are somewhat technical. However the basic geometric idea is transparent, and we will describe it here. A parametric h-principle predicts that the homotopy type of $\mathcal{G}_\mu$ should remain the same under certain deformations of the form μ. In [5] we show that the homotopy type of $\mathcal{G}_\mu$ is not stable in this sense. Some forms μ allow maps $f \in \mathcal{G}_\mu$ which collapse hypersurfaces while others do not. These collapsed maps possess extra symmetry which give obstructions to the parametric h-principle.

We consider the case where $M = N = S^2$. Let σ be a volume form on N such that

$$\int_N \sigma = 1$$

We now describe a family of forms on the domain M. Divide M into two open hemispheres, H_+ and H_-, along an equator given by a simple closed curve γ.

Definition 1.1. *Let μ_t be a family of forms, for $0 < t \leq 1$ on M such that:*

1. *Each is non degenerate on the hemispheres $H_\pm$.*
2. *The total area of each hemisphere satisfies:*

$$\int_{H_+} \mu_t = +t$$
$$\int_{H_-} \mu_t = -t$$

3. *For each t there is a neighborhood of γ where $\mu_t = \mu_1$*

Denote by $\mathcal{M}_{\mu_t}$ the space of maps $M \to N$ such that $f^*\sigma = \mu_t$.

The parametric principle would predict that the spaces $\mathcal{M}_{\mu_t}$ would all be homotopy equivalent. We show that for $0 < t < 1$ they all are, *however when $t = 1$ we have a new phenomena.* There are maps in $\mathcal{M}_{\mu_1}$ which map the equator γ to a point. This was impossible for $f \in \mathcal{M}_{\mu_t}$ for $0 < t < 1$, as such a map f must miss an open set in the range. The collapse of the equator allows a new symmetry; one may rotate the image of the upper hemisphere against that of the lower. This new S^1 of maps in $\mathcal{M}_{\mu_1}$ produces an obstruction to the parametric h-principle.

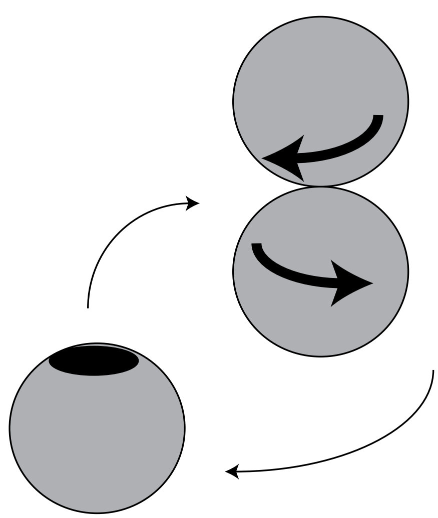

Figure 1.1.

Theorem 1.2. **(Coffey)** *Either the all of the inclusions i_t: $\mathcal{M}_{\mu_t} \to \mathcal{B}\,\mathcal{M}_{\mu_t}$, for $0 < t < 1$, are not homotopy equivalences or the inclusion i_1: $\mathcal{M}_{\mu_1} \to \mathcal{B}\,\mathcal{M}_{\mu_1}$ is not a homotopy equivalence. Thus the parametric h-principle fails for maps with prescribed Jacobian.*

This theorem brings forth a wealth of new questions. One might try to understand collapsing of spheres within other ambient manifolds, further deformations of the form μ, computing the full homotopy type of in low dimension etc...

2 Symplectomorphism groups and isotropic skeletons

2.1 Surface diffeomorphisms

A finite set $L = \{\gamma_i\}$ of simple, closed, transversally intersecting parameterized curves on a surface X *fills* if $X \backslash \{\gamma_i\}$ consists solely of discs. Endow X with a symplectic structure ω, and denote by $\mathcal{L}$ the orbit of L under the action of the symplectomorphism group $\mathcal{S}$. We have the following classical result:

Theorem 2.1. *Let (X, ω) be a symplectic surface. Then the orbit map: $\mathcal{S} \to \mathcal{L}$ is a homotopy equivalence.*

Proof. *(sketch)* We must show that the stabilizer of $\{\gamma_i\}$ in $\mathcal{S}$ -- the symplectomorphism of a disjoint union of discs, fixing their boundaries -- is contractible.

1. Moser's lemma allows us to replace symplectomorphisms with diffeomorphisms (The inclusion is a weak deformation retract.)

2. By the Riemann mapping theorem, these act transitively on the space of complex structures on the disc which are standard at the boundary.

3. This set of complex structures is contractible, and thus the homotopy type is reduced to that of the complex automorphisms of the disc which fix the boundary, a contractible set. □

In this paper we prove a 4 dimensional analog of this statement. The proof follows a similar outline, although it is of course more difficult to carry out each step. Our tool is again the Riemman mapping theorem, but this time augmented by the theory of J-holomorphic spheres in sphere bundles over surfaces, developed by Gromov, Lalonde and McDuff [10, 18, 19].

2.2 Biran Decompositions- a higher dimensional analog of filling systems of curves

Paul Biran [3] recently showed that every Kahler manifold M whose symplectic form lies in a rational cohomology class admits a decomposition

$$M = L \coprod E$$

where L is an embedded, isotropic cell complex and E is a symplectic disc bundle over a hypersurface Σ. L is called an isotropic skeleton of M.

We will argue that a Biran decomposition of a symplectic 4-manifold should be regarded as the 4-dimensional analog of a filling system of curves. Indeed, when M is a surface L is a filling system of curves, Σ is a union of points -- one in each disc inside $M \backslash L$ -- and E is the union of discs.

In higher dimensions we have less understanding of the possible singularities of the spine L and as a result we prove a weaker, more technical result. This requires a bit of machinery to state, however when L is given by a smooth submanifold it reduces to the following:

Definition 2.2. *Let (M, ω) be a symplectic 4-manifold with the decomposition $(L, E \to \Sigma)$ such that $L \hookrightarrow M$ is a smooth Lagrangian submanifold of M.*

1. *Let $\mathcal{L}^{\mathrm{sm}}$ denote the Lagrangian embeddings of $\phi: L \hookrightarrow M$ which extend to symplectomorphisms of M.*
2. *Let $\mathcal{L}\mathcal{E}^{\mathrm{sm}}$ denote the space of pairs (ψ, S) where $\psi \in \mathcal{L}^{\mathrm{sm}}$ and S is a symplectic embedded unparameterized surface which is abstractly symplectomorphic to Σ and disjoint from $\psi(L)$.*
3. *Let $\mathcal{E}mb_\omega(\Sigma, E)$ denote the space of unparameterized, embedded symplectic surfaces S in $E \subset M$ such that $\omega[S] = \omega[\Sigma]$.*

Theorem 2.3. (C-) *(Main theorem when L is smooth)Let (M, ω) be a symplectic 4-manifold with Biran decomposition $(L, E \to \Sigma)$ such that $\phi: L \hookrightarrow M$ is a smooth, Lagrangian submanifold. Then $\mathcal{S}(M)$ is homotopy equivalent to $\mathcal{L}\mathcal{E}^{\mathrm{sm}}$.*

Moreover there is a fibration $\mathcal{L}\mathcal{E}^{\mathrm{sm}} \to \mathcal{L}^{\mathrm{sm}}$ whose fiber is homotopy equivalent to $\mathcal{E}mb_\omega(\Sigma, E)$. When Σ has genus 0, $\mathcal{E}mb_\omega(\Sigma, E)$ is contractible and thus $\mathcal{S}(M)$ is homotopy equivalent to $\mathcal{L}^{\mathrm{sm}}$.

We note that in the case that M is a surface the corresponding symplectic embeddings $\mathcal{E}mb_\omega(\Sigma, E)$ are just the embeddings of each point inside the appropriate disc. Thus in the two dimensional case $\mathcal{E}mb_\omega(\Sigma, E)$ is always contractible, and one recovers (modulo concerns of L's smoothness) Theorem 2.1. We show that in dimension 4 $\mathcal{E}mb_\omega(\Sigma, E)$ is contractible if Σ is a sphere. Richard Hind and A.Iviri have a recent preprint [13], building on the methods of my paper, which claims that this space $\mathcal{E}mb_\omega(\Sigma, E)$ is always contractible for every genus. Using this result $\mathcal{S}(M)$ is homotopy equivalent to $\mathcal{L}^{\mathrm{sm}}$ regardless of the genus of Σ. When the spine L is not smooth, we give in [8] a generalizaton this statement where $\mathcal{L}^{\mathrm{sm}}$ is replaced by an appropriate Kan complex of Isotropic embeddings.

Understanding the higher homotopy groups of spaces of Lagrangian embeddings is quite hard; indeed prior to this paper the author knows of no such computation when both the domain and range are closed. Thus, while Theorem 2.3 and its more general counterpart provide a satisfying generalization it is, at present, difficult to use it to compute much about the symplectomorphism group of a 4 manifold. However we can use it to leverage our knowledge of symplectomorphism groups into an understanding of spaces of Lagrangian embeddings -- a result which should satisfy in proportion to our previous frustration. We obtain the following corollaries in [8]; each is obtained by combining our result with Gromov's computations of the symplectomorphism groups of $\mathbb{CP}^2$ and $S^2 \times S^2$.

Theorem 2.4. (C-) *The space of Lagrangian embeddings of $\mathbb{RP}^2 \hookrightarrow \mathbb{CP}^2$ isotopic to the standard one is homotopy equivalent to $\mathbb{P}\mathbb{U}(3)$*

Theorem 2.5. (C-) *Let ω be a symplectic form on $S^2 \times S^2$ such that $\omega[S^2 \times pt] = \omega[pt \times S^2]$, then the space of Lagrangian embeddings $S^2 \hookrightarrow S^2 \times S^2$ isotopic to the anti-diagonal is homotopy equivalent to $SO(3) \times SO(3)$.*

Richard Hind has recently proven that every Lagrangian sphere in $S^2 \times S^2$ (with the above symplectic structure) is isotopic to the anti-diagonal [12]. One suspects that his methods probably can be used to show that every Lagrangian $\mathbb{RP}^2$ is isotopic to the standard one. Thus both theorems above should actually compute the homotopy type of the full space of Lagrangian embeddings.

The following is also a direct consequence of the results of this paper, and is an answer to a question of Hind. Denote by $T_1^*(S^2)$ the bundle of cotangent vectors to S^2 of norm less than 1, with respect to the standard metric.

Theorem 2.6. (C-) *The space of parametrized Lagrangian embeddings $S^2 \to T_1^*(S^2)$ isotopic to 0-section is homotopy equivalent to* SO(3)*.*

Again Hind has proven that every Lagrangian sphere is isotopic to the 0-section. Since every Lagrangian sphere has a neighborhood conformally symplectomorphic to $T_1^*(S^2)$ this result show that there is no nontrivial "higher knotting" near an embedded Lagrangian 2-sphere. This last result has also been followed upon by results of Hind and Iviri on embedings in cotangent bundles of surfaces of higher genus [11].

3 Maps with Symplectic Graphs

The spaces $\mathcal{H}ol(\mathbb{CP}^1, \mathbb{CP}^1)$ of holomorphic maps of $\mathbb{CP}^1$ to itself have enjoyed a long and fruitful study, beginning with Segal's work [20]. One can view a holomorphic map in at least two ways: as a map preserving the complex structure, or as a map whose graph in $\mathbb{CP}^1 \times \mathbb{CP}^1$ is holomorphic. If one takes this second point of view we are led to consider maps whose graphs are otherwise constrained. In particular if one endows the domain Σ and the range Γ with volume forms σ_Σ and σ_Γ, one can consider the maps $\mathcal{M}^\sigma(\Sigma, \Gamma)$ whose graphs in $\Sigma \times \Gamma$ are symplectic submanifolds with respect to the product symplectic structure:

$$\sigma = \pi_\Sigma^* \sigma_\Sigma + \pi_\Gamma^* \sigma_\Gamma$$

This paper is motivated by the study of homotopy type of $\mathcal{M}^\sigma(\Sigma, \Gamma)$. However the problem has a more natural setting in the product of two distinct n-manifolds, each equipped with a volume forms. Then $\mathcal{M}^\sigma(\Sigma, \Gamma)$ will be the space of maps whose graph G is such that $\sigma|_G$ yields a volume form. We begin by by examining this more general problem.

In this context we construct a purely topological model of the homotopy type of $\mathcal{M}^\sigma(\Sigma, \Gamma)$ in terms of maps with constrained numbers of pre-images. For example:

Theorem. (C-Example of results) *Consider the product of two n-manifolds: $\Sigma \times \Gamma$, with a product n-form $\sigma = \pi_\Sigma^* \sigma_\Sigma + \pi_\Gamma^* \sigma_\Gamma$. Suppose $\frac{\sigma_\Sigma[\Sigma]}{\sigma_\Gamma[\Gamma]} \leq 1$, then $\mathcal{M}_0^\sigma(\Sigma, \Gamma)$ is homotopy equivalent to the space of non-surjective maps from Σ to Γ.*

In general, when the ratio $\frac{\sigma_\Sigma[\Sigma]}{\sigma_\Gamma[\Gamma]} > 1$ we will prove a similar result with the non-surjective maps replaced by "non Q-surjective" maps. We use this characterization to show that the dependence of the homotopy type of $\mathcal{M}^\sigma(\Sigma, \Gamma)$ on the form σ is quantized; it may jump only at discrete intervals, described when the ratio $\frac{\sigma_\Sigma[\Sigma]}{\sigma_\Gamma[\Gamma]}$ passes an integer. This "quantized" topology is reminiscent of that found in the symplectomorphism groups of ruled surfaces. In fact this paper was originally motivated by efforts to understand these groups by primarily soft methods.

We also prove certain (general) identities between different components of $\mathcal{M}^\sigma(\Sigma, \Gamma)$ for different forms σ. *Then we use combine these identities with the theory of J-holomorphic spheres to show that the homotopy type of the space of symplectic sections of a fibration must sometimes change as we deform the fibration.* This suggests both an interesting problem, and that the main results of this paper are probably out of reach of the standard methods of J-holomorphic curves.

We return to the case where both the domain and range are surfaces to compute the homotopy type of $\mathcal{M}^\sigma(\Sigma, \Gamma)$. We build concrete geometric models for these spaces, via a decomposition method which may be of independent interest. In an upcoming paper the we use these techniques to show the failure of the parametric h-principle for maps with prescribed Jacobian.

We succeed in providing a complete description of when the domain Σ is a sphere or torus, and further $\sigma_\Sigma(\Sigma) \leq \sigma_\Gamma(\Gamma)$. Under these conditions, when Σ is a sphere we see that the inclusion $\mathcal{H}ol(\mathbb{CP}^1,\Gamma) \hookrightarrow \mathcal{M}^\sigma(\mathbb{CP}^1, \Gamma)$ is a homotopy equivalence for maps of degree 0 or 1. However, when the domain is T^2 this fails dramatically:

Corollary 3.1. (C-) *Let (T^2, σ_Σ) and (S^2, σ_Γ) be such that $K(\sigma) \leq 1$ then $\mathcal{M}_0^\sigma(T^2, S^2)$ has infinitely generated π_1, and infinite dimensional homology in each dimension.*

4 Measured J-holomorphic foliations

There is a gulf in our understanding of symplectic 4-manifolds, lying between those which have symplectic 2-spheres whose non-negative self intersection and those which lack them. For those with such spheres the theory of J-holomorphic curves provides a great deal of geometric information well beyond that of Gromov-Witten invariants. For instance it provides for a complete classification of these manifolds, and pretty good understanding of the homotopy type of their symplectomorphism groups.

The root of this dichotomy lies in the existence of foliations by holomorphic spheres for such manifolds. Foliations by curves of higher genus are always non-generic - when we deform the almost complex structure these curves disappear.

4.1 What happens to non generic, high genus curves when we deform j?

To illustrate consider the two product manifolds: $S^2 \times S^2$ and $S^2 \times \Sigma$, where Σ is a surface of positive genus, and each manifold is equipped with a product complex structure j. Then each has 2 transverse foliations by holomorphic curves, given by the fibers of each projection.

However when we deform j to a new (generic) almost complex structure j_t the two manifolds behave quite differently: the product foliations on $S^2 \times S^2$ deform to become j_t holomorphic foliations, the foliation by spheres $S^2 \times \mathrm{pt}$ on $S^2 \times \Sigma$ also deforms, however the higher genus curves $\Sigma \times \mathrm{pt}$ are non generic and ‘‘disappear”. The purpose of this paper is to investigate the mysterious disappearance of these curves.

Integrable structures; curves disappear but foliations remain If the deformation j_t is through *integrable* complex structures which make the sections $0 \times \Sigma$ and $\infty \times \Sigma$ holomorphic, then the Narisham-Seshadri theorem provides an answer. *The curves* $\Sigma \times \mathrm{pt}$ *disappear but the foliation unwinds*: there is a unique holomorphic foliation, transverse to the fibration by 2-spheres $\mathrm{pt} \times S^2$ *which possesses an invariant metric* on its transversals. Gromov conjectured that this might also hold in the the almost complex category. In [7] we show that this was too optimistic:

Theorem 4.1. (Failure of Narisham Seshadri Theorem in almost complex category) *Let Σ be a surface of genus $g > 0$, and endow $S^2 \times \Sigma$ with a symplectic structure ω. Then there is a tamed almost complex structure j on $S^2 \times \Sigma$ such that the only holomorphic foliation possessing an invariant metric is that by the 2-spheres $\mathrm{pt} \times S^2$.*

We prove Theorem 4.1 by first leveraging positivity of intersection to show a **uniqueness** theorem for holomorphic foliations with smooth invariant *measures*. Then we construct a foliation $\mathcal{F}$, transverse to the spheres $\mathrm{pt} \times S^2$, with an invariant measure, but no invariant metric.

The ‘‘failure” of Theorem 4.1, while enlightening, is not disheartening. For the uniqueness of holomorphic foliations with invariant measures tells us that we were seeking an existence result in too narrow a category. We should discard metrics, and look instead for existence results with this weaker -- but still rigid! -- transverse structure.

4.2 Existence and uniqueness theorems for measured j-holomorphic foliations

In [7] we progress towards this goal; we prove an **existence** theorem for a less regular, but analogous object via Sullivan's Hahn Banach alternative. Let (M, J) be a compact almost complex manifold. Denote by $C_J \subset \bigwedge^2 T(M)$ the cone generated by positive linear combinations of j-complex planes. Then, following Sullivan, we make the following definition:

Definition 4.2. *A complex structure current is a 2-current represented by a measure μ on M and a μ-integrable section v of C_J. It acts on 2 forms by:*

$$\omega \to \int_M \omega(v) d\mu$$

Note that the function v and the measure μ are not unique, although the resulting measure $v_F\, d\mu_F$ with values in C_J is. When the resulting current F is a cycle we call these currents **complex cycles**, and denote the resulting homology class by $[F]$. Those such that $[F] \cdot [F] = 0$ can be viewed as less regular foliations, and we call them **complex foliation cycles**:

Proposition 4.3. (Characterization of foliations) *A complex foliation cycle which admits a representation where μ and v are smooth with support all of M is a holomorphic foliation with smooth invariant measure. Moreover, every such foliation yields a complex foliation cycle of this form.*

Proof. We first show that F describes a smooth plane field. For consider the self intersection with F with itself. This is given by

$$\int_M i_{v_F}\mu_F \wedge i_{v_F}u_F = 0$$

Here $i_{v_F}u_F$ denotes the contraction of the volume form μ_F with the complex bivector v_F. The integrand $i_{v_F}\mu_F \wedge i_{v_F}u_F \geq 0$ is everywhere positive by positivity of intersection of complex planes. Moreover it vanishes only where v_F is indecomposable- consists of a multiple of a single plane. The integrand must vanish everywhere, and we see that v_F yields a smooth plane field f.

By the Frobenius theorem to show that f is integrable it is sufficient to show that the space of differential forms which vanish on f is a differential ideal. However note that a form α vanishes on f if and only if $\alpha \wedge i_{v_F}\mu_F = 0$.

$$\begin{aligned} d\alpha \wedge i_{v_F}\mu_F &= d(\alpha \wedge i_{v_F}\mu_F) + \alpha \wedge d(i_{v_F}\mu_F) \\ &= \alpha \wedge d(i_{v_F}\mu_F) \\ &= 0 \end{aligned}$$

$i_{v_F}\mu_F$ is closed since F is a cycle. □

Sullivan's Hahn Banach alternative [21] uses the duality between forms and currents to produce existence results for complex cycles via the Hahn Banach theorem. (The author would like to thank Professor Lawson for redirecting his attention to this result.) We improve this method to gain better control over their homology classes.

Theorem 4.4. (Existence) *Let $M = S^2 \times \Sigma$, equipped with an almost complex structure j. Then there is a complex foliation cycle F such that $[F] = [\mathrm{pt} \times \Sigma]$.*

Theorem 4.5. (Uniqueness) *Let F be a complex foliation cycles such that the measure μ_F and section v_F are both smooth. Let F' be any other complex foliation cycle such that $[F] = [F']$. Then there is a set A of $\mu_{F'}$ measure zero, and a function $M \to [0, \infty)$ such that $v_{F'} = f \cdot v_F$ within $\mathrm{Supp}(F) \backslash A$.*

In conversation with Gromov we discovered that when an almost complex structure admits sufficient symmetry the objects we produce should be foliations with smooth invariant measures and thus unique.

Conjecture 4.6. *Let (M, J) be an almost complex manifold such that J is invariant under a transitive compact group action. Then there is a complex cycle with representation μ , v both smooth. Further, when M has dimension 4, and $H^2(M) = 0$, this cycle is a smooth foliation with invariant measure.*

One should be able to average the resulting complex cycles with the Haar measure on the group. The assumption $H^2(M) = 0$ can be weakened to something more technical. $[0, 1] \times S^3$ embeds in $\mathbb{C}^4$ via polar coordinates. The resulting complex structure is invariant along the interval. We can thus identify $0 \times S^3$, and $1 \times S^3$, to gain an $S1 \times S^3$ invariant complex structure on $S^1 \times S^3$. The resulting foliation is then that whose leaves are given by the Hopf circles cross the S^1 direction.

Problem 4.1. S^6 admits an almost complex structure J which is invariant under the action of G_2. What is the resulting smooth averaged complex cycle?

4.3 Complex cycles and deformations of symplectic structure

Gromov's, and Abreu-McDuff's work symplectomorphism groups of $S^2 \times S^2$, stemmed from the strong connection they established between deformations of symplectic structures and the existence of holomorphic curves in new classes. We now give a generalization of this connection within the context of complex cycles. Consider $S^2 \times \Sigma$, and call a symplectic structure there with cohomology class $[\omega_k] = k\,\mathrm{PD}(S^2) + \mathrm{PD}(\Sigma)$ a ω_k form. Here k ranges in $(0, \infty)$.

Theorem 4.7. *Let j be an almost complex structure on $S^2 \times \Sigma$ which is tamed by a ω_k form but not by a ω_{k_0} form for $k_0 < k$. Then there is a j-complex cycle which lies in the homology class $[S^2 \times \mathrm{pt}] - k\,[\mathrm{pt} \times \Sigma]$.*

2 Research in progress

1 Symplectic Field Theory: relating convex decompositions of and holomorphic curves in symplectizations

1.0.1 Contact manifolds

Let (M, α) be a three manifold equipped with a one form α such that $\alpha \wedge d\alpha > 0$. Then the $\ker(\alpha)$ is a two plane distribution which is nowhere integrable. We say that α is a **contact form**, and that $\ker(\alpha)$ is a **contact structure**.

Example 1.1. Let (N, j) be a Stein 4-manifold with plurisubharmonic exhaustion ϕ. Then the level sets of ϕ have a canonical contact structure given by the the j-invariant planes.

There are two major threads in the explosion of research in contact topology in the last decade. One -- the theory of convex surfaces -- provides a "cellular decomposition" in the contact category, the other -- contact homology -- provides a homology theory constructed out of holomorphic curves in the symplectization. This two threads have thus far proceeded separately, there is yet no way of computing contact homology from the convex decomposition. The major aim of this project is to fill this gap. This would have far reaching consequences, greatly increasing both our computational power and theoretical understanding. For instance it would provide a direct link between the three dimensional contact geometry of the level sets of ϕ and the ambient holomorphic geometry of (N, j).

Ultimately, given a convex decomposition of a contact 3-manifold -- that is enough convex surfaces Σ_i such that $M \setminus \bigcup_i \Sigma_i$ consists of disjoint 3-balls -- we would like to have a combinatorial definition of contact homology solely in terms of the dividing sets on the surfaces Σ_i.

1.1 Contact homology; Holomorphic curves in symplectizations

Let (M, α) be a contact manifold. Its **symplectization** $M \times I$ is then a symplectic manifold with symplectic form $\omega_\alpha = d\alpha + \alpha \wedge ds$ where s is the co-ordinate in the I direction. The **Reeb vector field** associated to α is the unique vector field r such that $d\alpha(r) = 0$ and $\alpha(r) = 1$. The orbits of r, called the Reeb orbits, form an algebra under a product of disjoint union and formal sum. One can then build a differential on this algebra.

To do this we equip the symplectization $M \times I$ with an almost complex structure j which is tamed by ω_α, leaves the planes of the contact structure invariant, and carries the Reeb vector field r to $\frac{\partial}{\partial s}$, the tangent vector in the I direction. We say that such a j is **adapted** to α. Then let $(f_M, f_I)\colon \Sigma \to M \times I$ be a j-holomorphic map of a punctured Riemann surface Σ into the symplectization which is proper and further such that

$$\int_\Sigma f_M^*(d\alpha) < \infty$$

Call such curves **finite energy curves**. These curves have well understood asymptotic behavior, as f_I tends towards $\pm\infty$ the curves approach **Reeb cylinders** -- trivial holomorphic curves of the form $r_o \times I$ for some Reeb orbit r_o. We now uses them to define the differential. Let $\coprod r_{o+}^i$ denote some collection of Reeb orbits. Their product forms an element in our algebra. (Strictly speaking one doesn't orbits with "bad" Conley-Zehnder index, but we will ignore this subtlety here.) The differential of this collection is given by a formal sum of disjoint unions of orbits:

$$d(\coprod r_{o+}^i) = \coprod r_{o-}^{i,1} + \coprod r_{o-}^{i,2} + \coprod r_{o-}^{i,3} + \ldots \coprod r_{o-}^{i,n}$$

where each term in the sum corresponds to an isolated finite energy curve whose boundary consists of $\coprod r_{o+}^i$ as f_I tends towards $+\infty$ and $\coprod r_{o-}^i$ as f_I tends towards $-\infty$.

Compactness results for finite energy curves imply that $d^2 = 0$, and thus this defines a differential graded algebra. Note that it uses both a choice of contact form α yielding the contact structure, and a choice of adapted almost complex structure j in its definition. However, the homology of this algebra, called **contact homology**, is conjecturally independent of the choices.

Unfortunately, except in the rare cases where both the contact form α and the almost complex structure j admit a lot of symmetry, computing contact homology has been proved difficult. However, in these special cases contact homology has shown its power: Most contact structures do not not admit defining contact forms with this much global symmetry. But one can always find useful local symmetry in the form of convex surfaces.

1.2 Convex decompositions-cellular decomposition in the contact category

A path in M is **Legendrian** if it is always tangent to ζ. A **convex surface** $\Sigma \hookrightarrow M$, is an embedded surface with Legendrian boundary which admits a vector field v which is transverse to Σ and whose flow preserves the contact structure $\ker(\alpha)$. Any embedded surface is C^∞ close to a convex one [?]. The singular foliation of Σ given by $\mathcal{F} = \ker(\alpha) \cap T(\Sigma)$ is called the **characteristic foliation** of Σ. *It determines the contact structure near* Σ. If one isotops the surface the characteristic foliation will change. The way in which it does is described by the **dividing set**

$$\Gamma = \{x \in \Sigma: v(x) \in \ker(\alpha(x))\}$$

a set of disjoint, simple, curves on Σ whose boundary is contained in $\partial\Sigma$. The isotopy class of Γ in Σ is independent of v. Γ divides Σ into two types of pieces; those in Σ_+ where the projection of the contact plane to Σ, away from v, preserves orientation and the other in Σ_- where the projection reverses orientation.

Giroux proved the following fundamental relationship:

Theorem 1.2. (Giroux Flexibility Theorem-[?]) *Let $\mathcal{F}_1$ be any singular foliation which is transverse to the the dividing set Γ, and on $\Sigma \backslash \Gamma$ is directed by a vector field which contracts an area form on each component. Then one can find an isotopy Σ_t of Σ such that $\ker(\alpha) \cap T(\Sigma_1) = \mathcal{F}_1$. Moreover one can ensure that the surfaces Σ_t are all transverse to the vector field v.*

Thus *the dividing set Γ determines all the essential contact information near the surface.* It serves as a label describing the set of characteristic foliations one can achieve by a "local" isotopy of the surface.

We say that a contact structure α is **overtwisted** if there is an embedded convex disc, with Legendrian boundary, and a dividing set consisting of a single circle. Otherwise we say the contact structure is **tight**. Overtwisted contact structures satisfy a parametric h-principle and can thus be understood by homotopy theory; tight contact structures do not and contain a wealth of geometry. All contact structures given in Example 1.1 are tight.

The possible dividing sets on a convex surface in a tight contact manifold are restricted. If the surface is a sphere there is unique dividing set. Moreover, if this sphere bounds a 3-ball the contact structure on the three ball is uniquely determined in the following sense:

Theorem 1.3. (Eliashberg-Giroux Uniqueness Theorem) *Let $S^2 \hookrightarrow M$ be a convex surface in a tight contact 3-manifold. Then the dividing set consists of a single closed curve. Moreover if Σ_1 and Σ_2 are two such convex spheres bounding 3-balls B_1, B_2, then one can isotop Σ_1 to Σ_1' and find a diffeomorphism of the resulting three balls $B_1' \to B_2$ which preserves the contact structures.*

Given a tight contact three manifold (M, α), we can find enough convex surfaces Σ_i such that $M \backslash \bigcup_i \Sigma_i$ consists of disjoint 3-balls. Then the resulting configuration of surfaces and dividing sets uniquely determines the 3-manifold up to contactomorphism, and is called a **convex decomposition**. It is thus directly analogous to cellular decomposition in the contact category. This is a deep and powerful theory, however it suffers the same defect as in the topological category: two manifolds with the same convex decomposition are contactomorphic, but a single contact manifold can admit many convex decompositions. It thus requires ingenuity to use convex decompositions to distinguish contact structures.

In topology we solve this problem by computing certain invariants, like homology, from the CW-complex. In contact topology we are blessed with a beautiful contact homology theory, however there is as yet no way to compute contact homology from a convex decomposition. It is this gap we aim to fill. Contact homology and convex decompositions have complementary strengths: one is concrete but non-invariant, the other is abstract but invariant. We hope to let each sides strength bolster the others weakness by laying down a bridge between the two.

1.3 Relations between convex decompositions and holomorphic curves in symplectizations

We now describe how to find a almost complex structure j and contact form α which build from a convex surface holomorphic finite energy, holomorphic, foliations in the symplectization.

Criterion for lifting a surface in M to a holomorphic curve in symplectization (Hofer): Let f_M: $S \subset M$ be an embedded surface whose boundary consists of a union of Reeb orbits, such that the linear projection of $T(S)$ to the contact structure ζ with kernel the Reeb vector field r has full rank away form the boundary. Denote the complex structure induced by this projection by j_S. *Then there is a finite energy, j-holomorphic curve* $(f_M, f_I)\colon (S, j_S) \rightarrow M \times I$ *which lifts* f_M *if and only if the form* $\alpha \circ j_S$ *is exact restricted to* S.

Theorem 1.4. (Coffey) *Let* (M, ζ) *be a contact* 3 *manifold and et* $\Sigma \subset M$ *be a convex surface. Then there is a isotopy of* Σ *to a convex surface* Σ'*, preserving its dividing set* Γ*, a contact form* α *compatible with* ζ*, and an almost complex structure* j *adapted to* α *such that the symplectization admits a pair* $\mathcal{F}_1, \mathcal{F}_2$ *of transverse, holomorphic finite energy foliations.*

The leaves of $\mathcal{F}_1$ *project to the connected components of* Σ'*, and the leaves of* $\mathcal{F}_2$ *project to the* ***characteristic surfaces*** $l \times I$*, where* l *is a leaf of the characteristic foliation of* $\alpha|_{\Sigma'}$.

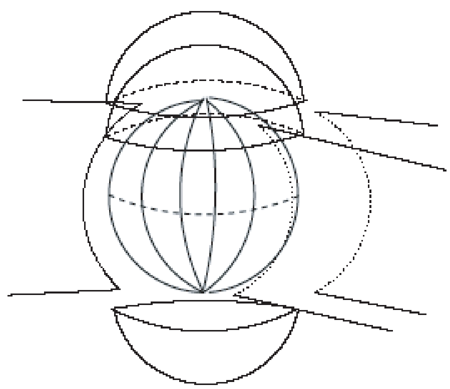

Figure 1.1. Dividing set, characteristic foliation of Σ' and the leaves of $\mathcal{F}_1, \mathcal{F}_2$.

Proof. (Sketch) Let α' be a contact structure on M s.t. $\ker(\alpha') = \zeta$. We construct a new contact form α on Σ, with the same dividing set by the following method: we find a complex structure j on Σ and function μ on Σ such that $\mu^{-1}(0) = \Gamma$, and further μ is sub-harmonic on $\Sigma_- = \mu^{-1}(x < 0)$ and sup-harmonic on $\Sigma_+ = \mu^{-1}(x > 0)$. Then we define a contact form α by:

$$\alpha = d\mu \circ j + \mu dt$$

Note that the characteristic foliation of $\alpha|_{\Sigma'}$ *is then given by the gradient flowlines of* μ with respect to the conformal structure j.

By Giroux's flexibility theorem we can find an isotopy of Σ to Σ' where the characteristic foliation of α is that induced by α'. Thus we can scale the contact form α near Σ' have the form α', and be invariant under the flow in the t direction.

Connected components of $\Sigma \backslash \Gamma$ lift to holomorphic curves The Reeb flow of α' is particularly simple: it travels along the level sets of μ, in the t direction in Σ'_+ and against it in Σ'_-. The curves in the dividing sets form Reeb orbits. They bound the various components of Σ'_+ and Σ'_-. We will now construct an adapted almost complex structure j in the symplectization which will make these components Σ_i lift to holomorphic curves. The restriction of α to each is given by

$$\beta = d\mu \circ j$$

We then adjust the almost complex structure j on the surface Σ, near the dividing set, so that the projection to the contact planes, away from the Reeb vector field, induces an almost complex structure

μ's gradient flowlines $\times I$ lift to j_1-holomorphic curves

The characteristic surfaces S_l are given by $l \times I$ where l is a gradient flowline of μ. At the critical points of μ, the Reeb vector field is a positive multiple of $\pm\frac{\partial}{\partial t}$, where the sign agrees with that of $\Sigma_{\pm}$. , Thus the boundaries of the surfaces S_l lift to Reeb segments. The restriction of the contact form α to each is:

$$\alpha|_{S_l} = \mu dt$$

Let j_S denote the almost complex structure induced on S_l by the j constructed above via Reeb projection. It is easy to see that $dt \circ j_S$ is a multiple of $d\mu$. Further, the almost complex structure j and the contact form α' are invariant in the t direction. So $dt \circ j_S$ can depend only on μ and thus

$$\mu dt \circ j_S = f(\mu)d\mu$$

and it is thus exact. Thus the characteristic surfaces S_l thus lift to holomorphic curves in the symplectization.

□

If ζ be is a contact structure on $M = S^1 \times \Sigma$ which is invariant under the S^1 action, the argument above can be globalized:

Theorem 1.5. (Coffey) *Let ζ be a contact structure on $M = S^1 \times \Sigma$ which is invariant under the S^1 action. Then there is a contact form α such that $\ker(\alpha) = \zeta$ and an almost complex structure j adapted to α such that the symplectization admits a pair $\mathcal{F}_1, \mathcal{F}_2$ of transverse, holomorphic finite energy foliations.*

These foliations provide the symplectization with pluri-harmonic coordinates which one may use to understand the holomorphic curves there. These generalize, and explain the coordinates Taubes uses in his paper [22] describing the holomorphic curves in $S^2 \times S^1$.

Denote by $\mathcal{J}_\alpha$ the almost complex structures on $M \times S^1$ which are tamed by $d\alpha + \alpha \wedge ds$ and adapted to α near the two ends. This is a contractible set. When the there is sufficiently many curves in the dividing set of the convex surfaces Σ_t the holomorphic leaves which lift the connected components of $\Sigma_t \backslash \Gamma_t$ should be "automatically generic" in the same way that j-holomorphic spheres in symplectic manifolds are. Those holomorphic curves lifting the characteristic surfaces are cylinders and thus should be similarity generic. Further, Richard Siefring has shown an adjunction equality in this context which should force the leaves to remain embedded. Thus these foliations should persist under deformation of the almost complex structure j.

For some finite co-dimension set within $\mathcal{J}_\alpha$ the foliations may degenerate, but we conjecture:

Conjecture 1.6. (Coffey) *Suppose that each connected component of $\Sigma_t \backslash \Gamma_t$ admits an embedding into S^2. Then there a set $\mathcal{J}_{\mathcal{F}} \subset \mathcal{J}_\alpha$ whose complement has finite co-dimension and such that for each almost complex structure $j \in \mathcal{J}_{\mathcal{F}}$ there is a a pair $\mathcal{F}_1^j, \mathcal{F}_2^j$ of transverse, holomorphic finite energy foliations of the symplectization, isotopic to the pair $\mathcal{F}_1$, $\mathcal{F}_2$, and which are "almost invariant under the $\mathbb{R}$ action on $M \times \mathbb{R}$ at infinity."*

1.3.1 Relations with symplectomorphism groups

$(S^2 \times S^2, \omega)$ possesses a similar structure. If you deform the product almost complex structure the two product foliations deform to two transverse, holomorphic foliations. Positivity of intersection forces each pair of leaves to intersect in a unique point. One can think of the foliations as "graph paper" on $S^2 \times S^2$, and each almost complex structure can be made to induce a diffeomorphism via these coordinates. This, along with Moser's Lemma, allowed Gromov, McDuff and Abreu to relate the homotopy types of the tamed almost complex structures on $S^2 \times S^2$ [2, 1, 10] with that of the symplectomorphism group.

We should seek to establish a similar relationship between $\mathcal{J}_{\mathcal{F}}$ and $\mathcal{C}_\alpha(M \times I)$, the symplectomorphisms of (M, α) which respect α at the two ends.

1.4 Excision in contact homology: contact homology limiting to Morse theory on the boundary

Theorem 1.4 implies that the function t, defined near Σ, is pluri-harmonic -- its restriction to any j_1-holomorphic curve satisfies the maximum principle. Thus it serves as a "barrier" for holomorphic curves. If a curve lies in one component of $M\backslash\Sigma$ for some almost complex structure j, which agrees with j_1 near Σ it must remain in that component as one varies the almost complex structure, so long as the variation is trivial near Σ. This gives some hope that one might decompose the theory of holomorphic curves in $M\times I$ into those in the symplectization of each component.

What remains is a "stretching" construction near Σ similar to the the stretching the neck construction of SFT, near a contact hypersurface [9]. We will degenerate both the contact structure $\alpha=\alpha_1$, and the almost complex structure $j=j_1$ of there so that the Reeb orbits limit onto critical points of $\mu=\mu_1$ and (conjecturally) the projections of holomorphic curves limit onto the gradient flow lines connecting them.

1.4.1 A degeneration of the contact form and almost complex structure near splitting surface Σ

We now consider the neighborhood $\Sigma\times(-1,1)\subset(M,\alpha)$, such that within the neighborhood α has the form form

$$\alpha = d\mu\circ j + \mu dt$$

where μ, j are as described above. We abbreviate $\beta = d\mu\circ j$ and consider of a one parameter family λ_κ: $\mathbb{R}\to(0,\infty)$ of functions becoming more and more peaked at 0, converging to a function λ_∞ which blows up there. Then we have the following deformation of contact structures

$$\alpha_\kappa = \lambda_\kappa(t)(\beta+\mu dt)$$

which can be induced by a symplectic cobordism.

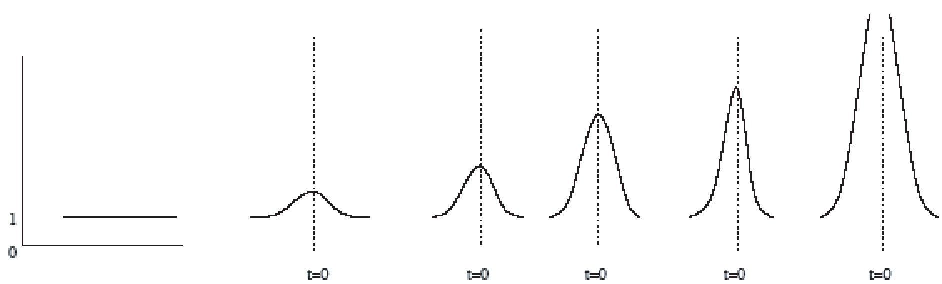

Figure 1.2. The functions λ_κ as $\kappa\to\infty$

Degeneration of Reeb orbits onto crit(μ)

As $\kappa\to\infty$ the Reeb orbits of α_k go to the critical points of the function μ, as they tend towards the splitting surface. For denote by r_t the t component of $r(\lambda_\kappa)$, $r_\beta=\beta(r(\lambda_k))$ and $r_\mu = d\mu(r(\lambda_k))$. Then r_t is independent of the function λ, r_β scales by $\frac{1}{\lambda}$ and r_μ by $\frac{\lambda'}{\lambda}$ as we deform α to α_λ. Thus as $t\to 0$, the Reeb vector field r_λ 's projection to the t-level sets becomes more and more tangent to the gradient flow lines of the characteristic foliation. As a result the Reeb orbits of α_∞ tend towards the critical points of μ. (in forward time in Σ_+ and in backward time in Σ_-).

Proposition 1.7. (Coffey-λ dependence of Reeb) *Let $x\in\Sigma\times\pm1\backslash\Gamma$. Then there is a Reeb chord o_κ for the form α_κ which passes through x and travels the whole $\Sigma\times I$. The integral $\int_{o_\kappa}\beta$ is bounded independently of κ. As $\kappa\to\infty$ the chords o_κ converge to o_∞ which divides into two pieces each, converging to the critical points of μ.*

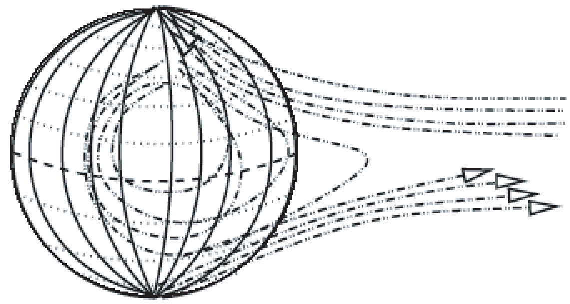

Figure 1.3.
λ deformed Reeb flow at splitting S^2

1.4.2 Holomorphic curves asymptotic to gradient flow lines

We now define a rescaling construction, similar to stretching the neck [9, 4], under which the finite energy curves in M should tend to gradient flowlines as they approach the splitting surface Σ_0. This construction is

We now pass to the limit as $\kappa \to \infty$, and rescale in the t direction to gain limiting contact structures away from the splitting surface Σ_0. For simplicitly lets assume that Σ_0 seperates $M\backslash\Sigma_0$ into two pieces M_- and M_+. Then in the limit M seperates into the following two contact manifolds with ends:

$$M_{\infty}^{+} = (M^{+}, \alpha) \cup (\Sigma \times [0, \infty), \alpha_{\infty}^{+})$$
$$M_{\infty}^{-} = (M^{-}, \alpha) \cup (\Sigma \times [0, -\infty), \alpha_{\infty}^{-})$$

Where:

$$\alpha_{\infty}^{+} = e^{t}\beta + \mu dt \text{ on } \Sigma \times [0, \infty)$$

$$\alpha_{-} = e^{-t}\beta + \mu dt \text{ on } \Sigma \times (-\infty, 0]$$

1.4.3 An appropriate energy

We now define an energy analogous to that used by Hofer, Wyzoski and Zehnder [16, 14, 17, 15] for finite energy curves in symplectizations, to tame our curves as they approach the splitting surface. Like his, ours will contain two parts. The first is analogous to the "$d\lambda$" energy, and is designed to ensure proper limiting behavior of our curves in M_+^{∞} and M_-^{∞} as they approach the ends at ∞. We give below the definitions for M_+^{∞}. Those for M_-^{∞} are entirely analogous.

$$E_{\beta}(f) := \sup_{K \subset I = (-\infty, \infty) \text{ of unit length}} \int_{f^{-1}(\Sigma \times K)} f^{*}(d\beta + \beta \wedge \mathrm{ds})$$

Here $I = (-\infty, \infty)$ denotes the symplectization direction. That the integral of β over the Reeb chords, $\int_{o_k} \beta$, remains bounded under our stretching should cause the E_β energy to remain bounded under our stretching construction.

The second part of the energy is necessary for control in the symplectization direction:

$$E_{\alpha}(f) := \sup_{K \subset [0, \infty) \text{ of unit length}} E_{\mathrm{hof}}(f|_{f^{-1}(\Sigma \times K)})$$

Where E_{hof} is Hofer's energy.

Then we wish to consider holomorphic curves $f: \Upsilon \to M_{\infty}^{+}$ such that the sum:

$$E(f) = E_{\beta}(f) + E_{\alpha}(f) + E_{\mathrm{hof}}(f|_{f^{-1}(M^{+})})$$

is bounded. That is our curves should be finite energy away from the end $(\Sigma \times [0, \infty), \alpha_{\infty}^{+})$, finite energy on compact pieces of the end, and further be finite E_β energy on compact slices in the symplectization direction.

1.4.4 Characteristic surfaces (analog of Reeb cylinders)

For a properly chosen almost complex structure $d\beta + \beta \wedge ds$ is semi-positive on complex planes. Moreover, while the E_β energy itself is not additive, its boundedness implies that the integrals

$$\int_{f^{-1}(\Sigma\times[0,\infty)\times K)} f^*(d\beta + \beta \wedge \mathrm{ds}) \qquad K \subset I$$

are also bounded for any compact slice K in the symplectization direction. Thus any curve which remains such a slice should spend all of its E_β energy as it travels towards ∞, and limit onto a curve of 0 E_β energy. These are precisely those holomorphic curves which project to Characteristic surfaces- gradient flowlines $\times\, [0,\infty)$. Thus we should have:

Conjecture 1.8. (Coffey) *As $t \to 0$, the projections f_M of finite E-energy holomorphic curves in $(M\backslash\Sigma \times 0, \alpha_{\lambda\infty}, j_\infty)$ are asymptotic to characteristic surfaces (gradient flowlines $\times I$), or have converge to to the critical points of μ_1.*

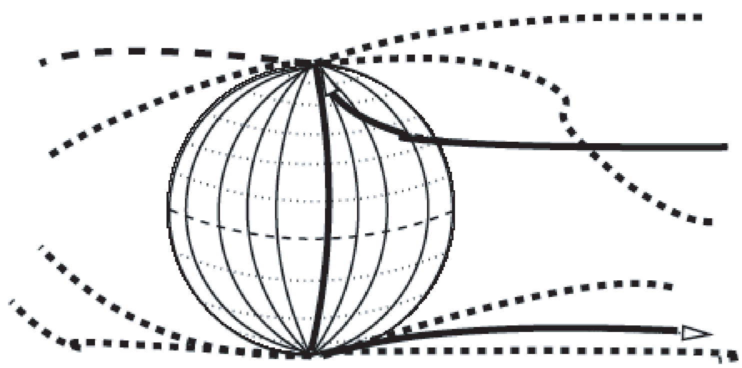

Figure 1.4. Projections of holomorphic curves converge either to gradient flowlines, or are pinched down at the critical points of μ.

1.4.5 A combinatorial contact homology ?

Ultimately given a convex decomposition of a contact 3-manifold -- that is enough convex surfaces Σ_i such that $M\setminus \bigcup_i \Sigma_i$ consists of disjoint 3-balls -- we would like to have a combinatorial definition of contact homology solely in terms of the dividing sets on the surfaces Σ_i. Such a program requires other components besides the above conjectures:

1. **A "stretching construction for surfaces with boundary"** In order to perform the convex decomposition one must consider cutting surfaces Σ_i with Legendrian boundary in one of the other surfaces Σ_j. Thus we must develop a stretching construction, perhaps along the above lines, for surfaces with boundary.

2. **Understanding holomorphic curves in 3 ball with convex boundary** One then needs to understand the holomorphic curves in certain basic pieces of the manifold, most pressingly those in the tight $3-$ball with convex boundary. Since we do understand the contact homology of S^3, it is possible that this can be computed from examining the splitting of S^3 into two convex balls.

3. **A gluing construction** One must develop a gluing construction for recovering finite energy curves from the broken curves produced by the stretching constructions.

4. **What is the algebraic structure** The possible degenerations occurring in the degenerations and gluings should be organized in terms of some algebraic structures, just as contact homology is given by a differential graded algebra. What is this structure?

Bibliography

[1] Miguel Abreu. Topology of symplectomorphism groups of $S^2 \times S^2$. *Invent. Math.*, 131(1):1--23, 1998.

[2] Miguel Abreu and Dusa McDuff. Topology of symplectomorphism groups of rational ruled surfaces. *J. Amer. Math. Soc.*, 13(4):971--1009 (electronic), 2000.

[3] P. Biran. Lagrangian barriers and symplectic embeddings. *Geom. Funct. Anal.*, 11(3):407--464, 2001.

[4] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder. Compactness results in symplectic field theory. *Geom. Topol.*, 7:799--888 (electronic), 2003.

[5] Joseph Coffey. Failure of the paramateric h-principle for maps with prescribed jacobian. *submitted for publication, www.math.nyu/ coffey*.

[6] Joseph Coffey. Maps with symplectic graphs. *submitted for publication, math.SG/0407221*.

[7] Joseph Coffey. Measured j-holomorphic foliations. *preprint, www.math.nyu/ coffey*.

[8] Joseph Coffey. Symplectomorphism groups and isotropic skeletons, *Geom. Topol.*, 9:935-970 (electronic), 2005.

[9] Y. Eliashberg, A. Givental, and H. Hofer. Introduction to symplectic field theory. *Geom. Funct. Anal.*, (Special Volume, Part II):560--673, 2000. GAFA 2000 (Tel Aviv, 1999).

[10] M. Gromov. Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307--347, 1985.

[11] R Hind and A Iviri. Isotopies of high genus lagrangian surfaces. *preprint*.

[12] Richard Hind. Lagrangian spheres in $S^2 \times S^2$. *math.SG/0311092*.

[13] Richard Hind and A. Iviri. Ruled 4-manifolds and isotopies of symplectic surfaces. *math.SG/0510108*.

[14] H. Hofer, K. Wysocki, and E. Zehnder. Properties of pseudo-holomorphic curves in symplectisations. II. Embedding controls and algebraic invariants. *Geom. Funct. Anal.*, 5(2):270--328, 1995.

[15] H. Hofer, K. Wysocki, and E. Zehnder. Properties of pseudoholomorphic curves in symplectisation. IV. Asymptotics with degeneracies. In *Contact and symplectic geometry (Cambridge, 1994)*, volume 8 of *Publ. Newton Inst.*, pages 78--117. Cambridge Univ. Press, Cambridge, 1996.

[16] H. Hofer, K. Wysocki, and E. Zehnder. Properties of pseudoholomorphic curves in symplectisations. I. Asymptotics. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 13(3):337--379, 1996.

[17] H. Hofer, K. Wysocki, and E. Zehnder. Properties of pseudoholomorphic curves in symplectizations. III. Fredholm theory. In *Topics in nonlinear analysis*, volume 35 of *Progr. Nonlinear Differential Equations Appl.*, pages 381--475. Birkhäuser, Basel, 1999.

[18] François Lalonde and Dusa McDuff. The classification of ruled symplectic 4-manifolds. *Math. Res. Lett.*, 3(6):769--778, 1996.

[19] Dusa McDuff. Symplectomorphism groups and almost complex structures. In *Essays on geometry and related topics, Vol. 1, 2*, volume 38 of *Monogr. Enseign. Math.*, pages 527--556. Enseignement Math., Geneva, 2001.

[20] Graeme Segal. The topology of spaces of rational functions. *Acta Math.*, 143(1-2):39--72, 1979.

[21] Dennis Sullivan. Cycles for the dynamical study of foliated manifolds and complex manifolds. *Invent. Math.*, 36:225--255, 1976.

[22] Clifford Henry Taubes. A compendium of pseudoholomorphic beasts in $\mathbb{R} \times (\mathbb{S}^1 \times \mathbb{S}^2)$. *Geom. Topol.*, 6:657--814 (electronic), 2002.