Measured J-holomorphic foliations

by Joseph Coffey

Abstract

Let Σ denote an oriented surface. The Narisham-Sheshadri theorem, suitably translated, says that on $S^2 \times \Sigma$ for any integrable complex structure j which makes the sections $0 \times \Sigma$ and $\infty \times \Sigma$ holomorphic there is a 1 complex dimensional foliation such that the leaves are j-invariant and the foliation has an invariant metric.

We show that this does not hold in the almost complex category. We produce a j on $S^2 \times \Sigma$ where there can be no such foliation, by proving a uniqueness theorem for foliations with smooth invariant *measures*. Then we prove an existence theorem for a less regular, but analagous object via Sullivan's Hahn Banach alternative.

We show that when an almost complex structure admits sufficient symmetry the objects we produce are foliations with smooth invariant *measures* and thus unique. However in general this uniqueness depends subtly on the regularity of the resulting "foliation". Finally we show that constructions relating J-holmorphic curves and almost complex structures on rational surfaces admit generalizations to products of surfaces in this context.

1 Introduction

There is a gulf in our understanding of symplectic 4-manifolds, lying between those which have symplectic 2-spheres whose non-negative self intersection and those which lack them. For those with such spheres the theory of J-holomorphic curves provides a great deal of geometric information well beyond that of Gromov-Witten invariants. For instance it provides for a complete classification of these manioflds, and pretty good understanding of the homotopy type of their symplectomorphism groups.

The root of this dichotomy lies in the dimension formula for the $\bar{\partial}$-operator. It predicts that a sphere with trivial normal bundle should foliate, such a torus should be isolated, and such a higher genus curve should not exist generically at all.

1.1 What happens to non genenric, high genus curves when we deform j?

To illustrate consider the two product manifolds: $S^2 \times S^2$ and $S^2 \times \Sigma$, where Σ is a surface of positive genus, and each manifold is equipped with a product complex structure j. Then each has 2 transverse foliations by holomorphic curves, given by the fibers of each projection.

However when we deform j to a new (generic) almost complex structure j_t the two manifolds behave quite differently: the product foliations on $S^2 \times S^2$ deform to become j_t holomorphic foliations; the foliation by spheres $S^2 \times \mathrm{pt}$ on $S^2 \times \Sigma$ also deform, however the higher genus curves $\Sigma \times \mathrm{pt}$ are non generic and "dissappear". The purpose of this paper is to investigate the mysterious dissapperance of these curves.

Integrable structures; curves dissappear but foliations remain

If the deformation j_t is through *integrable* complex structures which make the sections $0 \times \Sigma$ and $\infty \times \Sigma$ holomorphic, then the Narisham-Seshadri theorem provides an answer. *The curves* $\Sigma \times \mathrm{pt}$ *dissappear but the foliation unwinds*: there is a unique holomorphic foliation, transverse to the fibration by 2-spheres $\mathrm{pt} \times S^2$ *which posesses an invariant metric* on its transversals. Gromov conjectured that this might also hold in the the almost complex category. In [Cofa] we show that this was too optimistic:

Theorem 1.1. (Failure of Narisham Seshadri Theorm in almost complex category) *Let Σ be a surface of genus $g > 0$, and endow $S^2 \times \Sigma$ with a symplectic structure ω. Then there is a tamed almost complex structure j on $S^2 \times \Sigma$ such that the only holomorphic foliation posessing an invariant metric is that by the 2-spheres* $\mathrm{pt} \times S^2$.

We prove Theorem 1.1 by first leveraging positivity of intersection to show a **uniqueness** theorem for holomorphic foliations with smooth invariant *measures*. Then we construct a foliation $\mathcal{F}$, transverse to the spheres $\mathrm{pt} \times S^2$, with an invariant measure, but no invariant metric.

The "failure" of Theorem 1.1, while enlightening, is not disheartening. For the uniqueness of holomorphic foliations with invariant measures tells us that we were seeking an existence result in too narrow a category. We should discard metrics, and look instead for existence results with this weaker – but still rigid! – transverse structure.

1.2 Existence and uniqueness theorems for measured j-holomorphic foliations

In section 3 we progress towards this goal; we prove an **existence** theorem for a less regular, but analagous object via an enrichment of Sullivan's Hahn Banach alternative. Let $(M,\ j)$ be a compact almost complex manifold. Denote by $C_J\subset\bigwedge^2 T(M)$ the cone generated by positive linear combinations of j-complex planes. Then, following Sullivan, we make the following definition:

Definition 1.2. *A complex structure current is a 2-current represented by a measure μ on M and a μ-integrable section v of C_J. It acts on 2 forms by:*

$$\omega\rightarrow\int_M \omega(v)d\mu$$

Note that the function v and the measure μ are not unique, although the resulting measure $vd\mu$ with values in C_J is. When the resulting current F is a cycle we call these currents **complex cycles**, and denote the resulting homology class by $[F]$.

Just as one cannot, in general, directly define a product of distributions or measures, on cannot define the product of two currents. However, one can extend the wedge product on forms to a product on homology classes of currents, using DeRham's retraction. Those such that satisfy $[F]\cdot[F]=0$ can be viewed as less regular foliations, and we call them **complex foliation cycles**:

Proposition 1.3. (Characterization of foliations) *A complex foliation cycle which admits a representation where μ and v are smooth with support all of M is a holomorphic foliation with smooth invariant measure. Moreover, every such foliation yields a complex foliation cycle of this form.*

Proof. We first show that F describes a smooth plane field. Consider the self intersection with F with itself. This is given by

$$\int_M i_{v_F}\mu_F\wedge i_{v_F}u_F=0$$

Here $i_{v_F}u_F$ denotes the contraction of the volume form μ_F with the complex bivector v_F. The integrand $i_{v_F}\mu_F\wedge i_{v_F}u_F\geq 0$ is everywhere positive by positivity of intersection of complex planes. Moreover it vanishes only where v_F is indecomposable- consists of a multiple of a single plane. The integrand must vanish everywhere, aand we see that v_F yields a smooth plane field f.

By the Frobenius theorem to show that f is integrable it is sufficent to show that the space of differential forms which vanish on f is a differential ideal. However note that a form α vanishes on f if and only if $\alpha\wedge i_{v_F}\mu_F=0$.

$$\begin{aligned} d\alpha\wedge i_{v_F}\mu_F &= d(\alpha\wedge i_{v_F}\mu_F)+\alpha\wedge d(i_{v_F}\mu_F)\\ &= \alpha\wedge d(i_{v_F}\mu_F)\\ &= 0\end{aligned}$$

$i_{v_F}\mu_F$ is closed since F is a cycle. □

Sullivan's Hahn Banach alternative uses the duality between forms and currents to produce existence results for complex cycles via the Hahn Banach theorem. (The author would like to thank Professor Lawson for redirecting his attention to this result.) We improve this method to gain better control over their homology classes:

Theorem 1.4. (Existence) *Let $M=S^2\times\Sigma$, equipped with an almost complex structure j. Then there is . For each $p>1$, one can find a complex foliation cycle such that* $[F]=[\mathrm{pt}\times\Sigma]$.

The proof of this Theorem is in section 3.

Proposition 1.5. (Uniqueness) *Let F be a complex foliation cycles such that the measure μ_F and section v_F are both smooth. Let F' be any other complex foliation cycle such that $[F]=[F']$. Then there is a set A of $\mu_{F'}$ measure zero, and a function $M \to [0,\infty)$ such that $v_{F'} = f \cdot v_F$ within* $\mathrm{Supp}(F)\backslash A$.

Proof. Again the claim follows from positivity of intersection of complex planes.

$$\begin{aligned}[F]\cdot[F'] &= \int \mu_F(v_F \wedge v_{F'})\mu_{F'} \\ &= 0\end{aligned}$$

since $[F]=[F']$ and each are complex foliation cycles. But by positivity of intersection $\mu_F(v_F \wedge v_{F'}) \geq 0$ and vanishes only when v_F vanishes or $v_{F'} = f \cdot v_F$ for some function f.

□

Remark 1.6. This result can almost certainly be generalized to cases where F has lower regularity and F' has greater regularity. However such generalizations require some care. See remark for a cautionary example.

Corollary 1.7. *There can be only one holomorphic foliation, with a smooth invariant measure, supported over all of M in a given homology class.*

In conversation with Gromov we discovered that when an almost complex structure admits sufficient symmetry the objects we produce should be foliations with smooth invariant measures and thus unique.

Conjecture 1.8. *Let (M, J) be an almost complex manifold such that J is invariant under a transitive compact group action. Then there is a complex cycle with representation μ , v both smooth. Further, when M has dimension 4, and $H^2(M)=0$, this cycle is a smooth foliation with invariant measure.*

One should be able to average the resulting complex cycles with the Haar measure on the group. The assumption $H^2(M)=0$ can be weakend to something more technical. $[0,1]\times S^3$ embedds in $\mathbb{C}^4$ via polar coordinates. The resulting complex strucure is invariant along the interval. We can thus identify $0\times S^3$, and $1\times S^3$, to gain an $S1\times S^3$ invariant complex structure on $S^1\times S^3$. The resulting foliation is then that whose leaves are given by the Hopf circles cross the S^1 direction.

Problem 1.1. S^6 admits an almost complex structure J which is invariant under the action of G_2. What is the resulting smooth averaged complex cycle?

1.3 Complex cycles and deformations of symplectic structure

Gromov, and Abreu-Mcduffs work symplectomorphis groups of $S^2\times S^2$, stemmed from the strong connection they established between deformations of symplectic structures and the existence of holomorphic curves in new classes. We now give a generalization of this connection within the context of complex cycles. Consider $S^2\times\Sigma$, and call a symplectic structure there with cohomology class $[\omega_k] = k\,\mathrm{PD}(S^2) + \mathrm{PD}(\Sigma)$ a ω_k form. Here k ranges in $(0,\infty)$.

Theorem 1.9. *Let j be an almost complex structure which is tamed by a ω_k form but not by a ω_{k_0} form for $k_0 < k$. Then there is a j-complex cycle which lies in the homology class* $[S^2\times \mathrm{pt}] - k\,[\mathrm{pt}\times\Sigma]$.

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2 J with no foliation with a transverse invariant metric

We now prove Theorem 1.1, that there is a tamed almost complex structure j on $S^2\times\Sigma$ such that the only holomorphic foliation posessing an invariant metric is that by the 2-spheres $\mathrm{pt}\times S^2$.

Definition 2.1. *Let* $p: \pi_1(\Sigma) \rightarrow \mathrm{Diff}(S^2)$ *be a representation of* $\pi_1(\Sigma)$ *into the diffeomorphisms of* S^2. *Let* H *denote the universal cover of* Σ, *and denote by* p_H *the corresponding action of* $\pi_1(\Sigma)$ *on* H. *Then the* ***suspension*** *of* p *is given by:* $\Sigma \times H/\sim$ *where* $(s, h) \sim (s', h')$ *if there is a* $g \in \pi_1(\Sigma)$ such that $\left\{ \begin{smallmatrix} p_H(g)(s) = s' \\ p(g)(h) = h' \end{smallmatrix} \right.$. *This is a 2 dimensional foliation of* S^2 *by surfaces* F_p, *defined up to isotopy.*

The suspension of p has a invariant measure if and only if p preserves a measure on S^2. It has a a smooth invariant metric if and only if it preserves a metric g there.

Theorem 2.2. (Failure of Narisham-Seshadri) *There is tamed almost complex structure* j *on* $S^2 \times \Sigma$ *such that there is no foliation lying in the homology class* $[\mathrm{pt} \times \Sigma]$, *with* j*-invariants leaves and an invariant metric.*

Proof. We will now construct a foliation $\mathcal{F}$ with smooth invariant measure in this homology class, and a j which makes it and the spheres $S^2 \times \mathrm{pt}$ holomorphic.

We construct a symplectomorphism $f \neq \mathrm{id}$ of S^2 which fixes a neighborhood of a point x_0. $\mathcal{F}$ will then be the suspension of a representation built out of f.

Choose a function ρ which vanishes in U. Denote by χ_ρ the resulting Hamiltonian vector field defined by the equation $\omega(\chi_\rho, \cdot) = d\rho$. Then let f be the time one flow of χ_ρ. By choosing ρ generically we can ensure that $f^a = \mathrm{id}$ only when $a = 0$. I claim that f is not an isometry for any smooth metric g. For suppose it is and choose a point x_0. Then by conjugating the geodesic flow out of x_0 by f we see that f has to be the identity.

We now choose a set of generators α_i of $\pi_1(\Sigma)$, $\alpha \neq \mathrm{id}$ and define $p(\alpha_0) = f$ and $p(\alpha_i) = \mathrm{id}$ for $i \neq 0$. This defines a representation of $\pi_1(\Sigma) \rightarrow \mathrm{Diff}(S^2)$, and its suspension $\mathcal{F}$ has a smooth invariant measure, but no smooth invariant metric. We then choose a tamed j which makes both the leaves of $\mathcal{F}$ and the spheres $\mathrm{pt} \times S^2$ holomorphic. This is possible by Lemma 7.5 in [Cofb].

Any holomorphic foliation $\mathcal{F}_{\mathrm{met}}$ with an invariant metric would also have a smooth invariant measure. It would then describe a homology class $[\mathcal{F}_{\mathrm{met}}]$ with 0 - self intersection. There are only two possible rays of choices $\lambda[S^2 \times \mathrm{pt}]$ or $\lambda[\mathrm{pt} \times \Sigma]$. The only foliation in class $\lambda[S^2 \times \mathrm{pt}]$ is that by the holomorphic two spheres $S^2 \times \mathrm{pt}$. We thus can assume that $[\mathcal{F}_{\mathrm{met}}] = \lambda[\mathrm{pt} \times \Sigma]$. Thus by Corollary 1.7 $\mathcal{F}_{\mathrm{met}}$ would have to coincide with $\mathcal{F}$. Since $\mathcal{F}$ does not posess an invariant metric, $\mathcal{F}_{\mathrm{met}}$ does not either. □

3 Existence via Hahn-Banach

We now prove existence results, via Sullivan's Hahn Banach alternative. Our proof is an adaptation of that of Theorem in [Sul76]. We modify the argument to restrict the complex cycle we obtain to a subspace X. In particular one can sometimes restrict its homology class. This is useful as we want to find cycles in classes with 0-selfintersection. It also allows us generalize the relationship between almost complex structures an complex cycles on $S^2 \times S^2$ to $S^2 \times \Sigma$, where Σ is a surface of higher genus.

3.1 An enriched Sullivan Hahn-Banach alternative

3.1.1 Cones and currents

Definition 3.1. *Let* J *denote an almost complex structure and let* C_J *denote the cone of* J*-complex cycles. Denote by* σ *a (possibly non-closed) smooth 2 form taming* J. *Denote* $\sigma^{-1}(1) \cap C_J$ *by* C_σ.

One can construct σ via a partion of unity. C_σ is a weakly compact set, as it is closed and bounded. [Sul76]

Definition 3.2. (Minkowski functional) *Let* D *be a convex subset of a vector space* V *such that:*

1. *if* $x \in D$ *and* $0 \leq t \leq 1$, *then* $t x \in D$
2. $\bigcup_{t > 0} t \cdot D = V$

Then the Minkowski functional of D is the map $\rho: V \to [0, \infty)$ given by :

$$\rho(x) \;=\; \inf\{\lambda | x \in \lambda D\}$$

This Minkowski functional satisfies the following properties:

1. If $t \geq 0$, then $\rho(\text{tx}) = t\rho(x)$
2. $\rho(x+y) \leq \rho(x) + \rho(y)$
3. $\{x|\rho(x) < 1\} \subset D \subset \{x|\rho(x) \leq 1\}$

3.1.2 The alternative

Theorem 3.3. *Let $(M,\, J)$ be a compact almost complex manifold. Let X be a weakly closed subspace such that $B \subset X \subset Z$, and $X^\perp \cap C^\infty(\Omega)$ does not contain a smooth symplectic form taming J. Then there is a J-holmorphic cycle $\varsigma \in X \cap C_\sigma$. Further, either:*

1. *One such cycle is null homologous* ***or***
2. *There is a smooth symplectic structure $\omega \in X^\perp \cap C^\infty(\Omega)$ on M taming J*

Proof. We consider the how the cone C relates to the closed subspaces of cycles X and boundaries B. There are three possiblities:

1. $C_\sigma \cap X = \varnothing$ (there are no J-holomorphic cycles)
2. $C_\sigma \cap X \neq \varnothing$, $C_\sigma \cap B = \varnothing$ (there are J-holomorphic cycles, and none are null homologous)
3. $C_\sigma \cap B \neq \varnothing$ (there is a null homologous J-holomorphic cycle)

We will prove the theorem by showing that 1 cannot occur for it implies the existence of a symplectic structure $\omega \in X^\perp$ taming J, and that if we are in case 2 there is a symplectic structure ω on M taming J.

Case 1 $\Rightarrow \boldsymbol{\omega} \in \boldsymbol{X}^\perp$ Consider the set $C_\sigma - X$. Since C_σ is compact and both C_σ and X are weakly closed, , $C_\sigma - X$ is also closed. Since $C_\sigma \cap X = \varnothing$, $C_\sigma - X$ does not include 0. Thus there is a weakly open, convex neighborhood U of 0 which is disjoint from $C_\sigma - X$.

$C_\sigma + \frac{1}{2}U$ is then an open, convex set which is also disjoint from X. Let $c_0 \in C_\sigma$. Then $C_\sigma + \frac{1}{2}U - c_0$ is absorbing. Let ρ_C be the associated Minkowski functional. We define a functional on $\lambda \cdot c_0 \oplus X$ by $l(\lambda \cdot c_0, z) = \lambda$. l is then bounded by ρ_C. Thus by Hahn-Banach we can find an extension of l to all of $W^{-1,p}$ which is also bounded by ρ_C. It is thus weakly continious and strictly positive on C_σ.

The resulting functional $l \in X^\perp$. Since $X \supset B$, and thus l is given by an closed form. Its positivity on C_σ imples that this form is non-degenerate and σ_l gives a symplectic structure on M. However, we have assumed that there can be no such symplectic structure $\sigma_l \in X^\perp$. Thus case 1 does not occur, and there is a J-holomoprhic cycle in X. We are thus either in case 2 or case 3.

Case 2 $\Rightarrow \boldsymbol{\omega}$ taming $\boldsymbol{J}$ We apply the same argument as in case 1, this time replacing the subspace X by the boundaries B. We thus construct a function l which is strictly positive on C_σ and vanishes on B. The resulting form σ_l is a symplectic structure taming J, and is thus symplectic.

Case 3 Then by assumtion there is a null homologous J-holomorphic cycle.

□

Corollary 3.4. (Existence of complex foliation cycles) *Let Σ denote a surface and let $(M,\, J) = S^2 \times \Sigma$ equipped with an almost complex structure J. Then there is always a j-holomrophic cycles in C_σ which lies in the homology class $\lambda[\text{pt} \times \Sigma]$ for some $\lambda \in \mathbb{R}$. If j is tamed by a symplectic structure then $\lambda \neq 0$.*

Proof. Let $X = B \oplus \lambda(\Sigma_1 \times \text{pt})$. Then the cycles $z \in X^\perp$ are Poincare Dual to a multiple of $[\Sigma_1 \times \text{pt}]$. These have 0 self intersection, and thus cannot represent symplectic structures. Thus there is a complex cycle $F \in X$. By Stokes theorem there is a null homologous F only when j is not tamed by symplectic structure. □

Call a symplectic form in class Poincare dual to $k[S^2 \times \mathrm{pt}] + [\mathrm{pt} \times \Sigma]$ a ω_k form.

Corollary 3.5. (Connection between complex cycles and deformations of symplectic structure) *Let j be an almost complex structure which is tamed by a ω_{k_+} form for $k_+ > k_0$ but not by any ω_{k_-} form for $k_- \leq k_0$. Then there is a j-complex cycle which lies in the homology class $[S^2 \times \mathrm{pt}] - k\,[\mathrm{pt} \times \Sigma]$.*

Proof. Let X denote $\lambda(k_0[S^2 \times \mathrm{pt}] + [\mathrm{pt} \times \Sigma])$. Suppose there is no complex cycle in X. Then there is a smooth symplectic structure $\omega \in X^{\perp} \cap C^{\infty}(\Omega)$ on M taming J. Such a form is a ω_{k_0} form, contrary to our assumption. □

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