ON THE WEAK LIMITS OF SMOOTH MAPS FOR THE DIRICHLET ENERGY BETWEEN MANIFOLDS

FENGBO HANG

Abstract. We identify all the weak sequential limits of smooth maps in $W^{1,2}(M,N)$. In particular, this implies a necessary and sufficient topological condition for smooth maps to be weakly sequentially dense in $W^{1,2}(M,N)$.

1. Introduction

Assume M and N are smooth compact Riemannian manifolds without boundary and they are embedded into $\mathbb{R}^l$ and $\mathbb{R}^{\bar{l}}$ respectively. The following spaces are of interest in the calculus of variations:

$$\begin{aligned} W^{1,2}(M,N) &= \left\{u \in W^{1,2}\left(M,\mathbb{R}^{\bar{l}}\right) : u(x) \in N \text{ a.e. } x \in M\right\}, \\ H_W^{1,2}(M,N) &= \{u \in W^{1,2}(M,N) : \text{ there exists a sequence } u_i \in C^\infty(M,N) \\ &\qquad \text{such that } u_i \rightharpoonup u \text{ in } W^{1,2}(M,N)\}. \end{aligned}$$

For a brief history and detailed references on the study of analytical and topological issues related to these spaces, one may refer to [2, 3, 7]. In particular, it follows from theorem 7.1 of [3] that a necessary condition for $H_W^{1,2}(M,N) = W^{1,2}(M,N)$ is that M satisfies the 1-extension property with respect to N (see section 2.2 of [3] for a definition). It was conjectured in section 7 of [3] that the 1-extension property is also sufficient for $H_W^{1,2}(M,N) = W^{1,2}(M,N)$. In [1, 7], it was shown that $H_W^{1,2}(M,N) = W^{1,2}(M,N)$ when $\pi_1(M) = 0$ or $\pi_1(N) = 0$. Note that if $\pi_1(M) = 0$ or $\pi_1(N) = 0$, then M satisfies the 1-extension property with respect to N. In section 8 of [4], it was proved that the above conjecture is true under the additional assumption that N satisfies the 2-vanishing condition. The main aim of the present article is to confirm the conjecture in its full generality. More precisely, we have

Theorem 1.1. *Let M^n and N be smooth compact Riemannian manifolds without boundary ($n \geq 3$). Take a Lipschitz triangulation $h : K \to M$, then*

$$\begin{aligned} & H_W^{1,2}(M,N) \\ = & \left\{u \in W^{1,2}(M,N) : u_{\#,2}(h) \text{ has a continuous extension to } M \text{ w.r.t. } N\right\} \\ = & \left\{u \in W^{1,2}(M,N) : u \text{ may be connected to some smooth maps}\right\}. \end{aligned}$$

In addition, if $\alpha \in [M,N]$ satisfies $\alpha \circ h|_{|K^1|} = u_{\#,2}(h)$, then we may find a sequence of smooth maps $u_i \in C^\infty(M,N)$ such that $u_i \rightharpoonup u$ in $W^{1,2}(M,N)$, $[u_i] = \alpha$ and $du_i \to du$ a.e..

Here $u_{\#,2}(h)$ is the 1-homotopy class defined by White [8] (see also section 4 of [3]) and $[M,N]$ means all homotopy classes of maps from M to N. It follows from Theorem 1.1 that

Corollary 1.1. *Let M^n and N be smooth compact Riemannian manifolds without boundary and $n \geq 3$. Then smooth maps are weakly sequentially dense in $W^{1,2}(M,N)$ if and only if M satisfies the 1-extension property with respect to N.*

For $p \in [3, n-1]$ being an natural number, it remains a challenging open problem to find out whether the weak sequential density of smooth maps in $W^{1,p}(M,N)$ is equivalent to the condition that M satisfies the $p-1$ extension property with respect to N. This was verified to be true under further topological assumptions on N (see section 8 of [4]). However, even for $W^{1,3}\left(S^4, S^2\right)$, it is still not known whether smooth maps are weakly sequentially dense. Some very interesting recent work on this space can be found in [5].

The paper is written as follows. In Section 2, we will present some technical lemmas. In Section 3, we will prove the above theorem and corollary.

Acknowledgments. The research of the author is supported by National Science Foundation Grant DMS-0209504.

2. Some preparations

The following local result, which was proved by Pakzad and Riviere in [7], plays an important role in our discussion.

Theorem 2.1 ([7]). *Let N be a smooth compact Riemannian manifold. Assume $n \geq 3$, $B_1 = B_1^n$, $f \in W^{1,2}(\partial B_1, N) \cap C(\partial B_1, N)$, $f \sim \mathrm{const}$, $u \in W^{1,2}(B_1, N)$, $u|_{\partial B_1} = f$, then there exists a sequence $u_i \in W^{1,2}(B_1,N) \cap C\left(\overline{B}_1, N\right)$ such that $u_i|_{\partial B_1} = f$, $u_i \rightharpoonup u$ in $W^{1,2}(B_1,N)$ and $du_i \to du$ a.e.. In addition, if $v \in W^{1,2}(B_2 \backslash B_1, N) \cap C\left(\overline{B}_2 \backslash B_1, N\right)$ satisfies $v|_{\partial B_1} = f$ and $v|_{\partial B_2} \equiv \mathrm{const}$, then we may estimate*

$$\int_{B_1} |du_i|^2 \, d\mathcal{H}^n \leq c(n,N) \left(\int_{B_1} |du|^2 \, d\mathcal{H}^n + \int_{B_2 \backslash B_1} |dv|^2 \, d\mathcal{H}^n \right).$$

For convenience, we will use those notations and concepts in section 2, 3 and 4 of [3]. The following lemma is a rough version of Luckhaus's lemma [6]. For reader's convenience, we sketch a proof of this simpler version using results from section 3 of [3].

Lemma 2.1. *Assume M^n and N are smooth compact Riemannian manifolds without boundary. Let $e > 0$, $0 < \delta < 1$, $A > 0$, then there exists an $\varepsilon = \varepsilon(e, \delta, A, M, N) > 0$ such that for any $u, v \in W^{1,2}(M,N)$ with $|du|_{L^2(M)}, |dv|_{L^2(M)} \leq A$ and $|u - v|_{L^2(M)} \leq \varepsilon$, we may find a $w \in W^{1,2}(M \times (0,\delta), N)$ such that, in the trace sense $w(x,0) = u(x)$, $w(x,\delta) = v(x)$ a.e. $x \in M$ and*

$$|dw|_{L^2(M \times (0,\delta))} \leq c(M)\sqrt{\delta}\left(|du|_{L^2(M)} + |dv|_{L^2(M)} + e\right).$$

Proof. Let $\varepsilon_M > 0$ be a small positive number such that

$$V_{2\varepsilon_M}(M) = \left\{ x \in \mathbb{R}^l : d(x,M) < 2\varepsilon_M \right\}$$

is a tubular neighborhood of M. Let $\pi_M : V_{2\varepsilon_M}(M) \to M$ be the nearest point projection. Similarly we have ε_N, $V_{2\varepsilon_N}(N)$ and π_N for N. Choose a Lipschitz cubeulation $h : K \to M$. We may assume each cell in K is a cube of unit size. For $\xi \in B^l_{\varepsilon_M}$, $x \in |K|$, let $h_\xi(x) = \pi_M(h(x) + \xi)$. Assume ε_M is small enough such that all h_ξ's are bi-Lipschitz maps. Set $m = \left[\frac{1}{\delta}\right] + 1$, using $[0,1] = \cup_{i=1}^m \left[\frac{i-1}{m}, \frac{i}{m}\right]$, we may divide each k-cube in K into m^k small cubes. In particular, we get a subdivision of K, called K_m. It follows from section 3 of [3] that for a.e. $\xi \in B^l_{\varepsilon_M}$, $u \circ h_\xi, v \circ h_\xi \in \mathcal{W}^{1,2}(K_m, N)$. Applying the estimates in section 3 of [3] to each unit size k-cube in $\left|K^k_m\right|$, we get

$$\begin{aligned}
\int_{B^l_{\varepsilon_M}} d\mathcal{H}^l(\xi) \int_{|K^k_m|} \left|d\left(u \circ h_\xi|_{|K^k_m|}\right)\right|^2 d\mathcal{H}^k &\leq c(M)\,\delta^{k-n} |du|^2_{L^2(M)}, \\
\int_{B^l_{\varepsilon_M}} d\mathcal{H}^l(\xi) \int_{|K^k_m|} \left|d\left(v \circ h_\xi|_{|K^k_m|}\right)\right|^2 d\mathcal{H}^k &\leq c(M)\,\delta^{k-n} |dv|^2_{L^2(M)},
\end{aligned}$$

and

$$\begin{aligned}
&\left(\int_{B^l_{\varepsilon_M}} |u \circ h_\xi - v \circ h_\xi|^2_{L^\infty(|K^1_m|)} d\mathcal{H}^l(\xi)\right)^{\frac{1}{2}} \\
\leq\ & c(\delta, M)\left(|d(u-v)|^{\frac{3}{4}}_{L^2(M)} |u-v|^{\frac{1}{4}}_{L^2(M)} + |u-v|_{L^2(M)}\right) \\
\leq\ & c(\delta, A, M)\,\varepsilon^{\frac{1}{4}}.
\end{aligned}$$

By the mean value inequality, we may find a $\xi \in B^l_{\varepsilon_M}$ such that $u \circ h_\xi, v \circ h_\xi \in \mathcal{W}^{1,2}(K_m, N)$,

$$|u \circ h_\xi - v \circ h_\xi|_{L^\infty(|K^1_m|)} \leq c(\delta, A, M)\,\varepsilon^{\frac{1}{4}} < \varepsilon_N \quad \text{when } \varepsilon \text{ is small enough,}$$

and

$$\begin{aligned}
&\int_{|K^k_m|} \left[\left|d\left(u \circ h_\xi|_{|K^k_m|}\right)\right|^2 + \left|d\left(v \circ h_\xi|_{|K^k_m|}\right)\right|^2\right] d\mathcal{H}^k \\
\leq\ & c(M)\,\delta^{k-n}\left(|du|^2_{L^2(M)} + |dv|^2_{L^2(M)}\right)
\end{aligned}$$

for $1 \leq k \leq n$. Fix a $\eta \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $0 \leq \eta \leq 1$, $\eta|_{\left(-\infty, \frac{1}{3}\right)} = 1$ and $\eta|_{\left(\frac{2}{3}, \infty\right)} = 0$. Letting $f = u \circ h_\xi$, $g = v \circ h_\xi$, we will define $\phi : |K| \times [0, \delta] \to N$ inductively. First set $\phi(x, 0) = f(x)$ and $\phi(x, \delta) = g(x)$ for $x \in |K|$. For $\Delta \in K^1_m \backslash K^0_m$, on $\Delta \times [0, \delta]$, we let

$$\phi(x,t) = \pi_N\left(\eta\left(\frac{t}{\delta}\right) f(x) + \left(1 - \eta\left(\frac{t}{\delta}\right)\right) g(x)\right) \quad x \in \Delta, 0 \leq t \leq \delta.$$

For $\Delta \in K^2_m \backslash K^1_m$, let y_Δ be the center of Δ, and define ϕ on $\Delta \times [0, \delta]$ as the homogeneous degree zero extension of $\phi|_{\partial(\Delta \times [0,\delta])}$ with respect to $\left(y_\Delta, \frac{\delta}{2}\right)$. Next we handle each 3-cube, 4-cube, $\cdots$, n-cube in a similar way. Calculations show that

$$\begin{aligned}
&\int_{|K| \times [0,\delta]} |d\phi|^2 d\mathcal{H}^{n+1} \\
\leq\ & c(n) \sum_{k=1}^n \delta^{n+1-k} \int_{|K^k_m|} \left[\left|d\left(u \circ h_\xi|_{|K^k_m|}\right)\right|^2 + \left|d\left(v \circ h_\xi|_{|K^k_m|}\right)\right|^2\right] d\mathcal{H}^k + c(\delta, A, M)\,\varepsilon^{\frac{1}{2}} \\
\leq\ & c(M)\,\delta\left(|du|^2_{L^2(M)} + |dv|^2_{L^2(M)} + e^2\right)
\end{aligned}$$

when ε is small enough. Finally $w : M \times [0,\delta] \to N$, defined by $w(x,t) = \phi\left(h_\xi^{-1}(x), t\right)$, is the needed map. □

Lemma 2.2. *Assume N is a smooth compact Riemannian manifold, $n \geq 2$, $B_1 = B_1^n$, $u, v \in W^{1,2}(B_1, N)$ such that $u|_{\partial B_1} = v|_{\partial B_1}$. Define $w : B_1 \times (0,1) \to N$ by*

$$w(x,t) = \begin{cases} u(x), & x \in B_1 \backslash B_t; \\ u\left(\frac{t^2}{|x|}\frac{x}{|x|}\right), & x \in B_t \backslash B_{t^2}; \\ v\left(\frac{x}{t^2}\right), & x \in B_{t^2}; \end{cases}$$

then $w \in W^{1,2}(B_1 \times (0,1), N)$ and

$$|dw|_{L^2(B_1\times(0,1))} \leq c(n)\left(|du|_{L^2(B_1)} + |dv|_{L^2(B_1)}\right).$$

Proof. Note that

$$|dw(x,t)| \leq \begin{cases} |du(x)|, & t < |x|; \\ c(n)\left|du\left(\frac{t^2}{|x|}\frac{x}{|x|}\right)\right|\frac{t^2}{|x|^2}, & t^2 < |x| < t; \\ c(n)\left|dv\left(\frac{x}{t^2}\right)\right|\frac{1}{t^2}, & |x| < t^2. \end{cases}$$

Hence

$$\begin{aligned} &\int_{\substack{0<t<1 \\ t^2<|x|<t}} |dw(x,t)|^2 \, d\mathcal{H}^{n+1}(x,t) \\ \leq\ & c(n)\int_0^1 dt \int_{t^2}^t dr \int_{\partial B_r} \left|du\left(\frac{t^2}{r^2}x\right)\right|^2 \frac{t^4}{r^4} d\mathcal{H}^{n-1}(x) \\ =\ & c(n)\int_0^1 dt \int_t^1 ds \int_{\partial B_s} \frac{t^{2(n-2)}}{s^{2(n-2)}} |du(y)|^2 \, d\mathcal{H}^{n-1}(y) \\ \leq\ & c(n)\,|du|^2_{L^2(B_1)}, \end{aligned}$$

and

$$\begin{aligned} &\int_{\substack{0<t<1 \\ |x|<t^2}} |dw(x,t)|^2 \, d\mathcal{H}^{n+1}(x,t) \\ \leq\ & c(n)\int_0^1 dt \int_{B_{t^2}} \left|dv\left(\frac{x}{t^2}\right)\right|^2 \frac{1}{t^4} d\mathcal{H}^n(x) \\ \leq\ & c(n)\,|dv|^2_{L^2(B_1)}. \end{aligned}$$

The lemma follows. □

3. Identifying weak limits of smooth maps

In this section, we shall prove Theorem 1.1 and Corollary 1.1.

Proof of Theorem 1.1. Let $h : K \to M$ be a Lipschitz cubeulation. We may assume each cell in K is a cube of unit size. Let $\varepsilon_M > 0$ be a small number such that

$$V_{2\varepsilon_M}(M) = \left\{x \in \mathbb{R}^l : d(x,N) < 2\varepsilon_M\right\}$$

is a tubular neighborhood of M. Denote $\pi_M : V_{2\varepsilon_M}(M) \to M$ as the nearest point projection. For $\xi \in B^l_{\varepsilon_M}$, we let $h_\xi(x) = \pi_M(h(x) + \xi)$ for $x \in |K|$, the polytope of K. We may assume ε_M is small enough such that all h_ξ are bi-Lipschitz maps. Replacing h by h_ξ when necessary, we may assume $f = u \circ h \in \mathcal{W}^{1,2}(K, N)$.

Then we may find a $g \in C(|K|, N) \cap \mathcal{W}^{1,2}(K, N)$ such that $[g \circ h^{-1}] = \alpha$ and $g|_{|K^1|} = f|_{|K^1|}$ (see the proof of theorem 5.5 and theorem 6.1 in [4]). For each cell $\Delta \in K$, let y_Δ be the center of Δ. For $x \in \Delta$, let $|x|_\Delta$ be the Minkowski norm with respect to y_Δ, that is

$$|x|_\Delta = \inf\left\{t > 0 : y_\Delta + \frac{x - y_\Delta}{t} \in \Delta\right\}.$$

Step 1: For every $\Delta \in K^2 \backslash K^1$, we may find a sequence $\phi_i \in C(\Delta, N) \cap W^{1,2}(\Delta, N)$ such that $\phi_i|_{\partial\Delta} = g|_{\partial\Delta}$, $\phi_i \to f|_\Delta$ in $W^{1,2}(\Delta, N)$ and $d\phi_i \to d(f|_\Delta)$ a.e. (see lemma 4.4 in [3]). For $x \in \Delta$, let

$$f_i(x) = \begin{cases} \phi_i(x), & |x|_\Delta \geq \frac{1}{2^i}; \\ \phi_i\left(y_\Delta + \frac{1}{2^{2i}|x|_\Delta}\frac{x - y_\Delta}{|x|_\Delta}\right), & \frac{1}{2^{2i}} \leq |x|_\Delta \leq \frac{1}{2^i}; \\ g\left(y_\Delta + 2^{2i}(x - y_\Delta)\right), & |x|_\Delta \leq \frac{1}{2^{2i}}. \end{cases}$$

It is clear that $f_i \rightharpoonup f|_\Delta$ in $W^{1,2}(\Delta, N)$, $df_i \to d(f|_\Delta)$ a.e. on Δ,

$$|df_i|_{L^2(\Delta)} \leq c \cdot \left(|d\phi_i|_{L^2(\Delta)} + |d(g|_\Delta)|_{L^2(\Delta)}\right) \leq c(f, g)$$

and $f_i \in C(|K^2|, N)$. In addition, if we define $h_{2,i} : \Delta \times [0,1] \to N$ by

$$h_{2,i}(x,t) = \begin{cases} \phi_i(x), & |x|_\Delta \geq \frac{1}{2^i} + \frac{2^i - 1}{2^i}t; \\ \phi_i\left(y_\Delta + \frac{\left(\frac{1}{2^i} + \frac{2^i-1}{2^i}t\right)^2}{|x|_\Delta}\frac{x - y_\Delta}{|x|_\Delta}\right), & \left(\frac{1}{2^i} + \frac{2^i - 1}{2^i}t\right)^2 \leq |x|_\Delta \leq \frac{1}{2^i} + \frac{2^i - 1}{2^i}t; \\ g\left(y_\Delta + \frac{x - y_\Delta}{\left(\frac{1}{2^i} + \frac{2^i-1}{2^i}t\right)^2}\right), & |x|_\Delta \leq \left(\frac{1}{2^i} + \frac{2^i - 1}{2^i}t\right)^2. \end{cases}$$

Then by Lemma 2.2, we know $h_{2,i} \in W^{1,2}(\Delta \times [0,1], N)$,

$$|dh_{2,i}|_{L^2(\Delta \times [0,1])} \leq c \cdot \left(|d\phi_i|_{L^2(\Delta)} + |d(g|_\Delta)|_{L^2(\Delta)}\right) \leq c(f, g)$$

and $h_{2,i} \in C(|K^2| \times [0,1], N)$.

Step 2: Assume for some $2 \leq k \leq n-1$, we have a sequence $f_i \in C(|K^k|, N) \cap \mathcal{W}^{1,2}(K^k, N)$ and $h_{k,i} \in C(|K^k| \times [0,1], N)$ such that for each $\Delta \in K^k$, $f_i \rightharpoonup f|_\Delta$ in $W^{1,2}(\Delta, N)$, $h_{k,i} \in W^{1,2}(\Delta \times [0,1], N)$,

$$|d(f_i|_\Delta)|_{L^2(\Delta)} \leq c(f,g), \quad |dh_{k,i}|_{L^2(\Delta \times [0,1])} \leq c(f,g) \tag{3.1}$$

and $h_{k,i}(x, 0) = f_i(x)$, $h_{k,i}(x, 1) = g(x)$ for $x \in |K^k|$. Since for every $\Delta \in K^{k+1} \backslash K^k$, $f_i \rightharpoonup f|_{\partial\Delta}$ in $W^{1,2}(\partial\Delta, N)$, for fixed j by Lemma 2.1 we may find a $n_j \geq j$ such that for each $\Delta \in K^{k+1} \backslash K^k$, there exists a $w_j \in W^{1,2}(\partial\Delta \times [0, 2^{-j}], N)$ with $w_j(x, 0) = f(x)$, $w_j\left(x, \frac{1}{2^j}\right) = f_{n_j}(x)$ and

$$|dw_j|_{L^2\left(\partial\Delta \times \left(0, \frac{1}{2^j}\right)\right)} \leq \frac{c(n)}{2^{\frac{j}{2}}}\left(|d(f|_{\partial\Delta})|_{L^2(\partial\Delta)} + |df_{n_j}|_{L^2(\partial\Delta)} + 1\right) \leq \frac{c(f,g)}{2^{\frac{j}{2}}}.$$

Without loss of generality, we may replace f_i by f_{n_i} and $h_{k,i}$ by h_{k,n_i}. Fix a $\Delta \in K^{k+1} \backslash K^k$. For $x \in \Delta$, let

$$\psi_i(x) = \begin{cases} f\left(y_\Delta + \frac{2^i(x - y_\Delta)}{2^i - 1}\right), & |x|_\Delta \leq \frac{2^i - 1}{2^i}; \\ w_i\left(y_\Delta + \frac{x - y_\Delta}{|x|_\Delta}, |x|_\Delta - \frac{2^i - 1}{2^i}\right), & \frac{2^i - 1}{2^i} \leq |x|_\Delta \leq 1. \end{cases}$$

Then $\psi_i|_{|K^k|}=f_i$ and $\psi_i\to f|_\Delta$ in $W^{1,2}(\Delta,N)$ as $i\to\infty$ for each $\Delta\in K^{k+1}\backslash K^k$. By Theorem 2.1 and (3.1) (use $h_{k,i}$ and g for the needed "v" in Theorem 2.1, one may refer to lemma 9.8 of [4]), for every $\Delta\in K^{k+1}\backslash K^k$, we may find $\phi_i\in C(\Delta,N)\cap W^{1,2}(\Delta,N)$ such that $\phi_i|_{\partial\Delta}=f_i|_{\partial\Delta}$, $|\phi_i-\psi_i|_{L^2(\Delta)}<\frac{1}{2^i}$, $|d\phi_i|_{L^2(\Delta)}\le c(f,g)$ and

$$\int_M \frac{|d\phi_i-d\psi_i|}{1+|d\phi_i-d\psi_i|}d\mathcal{H}^{k+1}\le\frac{1}{2^i}.$$

After passing to subsequence, we may assume $d\phi_i\to d(f|_\Delta)$ a.e. on Δ. Fix a $\Delta\in K^{k+1}\backslash K^k$, for any $x\in\Delta$, define

$$\begin{aligned}
g_{k+1,i}(x) &= \begin{cases} h_{k,i}\left(y_\Delta+\frac{x-y_\Delta}{|x|_\Delta},1+2\left(\frac12-|x|_\Delta\right)\right), & \frac12\le|x|_\Delta\le1;\\ g\left(y_\Delta+2(x-y_\Delta)\right), & |x|_\Delta\le\frac12,\end{cases}\\
f_i(x) &= \begin{cases} \phi_i(x), & |x|_\Delta\ge\frac{1}{2^i};\\ \phi_i\left(y_\Delta+\frac{1}{2^{2i}|x|_\Delta}\frac{x-y_\Delta}{|x|_\Delta}\right), & \frac{1}{2^{2i}}\le|x|_\Delta\le\frac{1}{2^i};\\ g_{k+1,i}\left(y_\Delta+2^{2i}(x-y_\Delta)\right), & |x|_\Delta\le\frac{1}{2^{2i}},\end{cases}\\
\widetilde{h}_{k+1,i}(x,t) &= \begin{cases} \phi_i(x), & |x|_\Delta\ge\frac{1}{2^i}+\frac{2^i-1}{2^i}t;\\ \phi_i\left(y_\Delta+\frac{\left(\frac{1}{2^i}+\frac{2^i-1}{2^i}t\right)^2}{|x|_\Delta}\frac{x-y_\Delta}{|x|_\Delta}\right), & \left(\frac{1}{2^i}+\frac{2^i-1}{2^i}t\right)^2\le|x|_\Delta\le\frac{1}{2^i}+\frac{2^i-1}{2^i}t;\\ g_{k+1,i}\left(y_\Delta+\frac{x-y_\Delta}{\left(\frac{1}{2^i}+\frac{2^i-1}{2^i}t\right)^2}\right), & |x|_\Delta\le\left(\frac{1}{2^i}+\frac{2^i-1}{2^i}t\right)^2,\end{cases}\\
\widetilde{\widetilde{h}}_{k+1,i}(x,t) &= \begin{cases} h_{k,i}\left(y_\Delta+\frac{x-y_\Delta}{|x|_\Delta},1+2\left(\frac{1+t}{2}-|x|_\Delta\right)\right), & \frac{1+t}{2}\le|x|_\Delta\le1;\\ g\left(y_\Delta+\frac{2}{1+t}(x-y_\Delta)\right), & |x|_\Delta\le\frac{1+t}{2},\end{cases}
\end{aligned}$$

and

$$h_{k+1,i}(x,t)=\begin{cases}\widetilde{h}_{k+1,i}(x,2t), & 0\le t\le\frac12;\\ \widetilde{\widetilde{h}}_{k+1,i}(x,2t-1), & \frac12\le t\le1.\end{cases}$$

Simple calculations show that for any $\Delta\in K^{k+1}\backslash K^k$, $f_i\rightharpoonup f|_\Delta$ in $W^{1,2}(\Delta,N)$, $df_i\to d(f|_\Delta)$ a.e. on Δ, $h_{k+1,i}\in W^{1,2}(\Delta\times[0,1],N)$,

$$|df_i|_{L^2(\Delta)}\le c(f,g),\quad |dh_{k+1,i}|_{L^2(\Delta\times[0,1])}\le c(f,g)$$

and $h_{k+1,i}(x,0)=f_i(x)$, $h_{k+1,i}(x,1)=g(x)$ for $x\in\left|K^{k+1}\right|$. Hence we finish when we reach $f_i\in C(|K|,N)\cap\mathcal{W}^{1,2}(K,N)$ and $h_{n,i}\in C(|K|\times[0,1],N)$. Let $v_i=f_i\circ h^{-1}$. Then it is clear that $v_i\in C(M,N)\cap W^{1,2}(M,N)$, $[v_i]=\alpha$, $|v_i-u|_{L^2(M)}\to0$, $|dv_i|_{L^2(M)}\le c(u,g)$ and $dv_i\to du$ a.e. on M. Hence, we may find $u_i\in C^\infty(M,N)$ such that $|u_i-u|_{L^2(M)}\to0$, $|du_i|_{L^2(M)}\le c(u,g)$, $[u_i]=\alpha$ and $du_i\to du$ a.e. on M. In particular, this shows

$$H_W^{1,2}(M,N)\supset\left\{u\in W^{1,2}(M,N):u_{\#,2}(h)\text{ has a continuous extension to }M\text{ w.r.t. }N\right\}.$$

The other direction of inclusion was proved in section 7 of [3]. To see

$$H_W^{1,2}(M,N)=\left\{u\in W^{1,2}(M,N):u\text{ may be connected to some smooth maps}\right\},$$

we only need to use the above proved equality and proposition 5.2 of [3], which shows

$$\begin{aligned}&\left\{u\in W^{1,2}(M,N):u_{\#,2}(h)\text{ has a continuous extension to }M\text{ w.r.t. }N\right\}\\ =&\left\{u\in W^{1,2}(M,N):u\text{ may be connected to some smooth maps}\right\}.\end{aligned}$$

□

We remark that many constructions above are motivated from section 5 and section 6 of [4].

Proof of Corollary 1.1. This follows from Theorem 1.1 and corollary 5.4 of [3]. □

References

[1] P. Hajlasz. Approximation of Sobolev mappings. *Nonlinear Anal* **22** (1994), no. 12, 1579–1591.

[2] F. B. Hang and F. H. Lin. Topology of Sobolev mappings. *Math Res Lett* **8** (2001), no. 3, 321–330.

[3] F. B. Hang and F. H. Lin. Topology of Sobolev mappings II. *Acta Math* **191** (2003), no. 1, 55–107.

[4] F. B. Hang and F. H. Lin. Topology of Sobolev mappings III. *Comm Pure Appl Math* **56** (2003), no. 10, 1383–1415.

[5] R. Hardt and T. Riviere. Connecting topological Hopf singularities. *Annali Sc Norm Sup Pisa*, **2** (2003), no. 2, 287–344.

[6] S. Luckhaus. Partial Holder continuity for minima of certain energies among maps into a Riemannian manifold. *Indiana Univ Math J* **37** (1988), 349–367.

[7] M. R. Pakzad and T. Riviere. Weak density of smooth maps for the Dirichlet energy between manifolds. *Geom Func Anal* **13** (2003), no. 1, 223–257.

[8] B. White. Homotopy classes in Sobolev spaces and the existence of energy minimizing maps. *Acta Math* **160** (1988), no. 1–2, 1–17.

Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544, and, School of Mathematics, Institute for Advanced Study, 1 Einstein Drive, Princeton, NJ 08540

E-mail address: fhang@math.princeton.edu