Chapter 1, Problem 15.

(a) Let us compute the conditional probability of the complement event that \{no child or exactly one child has blue eyes\}. Intersecting with the event \{at least one child has blue eyes\}, we get \{exactly one child has blue eyes\}. Let the eye color of the children be $C_1$, $C_2$, $C_3$. The probability $P(\text{exactly one child has blue eyes}) = P(C_1 = \text{blue}, C_2 \neq \text{blue}, C_3 \neq \text{blue}) + P(C_1 \neq \text{blue}, C_2 = \text{blue}, C_3 \neq \text{blue}) + P(C_1 \neq \text{blue}, C_2 \neq \text{blue}, C_3 = \text{blue}) = \frac{3}{4} \left( \frac{3}{4} \right)^2 = \left( \frac{3}{4} \right)^3$.

We have to divide by $P(\text{at least one child has blue eyes}) = 1 - P(\text{no child has blue eyes}) = 1 - \left( \frac{3}{4} \right)^3$. So,

$$P(\text{no child or exactly one child has blue eyes} \mid \text{at least one child has blue eyes}) = \frac{\left( \frac{3}{4} \right)^3}{1 - \left( \frac{3}{4} \right)^3}.$$ 

Thus

$$P(\text{at least two children have blue eyes} \mid \text{at least one child has blue eyes}) = 1 - \frac{\left( \frac{3}{4} \right)^3}{1 - \left( \frac{3}{4} \right)^3} = \frac{10}{37}.$$ 

(b) Let the eye color of the children be $C_1$, $C_2$, $C_3$, where $C_3$ is the eye color of the youngest child. Define the event $B = \{C_1 = \text{blue}\}$. Now, we know that

$$P(\text{at least two children have blue eyes} \mid B) = 1 - P(\text{less than 2 children have blue eyes} \mid B).$$

Also $\{\text{less than 2 children have blue eyes}\} = \{\text{no child has blue eyes}\} \cup \{\text{one child has blue eyes}\}$. Using the definition of conditional probability, we get

$$P(\text{less than 2 children have blue eyes} \mid B) = \frac{P(\{\text{no child has blue eyes}\} \cup \{\text{one child has blue eyes}\} \cap B)}{P(B)}$$

$$= \frac{P(\{\text{no child has blue eyes}\} \cap B) \cup (\{\text{one child has blue eyes}\} \cap B))}{P(B)}.$$ 

Clearly, $\{\text{no child has blue eyes}\} \cap B = \emptyset$, so using independence in the third equality, we have:

$$P(\{\text{no child has blue eyes}\} \cap B) = P(\{\text{one child has blue eyes}\} \cap B)$$

$$= P(C_2 \neq \text{blue}, B)$$

$$= P(C_2 \text{ doesn't have blue eyes})P(C_3 \text{ doesn't have blue eyes})P(B)$$

$$= \frac{3}{4} \cdot \frac{3}{4} \cdot P(B) = \frac{9}{16} P(B).$$
So, $\mathbb{P}(\text{less than 2 children have blue eyes } | B) = \frac{9}{16}$. Therefore, $\mathbb{P}(\text{at least two children have blue eyes } | B) = 1 - \frac{9}{16} = \frac{7}{16}$.

Another way is to see that, by independence, $\mathbb{P}(\text{at least two children have blue eyes } | B)$ is the probability of the event that

$$A = \{C_2 = \text{blue}, C_3 = \text{blue}\} \cup \{C_1 \neq \text{blue}, C_3 = \text{blue}\} \cup \{C_1 = \text{blue}, C_3 \neq \text{blue}\}.$$  

The function $\mathbb{P}$ is additive over disjoint events, so using independence in the second equality

$$\mathbb{P}(A) = \mathbb{P}(C_2 = \text{blue}, C_3 = \text{blue}) + \mathbb{P}(C_2 \neq \text{blue}, C_3 = \text{blue}) + \mathbb{P}(C_2 = \text{blue}, C_3 \neq \text{blue})$$

$$= \mathbb{P}(C_2 = \text{blue})\mathbb{P}(C_3 = \text{blue}) + \mathbb{P}(C_2 \neq \text{blue})\mathbb{P}(C_2 = \text{blue}) + \mathbb{P}(C_2 = \text{blue})\mathbb{P}(C_3 \neq \text{blue})$$

$$= \frac{1}{4} \cdot \frac{3}{4} + 3 \cdot \frac{1}{4} \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{1}{4} = \frac{7}{16}.$$

Chapter 1, Problem 19.

Use abbreviations $W=\text{Windows}$, $M=\text{Mac}$, $L=\text{Linux}$, $I=\text{infected}$. By Bayes’ theorem

$$\mathbb{P}(W | I) = \frac{\mathbb{P}(I | W)\mathbb{P}(W)}{\mathbb{P}(I | W)\mathbb{P}(W) + \mathbb{P}(I | M)\mathbb{P}(M) + \mathbb{P}(I | L)\mathbb{P}(L)} = \frac{82}{100} \cdot \frac{82}{100} \cdot \frac{1}{10} + \frac{2}{10} = \frac{82}{141}.$$  

Chapter 2, Problem 9.

Let $X \sim \text{Exp}(\beta)$. Then $F(x) = \mathbb{P}(X \leq x) = \int_0^x \beta e^{-\beta t} dt = 1 - e^{-\beta x}$. So setting $q = 1 - e^{-\beta x}$, we have

$$x = \frac{\log(1-q)}{-\beta} = \log \left( \frac{1}{1-q} \right).$$

Therefore $F^{-1}(q) = \log \left( \frac{1}{1-q} \right)$.

Chapter 2, Problem 16.

Let $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$ independent. Given $X + Y = n$, $X$ can take values in $D = \{0, 1, \cdots, n\}$. Let $k \in D$, we compute the probability

$$\mathbb{P}(X = k | X + Y = n) = \frac{\mathbb{P}(X = k, X + Y = n)}{\mathbb{P}(X + Y = n)}.$$  

$$= \frac{\mathbb{P}(X = k) \mathbb{P}(Y = n-k)}{\mathbb{P}(X + Y = n)}$$

$$= e^{-\lambda} \frac{\lambda^k}{k!} e^{-\mu} \frac{\mu^{n-k}}{(n-k)!}$$

$$= e^{-(\lambda+\mu)} \frac{n!}{(k)(n-k)!} \lambda^k \mu^{n-k}.$$  

Define $\pi = \frac{\lambda}{\lambda+\mu}$. Noting that $\lambda^k \mu^{n-k} = (\lambda+\mu)^n \pi^k (1-\pi)^{n-k}$, we get $\mathbb{P}(X = k, X + Y = n) = C(n \choose k) \pi^k (1-\pi)^{n-k}$, where $C$ doesn’t depend on $k$. Since we know that in case $C = 1$, the above distribution is already normalized, we conclude that $\mathbb{P}(X + Y = n) = C$, and $X$ is $\text{Binom}(n, \pi)$ distributed given $X + Y$.

Chapter 2, Problem 17.

First of all $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$. The density of $Y$ is $f_Y(y) = \int_0^1 f_{X,Y}(x,y) dx = \int_0^1 c(x+y^2) dx = c\left(\frac{1}{2} + y^2\right)$, so $f_{X|Y}(x|y) = \frac{x + y^2}{2 + y^2}$. Now, we can compute

$$\mathbb{P}(X < 1/2 | Y = y) = \int_0^{1/2} f_{X|Y}(x|y) dx = \int_0^{1/2} \frac{x + y^2}{1/2 + y^2} dx = \frac{1 + 4y^2}{4 + 8y^2}.$$
Chapter 2, Problem 20.

Let \( X, Y \sim \text{unif}(0, 1) \) be independent random variables. We look for the distribution of \( X - Y \) and \( X/Y \). Clearly \( X - Y \) is distributed on \((-1, 1)\), and \( X/Y \) is distributed on \((0, \infty)\). Let us compute the cumulative distribution functions, for \( z \in [0, 1) \), we get

\[
\mathbb{P}(X - Y \leq z) = \mathbb{P}(X \leq Y + z) = 1 - \frac{(1 - z)^2}{2},
\]

since in the \((X, Y)\)-plane it is the area of the square \((0, 1)^2\), minus the area of the triangle with edges \((z, 0), (1, 1 - z), (1, 0)\). Similarly, for \( z \in (-1, 0] \), we get

\[
\mathbb{P}(X - Y \leq z) = \mathbb{P}(X \leq Y + z) = \frac{(1 + z)^2}{2},
\]

which is the area of the triangle with edges \((0, -z), (1 + z, 1), (0, 1)\). Differentiation gives the PDF \( f(z) = 1 - |z| \) for \( z \in (-1, 1) \).

Similarly for \( X/Y \) and \( z \in [0, 1) \), we compute

\[
\mathbb{P}(X/Y \leq z) = \mathbb{P}(X \leq zY) = \frac{z}{2}.
\]

For \( z > 1 \), we get

\[
\mathbb{P}(X/Y \leq z) = \mathbb{P}(X \leq zY) = 1 - \frac{1}{2z}.
\]

Differentiation gives the PDF \( f_{X/Y}(z) = \frac{1}{2}(1_{(0,1)} + \frac{1}{z^2} 1_{(1,\infty)}) \). I.e. \( f_{X/Y}(z) \) is constant \( \frac{1}{2} \) for \( z \in (0, 1) \) and takes the value \( \frac{1}{2z} \) for \( z > 1 \).

Chapter 2, Problem 21.

Let \( X_1, \ldots, X_n \) i.i.d. \( \text{Exp}(\beta) \) distributed random variables. Define \( Y = \max\{X_1, \ldots, X_n\} \). Let us compute

\[
\mathbb{P}(Y \leq y) = \mathbb{P}(X_1 \leq y, \ldots, X_n \leq y) = \prod_{k=1}^{n} \mathbb{P}(X_k \leq y) = \left( \int_0^y \beta e^{-\beta x} dx \right)^n = (1 - e^{-\beta y})^n.
\]

Differentiation in \( y \) gives \( f_Y(y) = n(1 - e^{-\beta y})^{n-1} \beta e^{-\beta y} \).

Chapter 3, Problem 4.

Let \( Y_k, k \in \mathbb{N} \) be i.i.d. random variables with \( \mathbb{P}(Y_1 = 1) = 1 - p \) and \( \mathbb{P}(Y_1 = -1) = p \). Then \( X_n = \sum_{k=1}^{n} Y_k \). The expectation is \( \mathbb{E}[X_n] = \sum_{k=1}^{n} \mathbb{E}[Y_k] = n\mathbb{E}[Y_1] = n(1 - p) + (-1)p = n(1 - 2p) \). For the variance, we get \( \text{var}(X_n) = \sum_{k=1}^{n} \text{var}(Y_k) = n\text{var}(Y_1) = 4np(1 - p) \), where we used independence in the first equality.
HW1.

1. Ch. 1, 15.
(a) at least one has blue eyes:
\[ P = 1 - (1 - \frac{3}{4})^3 = \frac{87}{64} \]

at least two have blue eyes:
\[ P = C_3^2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right) + C_3^3 \left(\frac{3}{4}\right)^3 = \frac{27}{64} + \frac{1}{4} = \frac{10}{14} \]

\[ P = \frac{P_2}{P_1} = \frac{\frac{10}{14}}{\frac{87}{64}} = \frac{40}{87} \]

(b) this will happen if at least one of the other two children has blue eyes.
\[ P = 1 - \left(\frac{3}{4}\right)^2 = \frac{7}{16} \]

2. Ch. 1, 19.
\[ P(M) = 0.3, \quad P(W) = 0.5, \quad P(L) = 0.2. \]
\[ P(V|M) = 0.65, \quad P(V|W) = 0.82, \quad P(V|L) = 0.5. \]

\[ P(W|V) = \frac{P(V|W) P(W)}{P(V|M) P(M) + P(V|W) P(W) + P(V|L) P(L)} \]
\[ = \frac{0.82}{14} \approx 0.59 \]

3. Ch 2, 9.
\[ X \sim \exp(\beta), \quad f(x) = \beta e^{-\beta x} \quad (x > 0) \]
\[ F(x) = \int_0^x \beta e^{-\beta y} dy = -e^{-\beta y} \bigg|_0^x = 1 - e^{-\beta x}, \quad x > 0. \]

let \[ F(y) = 1 - e^{-\beta y} \quad (0 \leq y < 1) \]

\[ e^{-\beta y} = 1 - F(y) \]

\[ y = \frac{\ln(1 - F)}{-\beta} \quad : \quad F^{-1}(y) = \frac{\ln(1 - y)}{-\beta}, \quad 0 \leq y < 1 \]
4. Ch 2.16 \[ X \sim \text{Poisson}(\lambda), \; Y \sim \text{Poisson}(\mu), \; X, Y \text{ ind. } \Rightarrow X+Y \sim \text{Poisson}(\mu+\lambda) \]

\[
P(X=m | X+Y=n) = \frac{P(X=m, X+Y=n)}{P(X+Y=n)} = \frac{P(X=m) P(Y=n-m)}{P(X+Y=n)}\]

\[
= \frac{e^{-\lambda} \frac{\lambda^m}{m!} e^{-\mu} \frac{\mu^{n-m}}{(n-m)!}}{P(X+Y=n)} = \frac{e^{-(\lambda+\mu)} \frac{(\lambda^{m} \mu^{n-m})}{m! (n-m)!}}{P(X+Y=n)} = \frac{C_n \frac{\lambda^{m}}{m!} \frac{\mu^{n-m}}{(n-m)!}}{P(X+Y=n)} = \frac{C_n \frac{\lambda^{m}}{m!} \frac{\mu^{n-m}}{(n-m)!}}{P(X+Y=n)} = \frac{C_n \frac{\lambda^{m}}{m!} \frac{\mu^{n-m}}{(n-m)!}}{P(X+Y=n)}
\]

this is the p.m.f of a Binomial \((n, \frac{\lambda}{\lambda+\mu})\). \(\Box\)

5. Ch 2.17

\[
\iint_{D} f(x, y) \, dx \, dy = \iint_{D} (x+y^2) \, dx \, dy
\]

\[
= \int_0^1 \int_0^y (x+y^2) \, dx \, dy = \int_0^1 \left[ \frac{1}{2} x^2 + \frac{1}{3} y^3 \right]_0^y \, dy = \int_0^1 \left[ \frac{1}{2} y^2 + \frac{1}{3} y^3 \right] \, dy = \frac{1}{2} \frac{y^3}{3} + \frac{1}{3} \frac{y^4}{4} \bigg|_0^1 = \frac{1}{2} + \frac{1}{12} = \frac{7}{12}
\]

\[
P(x \leq 1, Y \leq 2) = \int_0^1 \int_0^{1/2} (x+y^2) \, dx \, dy = \int_0^1 \left[ \frac{1}{2} x^2 + \frac{1}{3} y^3 \right]_0^{1/2} \, dy = \frac{1}{2} \frac{1}{4} + \frac{1}{3} \frac{1}{8} = \frac{2}{10} = \frac{1}{5}
\]

\[
p(x < 1, y < 1) = \int_0^1 \int_0^1 (x+y^2) \, dx \, dy = \int_0^1 \left[ \frac{1}{2} x^2 + \frac{1}{3} y^3 \right]_0^1 \, dy = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}
\]

\[
f(x, y) = \frac{f(x, y)}{f(y)} = \frac{\frac{6}{5} x + \frac{1}{5} y^2}{\frac{1}{5} y^2 + \frac{2}{5}}
\]

\[
P(x < 1, y < 1) = \int_0^1 f(x, y) \, dx = \int_0^1 \frac{6}{5} x + \frac{2}{10} \, dx = \frac{1}{5}
\]
6. \( (x, y) \sim (0, 1) \). \( f(x, y) = f(x, y) = 1 \), \( x \in [0, 1], y \in [0, 1] \)

\[ z = x - y \]

\[ F_z(z) = P(x - y \leq z) = \int \int_{x - y \leq z} f(x, y) \, dx \, dy \]

\[ f_z(z) = \begin{cases} 0, & z \leq -1, z \geq 1 \\ 1 + z, & -1 < z < 0 \\ 1 - z, & 0 < z < 1 \end{cases} \]

\[ W = \frac{x}{y} \]

\[ F_W(w) = P \left( \frac{x}{y} \leq w \right) = \int \int_{x/y \leq w} f(x, y) \, dx \, dy \]

\[ f_W(w) = \begin{cases} 0, & w \leq 0 \\ \frac{1}{w}, & 0 < w < 1 \\ \frac{1}{w^2}, & w > 1 \end{cases} \]

7. \( (x, y) \sim (\beta, \gamma) \)

\[ f(x, y) = \frac{1}{\beta \gamma} e^{-x \beta} e^{-y \gamma} \]

\[ f_x(x) = \frac{1}{\beta \gamma} \beta e^{-x \beta} \]

\[ f_y(y) = \frac{1}{\beta \gamma} \gamma e^{-y \gamma} \]

\[ \text{pdf} = \frac{1}{\beta \gamma} e^{-x \beta} e^{-y \gamma} \]

\[ \text{CDF} = 1 - e^{-x \beta} - e^{-y \gamma} + e^{-(x \beta + y \gamma)} \]

\[ \text{Expected value} = \frac{\beta}{\gamma} \]

\[ \text{Variance} = \frac{\beta^2}{\gamma^2} \]
\[ \text{Ch 3, V 1,} \quad 4x^2 = (\sqrt{x})^4 = (x^{1/2})^4 = x^2 \]

Define \( I_i = \begin{cases} -1, & \text{if } p = p_i \\ 1, & \text{if } p = 1 - p_i \end{cases} \)

\[ \mathbb{E}(I_i) = 1 - 2p \]

\[ \text{Var}(I_i) = 1 - (1 - 2p)^2 \]

\[ X_n = \sum_{i=1}^{n} I_i \]

\[ \mathbb{E}(X_n) = \sum_{i=1}^{n} \mathbb{E}(I_i) = n \cdot (-p + 1 - p) = 0(1 - 2p) \]

\[ \text{Var}(X_n) = \text{Var}(\sum_{i=1}^{n} I_i) = \sum_{i=1}^{n} \text{Var}(I_i) = n \left(1 - 1 - (2p)^2\right) \]

\[ = n \left(4p^2 - 4p^2\right) \]

\[ = 0 \]

\[ \text{With } X < \infty \]

\[ n \rightarrow \infty \]

\[ \frac{X}{n} \xrightarrow{p} \frac{X}{\infty} \]

\[ \mathbb{E}(X) = \frac{\mathbb{E}(X)}{\infty} = \frac{0}{\infty} = 0 \]

\[ \text{Var}(X) = \frac{\text{Var}(X)}{\infty} = \frac{0}{\infty} = 0 \]

\[ \text{With } X \sim \text{Poisson} (\lambda) \]

\[ \lambda = \mathbb{E}(X) = 0 \]

\[ \text{Var}(X) = \text{Var}(\lambda) = 0 \]

\[ \lambda = \frac{1}{m} \]

\[ \text{With } m = \text{constant} \]

\[ \lambda = \frac{1}{\text{constant}} = \frac{1}{\infty} = 0 \]

\[ \mathbb{E}(X) = \frac{\mathbb{E}(X)}{\infty} = \frac{0}{\infty} = 0 \]

\[ \text{Var}(X) = \frac{\text{Var}(X)}{\infty} = \frac{0}{\infty} = 0 \]