The Geostrophic Momentum Approximation and the Semi-Geostrophic Equations

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ABSTRACT

Consideration of flows in which the rate of change of momentum is much smaller than the Coriolis force suggests that the advected quantity momentum may be approximated by its geostrophic value but that trajectories cannot be so approximated. The resulting set of equations imply full forms of the equations for potential temperature, three-dimensional vorticity, potential vorticity and energy. A transformation of horizontal coordinates produces the "semi-geostrophic" system in which conservation of potential vorticity and potential temperature suffice to determine the motion. The system is capable of describing the formation of fronts, jets, and the growth of baroclinic waves into the nonlinear regime. It sheds some light on the success and failure of the quasi-geostrophic equations.

1. Introduction

The historical development of the equations that meteorologists use to study synoptic and large-scale systems may be traced from the barotropic vorticity equation through the quasi-geostrophic equations to the "primitive" equations. The latter set is used by most of those engaged in predictive integrations on the computer. However, the fact that these equations are so general leads to many complications in their application. It also means that the theoretician bent on understanding the development of the flow is unable to cope with their complexity. He is thus left with the quasi-geostrophic equations and the approximations inherent in them. In particular, nonlinearities in which the vertical component of relative vorticity is comparable with the Coriolis parameter are beyond his reach.

In this paper we are interested in atmospheric motions in which the Rossby number, defined as the ratio of the magnitudes of the rate of change of momentum and of the Coriolis force, is small. In symbols,

$$\mathrm{Ro}=\frac{\left|\dfrac{D\mathbf{v}}{Dt}\right|}{|f\mathbf{v}|}\ll 1.$$

This condition is much less restrictive than that obtained by assuming a velocity scale V, length scale L, and demanding

$$\mathrm{Ro}=\frac{\left|\dfrac{D\mathbf{v}}{Dt}\right|}{|f\mathbf{v}|}\sim\frac{V^2/L}{fV}=\frac{V}{fL}$$

be small. We can hope to describe the formation of jet streams and fronts, provided their curvature is not large in some sense, and also the growth of baroclinic waves into the nonlinear regime.

It will be shown that the small Rossby number condition suggests the geostrophic momentum approximation. This leads to a system of equations first suggested by Eliassen (1948) and used by Fjortoft (1962). This system and, more frequently, that of Charney (1962) have been described as the balance equations. They were introduced as one possible alternative for numerical integrations before the almost universal adoption of the primitive equations. The balance equations in common with the quasi-geostrophic equations do not describe gravity wave motion. They are, therefore, in principle, simpler for numerical or analytical work. Since less terms are ignored than in the latter equations, they are more general. One important quality of the equations with the geostrophic momentum approximation is the preservation of the form of the four "pseudo-conservation" relations for entropy, three-dimensional vorticity, Ertel's potential vorticity and energy. The equations are the natural extension to three dimensions of those used previously by the author (e.g., Hoskins and Bretherton, 1972) to examine the formation of straight fronts with trivial variation along them.

The equations are made much more amenable to analysis by a transformation of horizontal coordinates. These momentum or geostrophic coordinates are also the extension of those used previously. The horizontal ageostrophic velocities become implicit in the coordinate transformation. The time development of the fluid motion is determined by conservation of potential vorticity and potential temperature in the interior and of the latter on horizontal boundaries. We refer to this set of equations as the semi-geostrophic equations. Of particular interest is the case of uniform potential vorticity since the semi-geostrophic equations then simplify and provide much insight into the success and failure of the usual quasi-geostrophic equations. In a subsequent paper, integrations of the semi-geostrophic equations will be described.

2. The primitive equations and the quasi-geostrophic approximation

To keep the discussion as simple as possible, we consider the primitive equations in their hydrostatic, Boussinesq form:

$$\frac{Du}{Dt}-fv+\frac{\partial\phi}{\partial x}=0, \tag{1}$$

$$\frac{Dv}{Dt}+fu+\frac{\partial\phi}{\partial y}=0, \tag{2}$$

$$g\frac{\theta}{\theta_0}=\frac{\partial\phi}{\partial z},$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

$$\frac{D\theta}{Dt}=0.$$

Here the time derivative following a fluid particle is

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}.$$

The Coriolis parameter f is taken as constant. As boundary conditions we take

$$w=0 \text{ on } z=0,\ H.$$

We refer to z as height, though a more consistent derivation of the above equations is obtained by taking z proportional to p^κ (see Hoskins and Bretherton, 1972). Other symbols are the geopotential ϕ, the potential temperature θ, and a constant reference potential temperature θ_0.

The above set of equations provides four important pseudo-conservation relations:

(i) Conservation of potential temperature

$$D\theta/Dt=0$$

(ii) Three-dimensional vorticity equation

$$\frac{D\boldsymbol{\zeta}}{Dt}=(\boldsymbol{\zeta}\cdot\nabla)\mathbf{u}-\mathbf{k}\times\frac{g}{\theta_0}\nabla\theta,$$

where

$$\boldsymbol{\zeta}=\left(-\frac{\partial v}{\partial z},\ \frac{\partial u}{\partial z},\ f+\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

(iii) Potential vorticity conservation

$$\frac{Dq}{Dt}=0,$$

where

$$q=\boldsymbol{\zeta}\cdot\nabla\theta$$

(iv) Energy equation

$$\frac{D}{Dt}(K+P)=-\nabla\cdot(\mathbf{u}\phi),$$

where

$$K=\tfrac{1}{2}(u^2+v^2),$$

$$P=-\frac{g}{\theta_0}z\theta.$$

As stated in the Introduction, in the usual geostrophic and quasi-geostrophic theory, scales for velocity V and length L are postulated. The Rossby number

$$\mathrm{Ro}=V/(fL)$$

provides a measure of the ratio of the inertia terms to the Coriolis terms in the momentum equations. Taking

$$V=10\ \mathrm{m\ s^{-1}},\quad L=1000\ \mathrm{km},\quad f=10^{-4}\ \mathrm{s^{-1}},$$

gives Ro=0.1, so that an expansion in Rossby number is possible. The zero-order approximation gives the geostrophic relations

$$u=u_g=-\frac{1}{f}\frac{\partial\phi}{\partial y},\quad v=v_g=\frac{1}{f}\frac{\partial\phi}{\partial x}.$$

Combined with the hydrostatic relation these give the thermal wind relations

$$f\frac{\partial u_g}{\partial z}=-\frac{g}{\theta_0}\frac{\partial\theta}{\partial y},\quad f\frac{\partial v_g}{\partial z}=\frac{g}{\theta_0}\frac{\partial\theta}{\partial x}.$$

The first-order equations yield an approximation to the vertical component of the vorticity equation:

$$\left(\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right)\zeta_g=f\frac{\partial w}{\partial z},$$

where

$$\zeta_g=f+\frac{\partial v_g}{\partial x}-\frac{\partial u_g}{\partial y}. \tag{3}$$

The nonlinear stretching and twisting terms are neglected. The quasi-geostrophic system is completed by approximating the static stability as a function of z

only in the thermodynamic equation:

$$\left(\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right)\theta=-w\frac{\theta_0 N^2}{g},$$

where

$$N^2(z)=\frac{g}{\theta_0}\frac{d\Theta}{dz}. \qquad (4)$$

Combining (3) and (4) gives the quasi-potential vorticity equation

$$\left(\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right)\left[\phi_{xx}+\phi_{yy}+\left(\frac{f^2}{N^2}\phi_z\right)_z\right]=0. \qquad (5)$$

Integration by parts gives for a horizontally periodic domain

$$K+P=\text{constant},$$

where

$$K=\int\tfrac{1}{2}(u_g^2+v_g^2)dV,\quad P=\int\frac{1}{N^2}\frac{g^2}{\theta_0^2}[\theta-\Theta(z)]^2dV.$$

In the usual atmospheric situation of equatorward temperature gradient, potential energy is released by moving heat poleward, rather than upward as in the primitive equations.

3. The geostrophic momentum approximation

Quasi-geostrophic theory has been used with a fair measure of success in situations in which its validity is not clear. Cross sections of averaged zonal velocities show a jet stream with winds in excess of 30 m s^{-1}. Large shears are commonplace. The typical cyclone contains vorticities comparable with the Coriolis parameter, particularly in frontal regions, yet quasi-geostrophic models such as that of Phillips (1956) produce remarkably realistic results.

We now proceed to develop a theory based on the smallness of the Rossby number. We do not assume overall velocity and length scales but do take account of the fact that at each point in the atmosphere, the flow is in a definite direction. We use a rectangular Cartesian coordinate system with the ξ axis in the direction of the horizontal projection of the direction of flow, and the η axis perpendicular to it. If χ is the angle the ξ axis makes with some fixed direction and V the velocity, then

$$\frac{D\mathbf{v}}{Dt}=\left(\frac{DV}{Dt},\ -V\frac{D\chi}{Dt},\ 0\right).$$

Then the small Rossby number implies

$$\frac{DV}{Dt}\ll fV, \qquad (6)$$

$$\frac{D\chi}{Dt}\ll f. \qquad (7)$$

The magnitude and direction of the momentum of a fluid particle changes little in the time $1/f\approx 3$ h. If r is the radius of curvature of fluid particle paths, the latter condition is

$$\frac{V}{fr}\ll 1. \qquad (7')$$

The momentum equations in this coordinate system are

$$\frac{DV}{Dt}+\frac{\partial\phi}{\partial\xi}=0,$$

$$-V\frac{D\chi}{Dt}+fV+\frac{\partial\phi}{\partial\eta}=0.$$

The small Rossby number then implies

$$\frac{\partial\phi}{\partial\eta}\approx fV\gg\frac{DV}{Dt}=-\frac{\partial\phi}{\partial\xi}. \qquad (8)$$

Thus the momentum vector $(V,0)$ is approximately equal to the "geostrophic momentum" vector $(-f^{-1}\phi_\eta,\ f^{-1}\phi_\xi)$ in magnitude and direction.[1] Thus in this paper we approximate the momentum by the geostrophic momentum. This does not imply that we should approximate the horizontal advecting velocity by its geostrophic value. In the presence of large cross-stream gradients in momentum or temperature, the small angles between the directions of particle motion and the geopotential surfaces could be crucial. This treatment of horizontal ageostrophic motions is entirely analogous to the hydrostatic approximation in which the vertical component of momentum is neglected but vertical advection is retained.

An alternative but equivalent consideration of the geostrophic momentum approximation may be obtained by substituting for the momentum in (1) and (2) from the Coriolis terms in (2) and (1). The equations may then be rewritten

$$v=v_g+\mathfrak{D}u_g-\mathfrak{D}^2v,$$

$$u=u_g-\mathfrak{D}v_g-\mathfrak{D}^2u,$$

where

$$\mathfrak{D}=\frac{1}{f}\frac{D}{Dt}.$$

The geostrophic momentum approximation is obtained if the last terms in the two equations are neglected. This is clearly valid if

$$\mathfrak{D}^2u\ll u,\quad \mathfrak{D}^2v\ll v. \qquad (9)$$

It should be noted that the second derivatives of each

[1] Shapiro (1970) using similar considerations found this to be true in a case study of an upper level front.

component of the velocity is neglected compared with the *same* component of velocity. In the flows considered here, it is quite possible that, for example,

$$\mathfrak{D}^2 v \gg u.$$

From the above equations

$$v = v_g + \mathfrak{D}u_g - \mathfrak{D}^2(v_g + \mathfrak{D}u_g) + \mathfrak{D}^4(v_g + \mathfrak{D}u_g) - \ldots,$$
$$u = u_g - \mathfrak{D}v_g - \mathfrak{D}^2(u_g - \mathfrak{D}v_g) + \mathfrak{D}^4(u_g - \mathfrak{D}v_g) - \ldots.$$

Higher order approximations may be obtained from different truncations of these series.

The full equations with the geostrophic momentum approximation are

$$\left.\begin{aligned} &\frac{Du_g}{Dt} - fv + \frac{\partial\phi}{\partial x} = 0 \\ &\frac{Dv_g}{Dt} + fu + \frac{\partial\phi}{\partial y} = 0 \\ &\frac{g\theta}{\theta_0} = \frac{\partial\phi}{\partial z} \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \\ &\frac{D\theta}{Dt} = 0 \end{aligned}\right\}, \tag{10}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z},$$

$$fu_g = -\frac{\partial\phi}{\partial y}, \quad fv_g = \frac{\partial\phi}{\partial x}.$$

If a stationary front is forming approximately parallel to geopotential lines, then the above equations reduce to the system suggested by a scaling analysis and used in Hoskins and Bretherton (1972). The set (10) was produced by the author as the natural extension to three dimensions of the one used in the frontogenesis work. It was later found to have been previously introduced by Eliassen (1948) and Fjortoft (1962)[2]. They form a balanced system in that they cannot describe gravity wave propagation. They can, however, describe regions of large shear vorticity such as jets or fronts in an arbitrary direction, provided that the curvature vorticity is small compared with the Coriolis parameter. In contrast, the usual balance equations described by Charney (1962) include a full representation of curvature effects, but are not formally valid in regions of large shear or curvature vorticity. It is shown in the Appendix that, for the case of steady circular motion, the geostrophic momentum approximation is very accurate when the curvature vorticity is small.

[2] Fjortoft also mentioned the condition (9) and a form of the three-dimensional vorticity equivalent to that given below in (ii).

One of the strong advantages and indications of the consistency of the geostrophic momentum approximation is the retention of the four important pseudo-conservation relations derived from the primitive equations in only slightly modified form:

(i) Conservation of potential temperature

$$\frac{D\theta}{Dt} = 0$$

(ii) Three-dimensional vorticity equation

$$\frac{D\boldsymbol{\zeta}_g}{Dt} = (\boldsymbol{\zeta}_g \cdot \nabla)\mathbf{u} - \mathbf{k} \times \frac{g}{\theta_0}\nabla\theta,$$

where

$$\boldsymbol{\zeta}_g = \left(-\frac{\partial v_g}{\partial z}, \frac{\partial u_g}{\partial z}, f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right) + \left(\frac{1}{f}\frac{\partial(u_g,v_g)}{\partial(y,z)}, \frac{1}{f}\frac{\partial(u_g,v_g)}{\partial(z,x)}, \frac{1}{f}\frac{\partial(u_g,v_g)}{\partial(x,y)}\right)$$

(iii) Potential vorticity conservation

$$\frac{Dq_g}{Dt} = 0,$$

where

$$q_g = \boldsymbol{\zeta}_g \cdot \nabla\theta$$

(iv) Energy equation

$$\frac{D}{Dt}(K_g + P) = 0,$$

where

$$K_g = \tfrac{1}{2}(u_g^2 + v_g^2),$$

$$P = -\frac{g}{\theta_0}z\theta.$$

Apart from the additional Jacobian term in the definition of vorticity, these relations are identical with those for the primitive equations except that wherever the velocity occurs in the quantity which is advected, it is replaced by its geostrophic value. The approximation could also, therefore, be referred to as the geostrophic potential vorticity approximation, or the geostrophic kinetic energy approximation. The extra term in the definition of vorticity is necessary for mathematical consistency. However, in the flows of interest in this paper it is only a small correction. This may be seen by taking the x axis parallel to the local geostrophic velocity vector. Thus, locally $v_g=0$, and

$$\boldsymbol{\zeta}_g=\left(-\frac{\partial v_g}{\partial z},\frac{\partial u_g}{\partial z},f+\frac{\partial v_g}{\partial x}-\frac{\partial u_g}{\partial y}\right)+\left(-\frac{\alpha}{f}\frac{\partial u_g}{\partial z},-\frac{\alpha}{f}\frac{\partial v_g}{\partial z},-\frac{\alpha^2}{f}\right)+\left(\frac{1}{f}\frac{\partial v_g}{\partial z}\frac{\partial u_g}{\partial y},\frac{1}{f}\frac{\partial u_g}{\partial z}\frac{\partial v_g}{\partial x},-\frac{1}{f}\frac{\partial u_g}{\partial y}\frac{\partial v_g}{\partial x}\right),$$

where

$$\alpha=\frac{\partial v_g}{\partial y}=-\frac{\partial u_g}{\partial x}.$$

The last two terms are small consistent with the geostrophic momentum approximation and the implied conditions:

1) Convergence of the geostrophic wind field (α) small compared with f.
2) Where the horizontal components of vorticity are not negligible (in their contribution to the potential vorticity) then $|\partial u_g/\partial z|\gg|\partial v_g/\partial z|$, i.e., in regions of large gradients, θ lines are nearly parallel to the direction of flow.
3) $\partial v_g/\partial x\ll f$ [from (7′)].

Despite the simple and consistent nature of the primitive equations including the geostrophic momentum approximation, and the beauty of the conservation relations derived from them, they are not amenable for use in the form (10). Prognostic equations for u_g and v_g cannot be solved separately. The ageostrophic motion is only implied in the equations. We now show how progress may be made using a coordinate transformation and the conservation of potential temperature and potential vorticity.

4. A transformation of coordinates

We use as independent variables in the x and y directions

$$X=x+v_g/f,\quad Y=y-u_g/f. \tag{11}$$

Since

$$\frac{DX}{Dt}=u+\frac{1}{f}\frac{Dv_g}{Dt}=u_g,\quad \frac{DY}{Dt}=v-\frac{1}{f}\frac{Du_g}{Dt}=v_g, \tag{12}$$

these may be referred to as geostrophic coordinates. They are the positions particles would have had if they had moved with their geostrophic velocity at every instant. For convenience, when using X and Y as independent variables, we use

$$Z=z,\quad T=t.$$

We also set

$$a=\frac{1}{f^2}\phi_{xx},\quad b=\frac{1}{f^2}\phi_{xy},\quad c=\frac{1}{f^2}\phi_{yy},\quad \alpha=\frac{1}{f^2}\phi_{xz},\quad \beta=\frac{1}{f^2}\phi_{yz}.$$

Then

$$\left.\begin{aligned}\frac{\partial}{\partial x}&=(1+a)\frac{\partial}{\partial X}+b\frac{\partial}{\partial Y}\\ \frac{\partial}{\partial y}&=b\frac{\partial}{\partial X}+(1+c)\frac{\partial}{\partial Y}\\ \frac{\partial}{\partial z}&=\alpha\frac{\partial}{\partial X}+\beta\frac{\partial}{\partial Y}+\frac{\partial}{\partial Z}\end{aligned}\right\}.$$

The Jacobian of the transformation is the nondimensional vertical component of absolute vorticity:

$$J=(1+a)(1+c)-b^2=\mathbf{k}\cdot\boldsymbol{\zeta}_g/f. \tag{13}$$

The inverse transformation is

$$J\frac{\partial}{\partial X}=(1+c)\frac{\partial}{\partial x}-b\frac{\partial}{\partial y}, \tag{14}$$

$$J\frac{\partial}{\partial Y}=-b\frac{\partial}{\partial x}+(1+a)\frac{\partial}{\partial y}, \tag{15}$$

$$\begin{aligned}J\frac{\partial}{\partial Z}&=-[\alpha(1+c)-\beta b]\frac{\partial}{\partial x}-[\beta(1+a)-\alpha b]\frac{\partial}{\partial y}+J\frac{\partial}{\partial z},\\ &=\frac{1}{f}\boldsymbol{\zeta}_g\cdot\nabla.\end{aligned} \tag{16}$$

If we now define

$$\Phi=\phi+\tfrac{1}{2}(u_g^2+v_g^2), \tag{17}$$

it is easily verified that

$$\left(\frac{\partial\Phi}{\partial X},\frac{\partial\Phi}{\partial Y},\frac{\partial\Phi}{\partial Z}\right)=\left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z}\right). \tag{18}$$

Thus the geostrophic velocity and potential temperature may be represented in terms of one function in the transformed coordinates, just as in physical coordinates.

Direct application of (14) and (15) gives

$$\left.\begin{aligned}J\left(1-\frac{1}{f}\frac{\partial v_g}{\partial X}\right)&=1+c,\\ \frac{J}{f}\frac{\partial v_g}{\partial Y}&=-\frac{J}{f}\frac{\partial u_g}{\partial X}=b\\ J\left(1+\frac{1}{f}\frac{\partial u_g}{\partial Y}\right)&=1+a\end{aligned}\right\}. \tag{19}$$

The product of the first and third equations minus the square of the second gives

$$J^2\left[\left(1-\frac{1}{f^2}\Phi_{XX}\right)\left(1-\frac{1}{f^2}\Phi_{YY}\right)-\frac{1}{f^4}\Phi_{XY}{}^2\right]=J,$$

so that

$$J^{-1}=1-\frac{1}{f^2}(\Phi_{XX}+\Phi_{YY})+\frac{1}{f^4}(\Phi_{XX}\Phi_{YY}-\Phi_{XY}{}^2). \quad (20)$$

Finally, we note that the potential vorticity is

$$q_g=\zeta_g\cdot\nabla\theta=fJ\frac{\partial\theta}{\partial Z}. \quad (21)$$

Following a fluid particle the time derivative is

$$\frac{D}{DT}=\mathfrak{D}+w\frac{\partial}{\partial Z}, \quad (22)$$

where from (12),

$$\left.\begin{aligned}\mathfrak{D}&=\frac{\partial}{\partial T}+u_g\frac{\partial}{\partial X}+v_g\frac{\partial}{\partial Y}\\ u_g&=-\frac{1}{f}\Phi_Y,\quad v_g=\frac{1}{f}\Phi_X\end{aligned}\right\}.$$

Conservation of potential temperature is

$$\left(\mathfrak{D}+w\frac{\partial}{\partial Z}\right)\theta=0, \quad (23)$$

where

$$\frac{g}{\theta_0}\theta=\Phi_Z.$$

Conservation of potential vorticity is

$$\left(\mathfrak{D}+w\frac{\partial}{\partial Z}\right)q_g=0, \quad (24)$$

where from (18), (20) and (21),

$$\frac{1}{f^2}(\Phi_{XX}+\Phi_{YY})-\frac{1}{f^4}(\Phi_{XX}\Phi_{YY}-\Phi_{XY}{}^2)+\frac{f\theta_0}{gq_g}\Phi_{ZZ}=1. \quad (25)$$

On horizontal boundaries,

$$\mathfrak{D}\theta=0. \quad (26)$$

The horizontal ageostrophic velocities are implicit in the coordinate transformation. Vertical advection remains in the equations but the vertical velocity is not determined explicitly. It is implied by the advection of potential temperature and potential vorticity, both of which are determined by one function. Given a solution in transformed space, the position of fluid particles in physical space may be obtained from the inverse coordinate transformation. The geostrophic velocity and potential temperature are known on these fluid particles. The actual velocities may be obtained from identification of the same fluid particle at different instants. It is clear that the whole analysis is an extension of that given in Hoskins and Bretherton (1972). The equations there are obtained if Y derivatives are neglected in the above equations.

In this paper we shall refer to the above set of equations (22)–(26) as the semi-geostrophic equations. Since no further approximation has been introduced, they are formally equivalent to the primitive equations with the geostrophic momentum approximation (10).

If the potential vorticity is uniform at some instant, it remains so at all times:

$$q_g=\frac{f\theta_0}{g}N^2,$$

where N is the Brunt-Väisälä frequency in the fluid if no velocity gradients are present. In the interior of the fluid for all time, (24) gives

$$\frac{1}{f^2}(\Phi_{XX}+\Phi_{YY})+\frac{1}{N^2}\Phi_{ZZ}-\frac{1}{f^4}(\Phi_{XX}\Phi_{YY}-\Phi_{XY}{}^2)=1. \quad (27)$$

The time development is forced by the advection of potential temperature on horizontal boundaries:

$$\left(\frac{\partial}{\partial T}-\frac{1}{f}\frac{\partial\Phi}{\partial Y}\frac{\partial}{\partial X}+\frac{1}{f}\frac{\partial\Phi}{\partial X}\frac{\partial}{\partial Y}\right)\frac{\partial\Phi}{\partial Z}=0,\quad \text{on } Z=0,H. \quad (28)$$

The potential temperature equation may be used in the interior to determine the vertical motion of particles, though this is not necessary for the solution of the problem.

A possible numerical scheme for integration of the semi-geostrophic equations when the potential vorticity is not uniform is sketched below (the potential temperature equation in the interior is used as a diagnostic equation for w; the superfix refers to the time step):

$$q_g^{n+1}=q_g^{n-1}+2\Delta t\left(-u_g\frac{\partial q_g}{\partial X}-v_g\frac{\partial q_g}{\partial Y}-w\frac{\partial q_g}{\partial Z}\right)^n$$

in the interior,

$$\theta^{n+1}=\theta^{n-1}+2\Delta t\left(-u_g\frac{\partial\theta}{\partial X}-v_g\frac{\partial\theta}{\partial Y}-w\frac{\partial\theta}{\partial Z}\right)^n$$

on horizontal boundaries.

With $\Phi_Z^{n+1}=g\theta^{n+1}/\theta_0$ on the boundaries, we solve (25) for Φ^{n+1}. This determines u_g^{n+1}, v_g^{n+1} and θ^{n+1}; w^{n+1} is determined from

$$\theta^{n+1}-\theta^n=\frac{\Delta t}{2}\left[\left(-u_g\frac{\partial\theta}{\partial X}-v_g\frac{\partial\theta}{\partial Y}-w\frac{\partial\theta}{\partial Z}\right)^n+\left(-u_g\frac{\partial\theta}{\partial X}-v_g\frac{\partial\theta}{\partial Y}-w\frac{\partial\theta}{\partial Z}\right)^{n+1}\right].$$

The transformation to physical space is

$$x^{n+1}=X^{n+1}-v_g^{n+1}/f,\quad y^{n+1}=Y^{n+1}+u_g^{n+1}/f.$$

5. Comparison with the quasi-geostrophic equations and simple deductions concerning nonlinear baroclinic waves

The semi-geostrophic equations include the advection of an approximation to the full potential vorticity, as opposed to the quasi-potential vorticity advected in the quasi-geostrophic equations. Ageostrophic advection of potential vorticity and potential temperature is included in the former system. In quasi-geostrophic theory the only ageostrophic advection is by the vertical velocity where it acts on a standard vertical temperature gradient.

From this point on, we simplify the comparison by considering only the uniform potential vorticity case. The quasi-geostrophic equations would be identical with the semi-geostrophic equations (27) and (28) except that Φ, X, Y and Z would be replaced by ϕ, x, y and z, and the nonlinear term in (26) would not appear. From (19), we have

$$J\left[1-\frac{1}{f^2}(\Phi_{XX}+\Phi_{YY})\right]=1-(ac-b^2).$$

Thus the nonlinear term is negligible if the correction to the geostrophic vorticity discussed in Section 3 is negligible.

The more important difference is that the geostrophic velocities and potential temperature are predicted at (X,Y,Z) not (x,y,z). From the nature of the coordinate transformation it is easily seen (e.g., Fig. 1) that positive relative vorticity is increased and the region in which it occurs is decreased. Negative relative vorticity is decreased in magnitude and the region in which it occurs is increased. Thus the semi-geostrophic theory allows the production of sharp fronts, small vigorous low pressure cells, and broad weak high pressure cells. This clearly depends on the inclusion of advection by the convergent or divergent wind field and the nonlinearity in the stretching of vorticity. Using the semi-geostrophic equations, systems that are vertical using quasi-geostrophic theory tend to orient themselves along absolute vortex lines [from (16)]. This was commented on by Fjortoft. This is exactly the sloping of frontal regions found in the frontal studies.

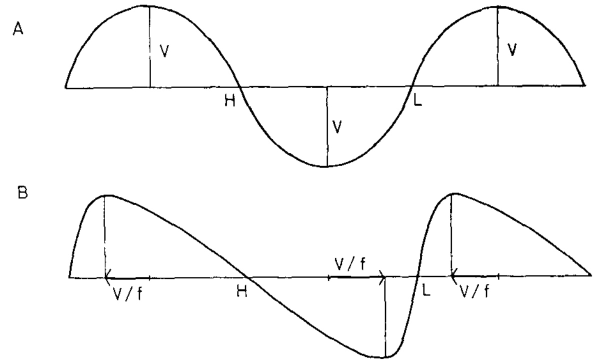

FIG. 1. A y-velocity pattern that is sinusoidal on quasi-geostrophic theory, or on semi-geostrophic theory in X space, is represented in A. In B is shown the physical space velocity pattern. The anticyclonic center H is broadened and weakened. The cyclonic center L is tightened and strengthened.

Another property of nonlinear baroclinic waves as described by the semi-geostrophic equations may be simply inferred. The phases of the temperature and pressure waves as given by quasi-geostrophic theory are always such that near the surface, the temperature perturbation maximum occurs in the cyclonic region and the minimum in the anticyclonic region. Thus the semi-geostrophic equations imply that the area of warm anomaly is decreased and that of cold anomaly is increased. Higher up in the atmosphere the reverse is true. This is clearly the occlusion process in which warm air is moved upward, thus releasing potential energy. As remarked previously, in quasi-geostrophic theory, potential energy is released by moving warm air poleward and the occlusion process is not described.

Despite the much less stringent approximations made in the derivation of the semi-geostrophic equations, they predict merely a distortion of the quasi-geostrophic solution in a range of parameter space in which the derivation of the latter is not consistent. This may go some way to explaining the point commented on earlier: that the quasi-geostrophic equations have been successfully used in situations in which their validity is not clear.

6. The Eady wave solution of the semi-geostrophic equations

We now consider perturbations to a basic state of uniform potential vorticity in which the zonal wind increases linearly in height associated with a linear north-south temparature gradient:

$$u_g=\frac{Uz}{H},\quad \theta=\theta_0+\frac{N^2\theta_0}{g}z-\frac{f\theta_0}{g}\frac{U}{H}y. \tag{29}$$

The exponentially growing quasi-geostrophic solutions are the classic Eady (1947) baroclinic waves. We use the semi-geostrophic equations (27) and (28). The basic state corresponding to (29) is

$$\Phi=gZ+\frac{N^2Z^2}{2}-\frac{fU}{H}YZ, \tag{30}$$

where

$$\left.\begin{aligned} X&=x\\ Y&=y-\frac{Uz}{fH}\end{aligned}\right\}.$$

The linearized equations for perturbations Φ' are

$$\frac{1}{f^2}(\Phi_{XX}'+\Phi_{YY}')+\frac{1}{N^2}\Phi_{ZZ}'=0,$$

with

$$\left(\frac{\partial}{\partial T}+\frac{UZ}{H}\frac{\partial}{\partial X}\right)\Phi_Z'-\frac{U}{H}\Phi_X'=0.$$

The Eady wave solutions are

$$\Phi'=e^{\sigma T^*}(a\cosh mZ^*\cos kX^* +b\sinh mZ^*\sin kX^*)\sin lY^*, \tag{31}$$

where

$$\left.\begin{aligned} &T^*=\frac{Uf}{NH}T,\quad X^*=\left(X-\frac{U}{2}T\right)\frac{f}{NH},\quad Y^*=\frac{f}{NH}Y\\ &Z^*=\frac{Z-H/2}{H},\quad m^2=k^2+l^2\end{aligned}\right\}, \tag{32}$$

$$\sigma\frac{a}{b}=\frac{k}{m}\left(1-\frac{m}{2}\coth\frac{m}{2}\right),\quad \sigma\frac{b}{a}=\frac{k}{m}\left(\frac{m}{2}\tanh\frac{m}{2}-1\right). \tag{33}$$

The growth rates are unchanged from quasi-geostrophic theory. The pressure pattern, geostrophic velocities and potential temperature are also identical except that they are distributed along lines $Y=$ constant, i.e.,

$$y-\frac{Uz}{fH}=\text{constant}.$$

These lines are parallel to the absolute vorticity vector of the basic state. They slope upward and northward at an angle $\tan^{-1}(U/fH)$ with the vertical. Taking

$$U\approx 30\text{ m s}^{-1},\quad H\approx 10\text{ km},\quad f\approx 10^{-4}\text{ s}^{-1},$$

this angle is approximately 88°. The lines are displaced 300 km in the northward direction from surface to lid.

To obtain the value of v, we note that

$$v=v_g+\frac{1}{f}\frac{Du_g}{Dt}.$$

If we denote

$$\frac{1}{NH}e^{\sigma T^*}(a\cosh mZ^*\cos kX^*+b\sinh mZ^*\sin kX^*).$$

by $\hat{\Phi}$, then from (31)

$$v=\hat{\Phi}_{X^*}\sin lY^*-\frac{UZ^*}{NH}l\hat{\Phi}_{X^*}\cos lY^*-\frac{U\sigma}{NH}l\hat{\Phi}\cos lY^*,$$

$$=\hat{\Phi}_{X^*}\left[\sin l\left(Y^*-\frac{UZ^*}{NH}\right)+O\left(\frac{U^2l^2}{4N^2H^2}\right)\right]-\frac{Ul}{NH}\sigma\hat{\Phi}\cos lY^*.$$

Since the maximum value of σ is $0.219(1-l^2/m^2)^{\frac{1}{2}}$, the last term is a very small correction which is 90° out of phase with the dominant term in both directions. Assuming that $Ul/(2NH)$ is small, the first correction term combines with the dominant term to produce a tilt in the northward direction. Using nondimensional variables, the v pattern tilts a small angle U/NH from the v_g pattern and Y^* lines. But the v_g pattern and Y^* lines themselves tilt an angle U/NH from the vertical in nondimensional physical space. Thus the v pattern tilts by $2U/NH$. In dimensional space, the tilt is $\tan^{-1}(2U/fH)$. This result was suggested by the work of McIntyre (1965) and Derome and Dolph (1970) using expansions in Rossby numbers to an order beyond quasi-geostrophic theory.

7. Conclusion

We have been investigating flows in which the magnitude of the rate of change of momentum vector is small compared with the magnitude of the Coriolis force. Reference to coordinates parallel to the direction of flow suggested that the momentum could be approximated by its geostrophic value. Trajectories may not be approximated in such a manner without further assumptions. The equations with the geostrophic momentum approximation preserve the four pseudo-conservation relations for potential temperature, three-dimensional vorticity, potential vorticity and energy in only slightly modified form. A transformation of horizontal coordinates produces the semi-geostrophic equations. The fluid motion is specified by conservation of potential temperature and potential vorticity. The horizontal ageostrophic velocities are implicit in the coordinate transformation.

The semi-geostrophic equations fill a position between the primitive equations and the quasi-geostrophic equations. In the general non-uniform potential vorticity case, their solution is scarcely more complicated than that of the latter system. Unlike the quasi-geostrophic equations or the usual balance equations (which have always proved difficult to integrate), they are able to describe the formation of regions in which the vorticity is large but particle accelerations are small, e.g., fronts and jet streams. Unlike the primitive equations, their simplicity allows dynamical insight into these nonlinear phenomena. This is particularly true when the potential vorticity is uniform. In a subsequent paper, examples will be given of the nonlinear development of baroclinic waves, including the formation of realistic warm and cold fronts and the occlusion process.

In this paper, effects due to the variation of the Coriolis parameter with latitude have been ignored. For systems of meridional scale much smaller than the radius of the earth, to the same accuracy as the β-plane approximation in quasi-geostrophic theory, it would appear that the variation of f should be included only where it occurs explicitly in the potential vorticity equation (21) and, equivalently, in the last term on the left-hand side of (25). The method of inclusion of physical processes such as precipitation and boundary layer processes is not straightforward. It may be possible to revert to the primitive equation form (10) and include them in a viable numerical procedure. These topics deserve further study.

There is clearly some similarity with the equations used by Bleck (1973a). He approximated potential vorticity by its geostrophic value and used coordinates which stretch baroclinic zones in the vertical. He advected the potential vorticity with the geostrophic velocity. In Section 3 we showed that the geostrophic momentum approximation leads to a form of potential vorticity almost identical with that obtained by inserting geostrophic values everywhere. In this paper we stretch baroclinic zones in the horizontal using (X,Y) coordinates. This has the advantage of giving small gradients on the boundaries also and simple conditions to be imposed there. In a later version, to achieve more realistic results, Bleck (1973b) estimated the ageostrophic velocities and advected with these also. In this form, his equations are almost equivalent to ours. The success of his model, particularly in predicting cases of strong development, suggests the power and validity of the equations when applied to real situations, even when no more "physics" is included.

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APPENDIX

Steady Circular Flow

The horizontal equations of motion for steady circular flow $v(r)$ reduce to the gradient wind formula

$$\frac{v^2}{r}+fv=fv_g.$$

Introducing two Rossby numbers

$$R=\frac{v}{fr},\quad R_g=\frac{v_g}{fr},$$

the gradient wind formula may be written in the non-

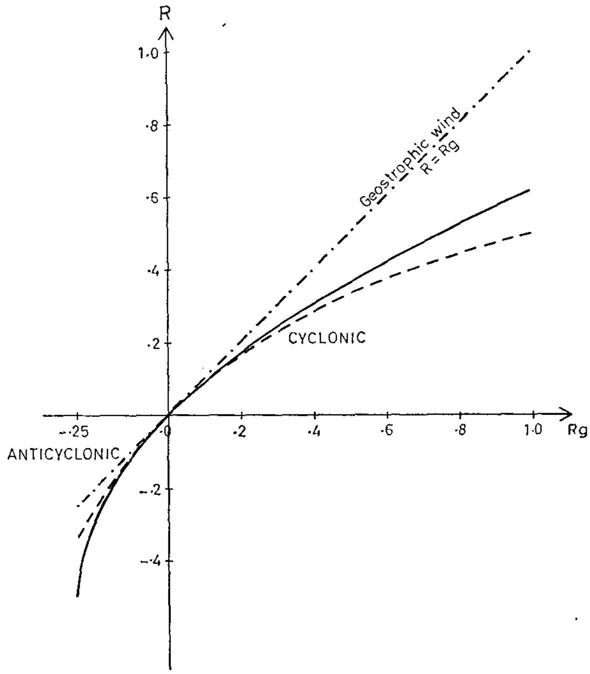

FIG. 2. A comparison of the geostrophic momentum approximation (dashed line) with the gradient wind solution (continuous line) for steady circular motion: $R=v/fr$, $R_g=v_g/fr$.

dimensional form

$$R^2+R-R_g=0,$$

so that

$$R=-0.5+(R_g+0.25)^{\frac{1}{2}}.$$

The geostrophic momentum approximation is

$$\frac{vv_g}{r}+fv=fv_g,$$

so that

$$R=\frac{R_g}{1+R_g}.$$

Fig. 2 shows the accuracy of the approximation (previously commented on by Fjortoft). The error in R is less than 10% for a surprisingly large range, $-0.2<R_g<0.55$.

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