Easy bounds for \( n! \)

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**Theorem 1.1**

\[ n^n \leq e^n n! \leq (n + 1)^{n+1} \]

We can derive this inequality from the simpler

\[ x \leq \lceil x \rceil \leq x + 1 \quad (*) \]

using the idea of a product integral.

Define \( \pi \int \) to be \( \exp \int \log f \) for any positive function \( f : \mathbb{R} \to \mathbb{R} \). We can think of \( \pi \int \) as a way of multiplying the values of \( f(x) \) over a range in contrast to the standard \( \int \), which adds these values.

For any positive sequence \( (a_k)_{k=1}^n \), we can see that

\[
\pi \int_0^n a_{\lceil x \rceil} dx = \pi \int_0^1 a_1 \cdot \pi \int_1^2 a_2 \cdots \pi \int_{n-1}^n a_n = \prod_{k=1}^n a_k,
\]

so that \( \pi \int_0^n \lceil x \rceil dx = n! \) It is clear from the definition of \( \pi \int \) that

\[
\pi \int_0^y x dx = \left( \frac{y}{e} \right)^y.
\]

We may extend one version of the fundamental theorem of calculus to its product analog: if \( \partial F = f \), then

\[
\pi \int_a^b f = F(b) \div F(a),
\]

where we define the product derivative \( \partial F = \exp \partial \log F \). If \( a = g(c) \) and \( b = g(d) \), then

\[
\pi \int_a^b f(x) dx = \pi \int_c^d [f \circ g(y)]^{\partial g(y)} dy
\]

represents the substitution \( x = g(y) \). Now we can see that

\[
\pi \int_0^y (x + 1) dx = \pi \int_1^{y+1} x dx = \left( \frac{y + 1}{e} \right)^{y+1} \div \left( \frac{1}{e} \right)^1 = \frac{(y + 1)^{y+1}}{e^y}
\]

To prove the theorem, just take \( \pi \int_0^n dx \) of \( (*) \) and multiply through by \( e^n \).  

\[ \square \]