Large simplices determined by finite point sets

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Abstract

Given a set P of n points in $\mathbb{R}^d$, let $d_1 > d_2 > \dots$ denote all distinct inter-point distances generated by point pairs in P. It was shown by Schur, Martini, Perles, and Kupitz that there is at most *one* d-dimensional regular simplex of edge length d_1 whose every vertex belongs to P. We extend this result by showing that for any k the number of d-dimensional regular simplices of edge length d_k generated by the points of P is bounded from above by a constant that depends only on d and k.

1 Introduction

The investigation of various properties of graphs of distances generated by a finite set of points in Euclidean space was initiated by Erdős in 1946, and it has become a classical topic in discrete and computational geometry, with applications in combinatorial number theory, the theory of geometric algorithms, pattern recognition, etc. Among the problems that have drawn a lot of attention for decades are: Erdős's problem on unit distances [4, 18], Erdős's problem on distinct distances [4, 10], Borsuk's conjecture on the chromatic number of diameter graphs [2, 13], the Hadwiger-Nelson coloring problem [11]. Consult [3] for many other problems of this kind.

In the present paper, we concentrate on graphs of large distances. Given a set P of n points in $\mathbb{R}^d$, consider all $\binom{n}{2}$ distances between pairs of points in P. The same distance may occur several times. Throughout this paper, we will use the convention that the sequence of *distinct* distances in decreasing order will be denoted by $d_1 > d_2 > \dots$. In other words, d_k is the *k-th largest distance* generated by P. For a fixed k, we can study the *graph of k-th largest distances* generated by P. The vertex set of this graph is P, and two vertices are connected by an edge if and only if their distance is d_k. The most frequently studied and perhaps most interesting case is $k = 1$, when we have a *graph of diameters*. One of the basic results concerning graphs of diameters was obtained by Hopf and Pannwitz in 1934 [8]: the maximum number of diameters among n points in the plane is n. Later a similar result was conjectured by Vázsonyi and proved by Grünbaum [9], Heppes [12], and Straszewicz [19]: the maximum number of diameters generated by n points in $\mathbb{R}^3$ is $2n - 2$. In higher dimensions, the analogous problem turned out to have a different flavor: Lenz found some simple constructions with a quadratic number of diameters. For more exact bounds, see Avis, Erdős and Pach [1], Erdős and Pach [7], and Swanepoel [20].

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For larger values of k, Vesztergombi showed that the *second largest* distance d_2 can occur at most $\frac{3}{2}n$ times among n points in the plane [22, 23], and at most $\frac{4}{3}n$ times if the points are in convex position [23]. She also observed that the number of k-th largest distances in the plane is always smaller than $2kn$. For small values of k, this was improved in [15]. While the majority of the results on graphs of large distances provide upper bounds on the number of edges, some other properties have also been explored. For example, Erdős, Lovász, and Vesztergombi [5, 6] obtained some results concerning the chromatic number of the graph generated by the top-k largest distances, i.e, the graph in which two points are connected if and only if their distance is *at least* d_k.

In [17], instead of counting the number of edges, Schur, Perles, Martini, and Kupitz initiated the investigation of the number of cliques in a graph of diameters. This paper is the starting point of our investigations. A k-*clique*, that is, a complete subgraph of k vertices in the graph of diameters of P corresponds to a regular $(k-1)$-dimensional simplex (or, in short, $(k-1)$-*simplex*) of side length d_1 generated by P.

Theorem A (Schur et al.). *Any finite subset $P \subset \mathbb{R}^d$ contains the vertices of at most one regular d-simplex of edge length d_1.*

The aim of this paper is to show that this beautiful statement marks the tip of an iceberg: for any k, the number of d-simplices of edge length d_k generated by P can be bounded from above by a constant depending only on d and k.

Theorem 1. *For any $k \geq 1$ and $d \geq 2$, there exists a constant $c(d,k)$ satisfying the following condition. Any finite set P of points in $\mathbb{R}^d$ can generate at most $c(d,k)$ regular d-simplices of edge length d_k.*

In Section 3, we give a construction with $d(k-1)+1$ regular simplices of edge length d_k. The proof of Theorem 1 presented in Section 2 uses extremal graph theory and provides an enormously huge bound for the constant $c(d,k)$, which can be surely improved a lot.

The main result in [17] is the following.

Theorem B (Schur et al.). *Any set of n points in $\mathbb{R}^3$ can generate at most n equilateral triangles of side length d_1.*

Again, we show an analogous result for the k-th largest equilateral triangles. The proof of this statement is given in Section 4.

Theorem 2. *For any $k \geq 1$, there exists a constant $c_k = k^{O(k)}$ such that the number of equilateral triangles of side length d_k generated by any set of n points in $\mathbb{R}^3$ is at most $c_k n$.*

Theorem B can be regarded as a 3-dimensional generalization of the Hopf-Pannwitz result mentioned above, according to which any set of n points in the plane has at most n diameters. It was conjectured by Z. Schur (see [17]) that this result can be extended to all dimensions d.

Conjecture 1 (Schur). *The number of d-cliques in a graph of diameters on n points in $\mathbb{R}^d$ is at most n.*

The fact that this bound can be attained can be shown by a simple construction; see [17].

In Section 5, we present the following theorem about the number of k-th largest distances in $\mathbb{R}^3$, generalizing the analogous observation of Vesztergombi in the plane.

Theorem 3. *For every $k \geq 1$, there is a constant c_k such that the following holds: the number of k-th largest distances generated by any set of n points in $\mathbb{R}^3$ is at most $c_k n$.*

Finding good bounds for c_k, at least for small values of $k \geq 2$, is a challenging open problem.

2 Proof of Theorem 1

First, we collect several auxiliary results needed for the proof. The following result was proved in [16].

Lemma 2.1. *There exists a constant $c > 0$ such that for any set of n distinct points $p_1, \ldots, p_n \in \mathbb{R}^d$ and for any $\epsilon > 0$, the number of triples $i < j < k$ for which $\angle p_i p_j p_k > \pi - \epsilon$, is at least $\lfloor n^3 / 2^{(c/\epsilon)^{d-1}} \rfloor$.*

By a *spherical cone* in a linear subspace $L \subset \mathbb{R}^d$ we understand a cone generated by a ball, i.e., a set C of the form $C = \{tx : t \geq 0, x \in B\}$, where $B = B(a, r)$ is a full-dimensional ball in L, that does not contain the origin (i.e., the dimension of B is the same as the dimension of L). In the sequel, unless indicated otherwise, we will use the term *cone* to refer to a spherical cone. The translate of a (spherical) cone C by a vector v we call a *cone with apex* v (see Figure 1(a)). The *angle* of a (spherical) cone C is defined as $2 \arcsin \frac{r}{\|a\|}$. When the subspace L is not specified (as in the next lemma), we assume $L = \mathbb{R}^d$.

To prove the next fact we use the well-known Kővári-Sós-Turán theorem: every bipartite graph that has m vertices in one part, n vertices in the other part, and at least $(r-1)^{1/s}(n-s+1)m^{1-1/s} + (s-1)m$ edges contains a subgraph isomorphic to $K_{r,s}$. In fact, we need only the following simple consequence of the theorem: for any $c_1 > 0$, there is $c_2 > 0$ such that any graph on n vertices with at least $c_1 n^2$ edges contains $K_{c_2 \log n, c_2 \log n}$ as a subgraph.

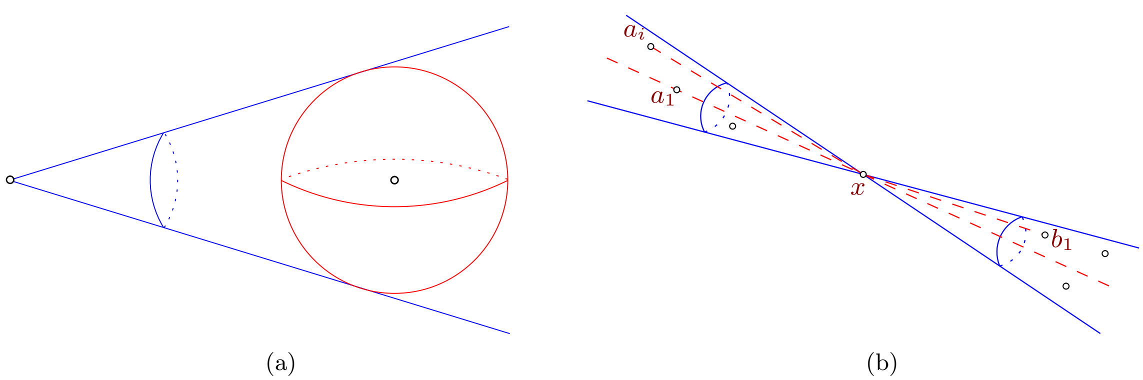

Figure 1: (a) Cone generated by a ball; (b) Proof of Lemma 2.2

Lemma 2.2. *For any $d \geq 2$, $n \geq 1$ and $\epsilon > 0$, there exists $c = c(d, \epsilon, n) > 0$ such that for any set T of c points in $\mathbb{R}^d$, one can find a point $x \in T$ and a cone C with apex x and angle ϵ such that both C and its opposite cone contain at least n points of T.*

Proof. We will show that $c(d, \epsilon, n) = 2^{c_1 n}$ is a good choice, for large enough c_1. Suppose we have a set T of $N = 2^{c_1 n}$ points in $\mathbb{R}^d$. From Lemma 2.1 it follows that there are fN^3 angles generated by these points of size at least $\pi - \frac{\epsilon}{4}$, for some $f = f(d, \epsilon) > 0$. Hence, there is a point $x \in T$ which is the apex of fN^2 angles of size at least $\pi - \frac{\epsilon}{4}$. Define a graph G with vertex set $T - \{x\}$ in which two points $q, r \in T - \{x\}$ are connected by an edge if and only if $\angle qxr > \pi - \frac{\epsilon}{4}$. Since this graph on $N - 1$ vertices has at least fN^2 edges, by the above observation we conclude that G contains a subgraph isomorphic to $K_{n,n}$, provided we choose large enough c_1. In other words, there are points

$a_1, \dots, a_n, b_1, \dots, b_n \in T$ such that $\angle a_i x b_j > \pi - \frac{\epsilon}{4}$ for any $i, j \in \{1, \dots, n\}$ (Figure 1(b)). Now we have that for any $i \in \{1, \dots, n\}$

$$\angle a_i x a_1 + 2\left(\pi - \frac{\epsilon}{4}\right) < \angle a_i x a_1 + \angle a_1 x b_1 + \angle b_1 x a_i \le 2\pi\,,$$

and, therefore, $\angle a_i p a_1 < \frac{\epsilon}{2}$ (here we used a small lemma, which is not difficult to show: for any four points $a, b, c, x \in \mathbb{R}^d$ we have $\angle axb + \angle bxc + \angle cxa \le 2\pi$). Finally, we can take for C the cone with apex x, axis xa_1 and angle ϵ. □

We introduce a notion that we need in order to formulate the next fact. In a linear subspace $L \subset \mathbb{R}^d$, consider a cone C, whose apex is the origin. Define the set

$$S(C) = \{x \in L \,:\, \text{there exists } v \in C \text{ such that } x \cdot v = 0\}\,.$$

We call $S(C)$ the *co-cone* of C (Figure 2(a)). Note that the angle of any cone that lies in $S(C)$ cannot exceed the angle of C.

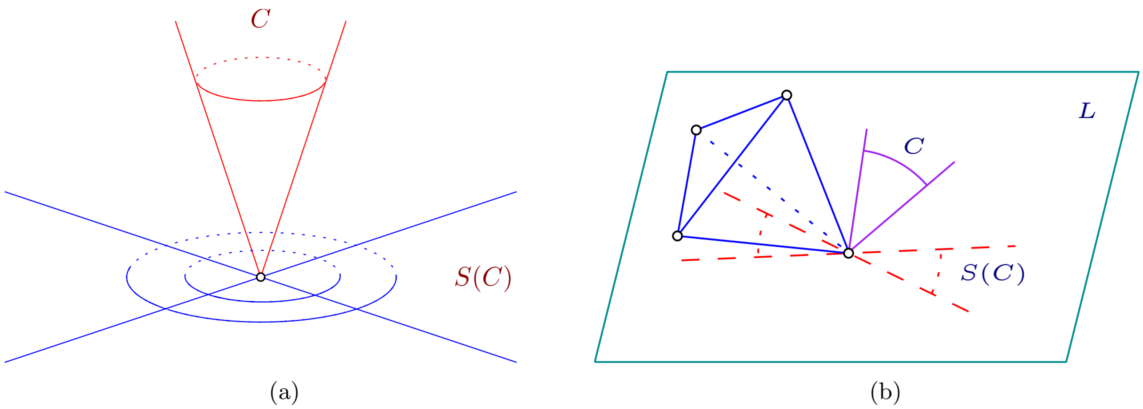

Figure 2: (a) The co-cone $S(C)$ of a cone C in $\mathbb{R}^3$; (b) Lemma 2.3: the projection of the simplex cannot fit in $S(C)$

Lemma 2.3. *For every $d \ge 2$, there exists $\epsilon(d) > 0$ with the following property. Let C be a cone whose apex is the origin, whose angle is at most ϵ, and which lies in a linear subspace $L \subset \mathbb{R}^d$. Then the orthogonal projection to L of any regular d-simplex whose one vertex is the origin, cannot fit into $S(C)$.*

Proof. The situation is illustrated in Figure 2(b). Denote by $r_d(a)$ the radius of the inscribed sphere of a regular d-simplex of edge length a and by $s_d(a)$ the distance between its vertex and the center of the inscribed sphere. By similarity, the ratio $r_d(a) : s_d(a)$ depends only on d (and not on a). We claim that the statement holds with $\epsilon = \arcsin \frac{r_d(a)}{s_d(a)}$. Suppose the contrary, i.e., that the projection to L of a d-simplex S' having the origin as its vertex is contained in $S(C)$, while cone C has angle at most ϵ. Let a be the edge length of S'. The projection of the inscribed ball of S' is a ball B of the same radius $r_d(a)$ that lies in $S(C)$. Denote its center by p. Since $\|p\| \le s_d(a)$, the angle of the cone generated by B is

$$2 \arcsin \frac{r_d(a)}{\|p\|} \ge 2 \arcsin \frac{r_d(a)}{s_d(a)} = 2\epsilon\,,$$

which is a contradiction. □

Now we move on with the proof of Theorem 1. Recall that we are given a finite set P of points in $\mathbb{R}^d$ and we want to upper-bound the number of regular d-simplices of edge length d_k generated by P. We can assume that every point in P is a vertex of at least one simplex, since otherwise we can delete non-interesting vertices.

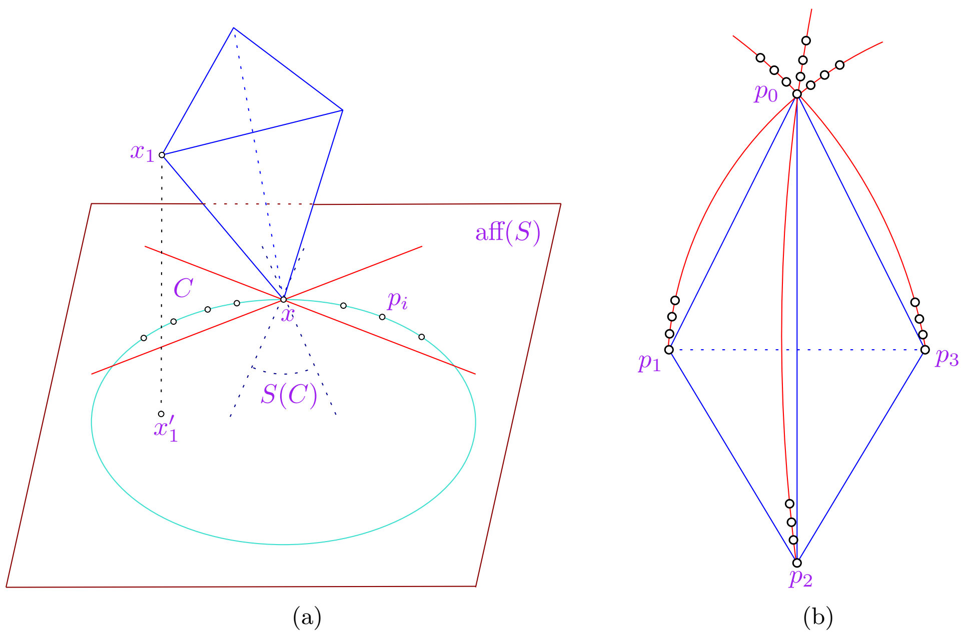

Figure 3: (a) Proof of Lemma 2.4; (b) a construction with many large simplices

Lemma 2.4. *For any $d \geq 2, k \geq 1, d' \leq d-1$, there exists $c'(d,d',k)$ such that the total number of points of P that can lie on a d'-sphere in $\mathbb{R}^d$ is at most c'.*

Proof. The proof is by induction on d'. For $d'=0$, the statement is trivially true with $c'(d,0,k)=2$ (since "0-sphere" is a set of two points). Now let $d' \geq 1$ and let S be a d'-sphere in $\mathbb{R}^d$. By induction, we assume that the statement is correct for all smaller values of d'. Let $N=(k-1)\cdot c'(d,d'-1,k)+1$. By Lemma 2.2, we choose a constant $c(d'+1,\epsilon,N)$, where $\epsilon=\epsilon(d)$ is given by Lemma 2.3. We claim that $c'(d,d',k)=c(d'+1,\epsilon,N)$ will work. We prove this by contradiction. Suppose that there are more than $c(d'+1,\epsilon,N)$ points of P on the sphere S. We can consider S as being embedded in a $(d'+1)$-dimensional space (the affine hull of S). By the choice of c (Lemma 2.2), we can find a point $x \in P \cap S$ such that in a cone C with apex x and angle ϵ as well as in its opposite cone, there are at least N points of $P \cap S$ (Figure 3(a)). Let $xx_1 \ldots x_d$ be a simplex with vertex x (and other vertices in P) and let $x'_1,\ldots,x'_d$ be the orthogonal projections of points $x_1,\ldots,x_d$ to the affine hull of S. By the choice of ϵ (according to Lemma 2.3), at least one of the vertices $x'_1,\ldots,x'_d$ lies outside the co-cone $S(C)$. Without loss of generality assume that $x'_1 \notin S(C)$ and, moreover,

$$(x'_1-x)\cdot(v-x)>0 \text{ for every } v \in C\,,$$

or, equivalently,

$$(x'_1-x)\cdot(v-x)<0 \text{ for every } v \in 2x-C$$

(note that $C' = 2x - C$ is the cone opposite to C). Let $p_1, \ldots, p_N$ be some points of $P \cap S$ that lie in the cone C'. Since the angles $\angle x_1' x p_i$ are all obtuse, we conclude that $|x_1' p_i| > |x_1' x|$ and by the Pythagorean theorem

$$|x_1 p_i| = \sqrt{|x_1 x_1'|^2 + |x_1' p_i|^2} > \sqrt{|x_1 x_1'|^2 + |x_1' x|^2} = |x_1 x| = d_k\,,$$

for all $i = 1, 2, \ldots, N$. The distance $|x_1 p_i|$ for every i can, thus, take one of the $k-1$ values $d_1, \ldots, d_{k-1}$. However, for every $j \in \{1, \ldots, k-1\}$ all points p_i that satisfy $|x_1 p_i| = d_j$ lie at the intersection of S (which is a d'-sphere) with the $(d-1)$-dimensional sphere with center x_1 and radius d_j, which is either empty or a $(d'-1)$-sphere. It follows that for every j there are at most $c'(d, d'-1, k)$ points p_i satisfying $|x_1 p_i| = d_j$. This contradicts the assumption that $N > (k-1) \cdot c'(d, d'-1, k)$, which completes the proof of the lemma. □

As an easy consequence we get the following fact.

Lemma 2.5. *No more than $(c'(d, d-1, k))^d$ simplices can share a vertex.*

Proof. Indeed, all simplices that have $p \in P$ as their vertex have the other d vertices on the sphere with center p and radius d_k. By Lemma 2.4, there are at most $c'(d, d-1, k)$ such vertices and at most $(c'(d, d-1, k))^d$ ways to choose d of them. □

Now we are in a position to complete the proof of Theorem 1. Let s be a d-simplex with vertices from P (if there is no such a simplex, we are done). Let r be the *Reuleaux simplex* of s, which is defined as the intersection of $d+1$ balls with centers at the vertices of s and with radius d_k. Observe that any simplex s' different from s has at least one vertex outside of r, by Theorem A. On the other hand, any point $p \in P$ that does not belong to r must lie on one of $(d+1)(k-1)$ spheres, each of them having one of the vertices of s as its center, and the radius in $\{d_1, \ldots, d_{k-1}\}$. Since any such sphere contains at most $c'(d, d-1, k)$ points from P, there are at most $(d+1)(k-1)c'(d, d-1, k)$ points from P lying outside of r, while every simplex $s' \neq s$ has at least one vertex among these points. Since no more than $(c'(d, d-1, k))^d$ simplices can share a vertex, we have that the theorem holds with $c(d, k) = (d+1)(k-1)(c'(d, d-1, k))^{d+1} + 1$. This completes the proof of Theorem 1.

Remark. Going through the proof, one can see that it produces an extremely fast-growing function $c(d, k)$: a tower exponential function with $\Omega(d)$ levels of the form $\Omega(k)$.

3 A construction

In this section, we describe a finite set of points in $\mathbb{R}^d$ that spans $d(k-1)+1$ regular d-simplices of edge length d_k. There is a huge gap between this lower bound and the upper bound in Theorem 1, but the construction shows that the maximum number of k-th largest simplices indeed grows both with k and d. The construction is inspired by the corresponding construction for maximal $(d-1)$-simplices, given in [17].

Let $p_0, p_1, \ldots, p_d$ be the vertices of a regular unit simplex S in $\mathbb{R}^d$ centered at the origin (Figure 3(b)). For $i \neq j$ denote by c_{ij} the center of the $(d-2)$-face of S complementary to the edge $p_i p_j$, i.e.,

$$c_{ij} = -\frac{1}{d-1}(p_i + p_j)\,.$$

It is easy to check that

$$|p_i c_{ij}| = |p_j c_{ij}| = \sqrt{\frac{d}{2(d-1)}}$$

and that the vectors $p_i - c_{ij}$ and $p_j - c_{ij}$ are orthogonal to $p_k - c_{ij}$ for all $k \neq i, j$. Denote by C_{ij} the circle centered at c_{ij} that passes through p_i and p_j. Then p_k is equidistant from all points of C_{ij} for $k \neq i, j$.

Now for all $j = 1, \ldots, d$ put on circle C_{0j} new points r_i^j, s_i^j $(i = 1, \ldots, k-1)$ with the following order

$$p_j, r_1^j, r_2^j, \ldots, r_{k-1}^j, p_0, s_1^j, s_2^j, \ldots, s_{k-1}^j\,,$$

so that

$$|p_j r_1^j| = |r_1^j r_2^j| = \cdots = |r_{k-2}^j r_{k-1}^j| = |p_0 s_1^j| = |s_1^j s_2^j| = \cdots = |s_{k-2}^j s_{k-1}^j| = \epsilon\,,$$

where $\epsilon > 0$ is very small.

Thus, we have $d+1+2(k-1)d$ points in total and we claim that the largest distances generated by these points are

$$|p_j p_0| = 1 = d_k, |p_j s_1^j| = d_{k-1}, \ldots, |p_j s_{k-1}^j| = d_1 \quad \text{for all } j = 1, \ldots, d\,.$$

To verify this, it is enough to check that $|s_m^i r_n^j| < 1$ for all $i \neq j$ and all m, n. Let s be the projection of s_m^i on $\operatorname{aff}(C_{0j})$. Then s lies in a small neighborhood of p_0 and we have that $|sr_n^i| < |sp_j|$. By the Pythagorean theorem we get

$$|s_m^i r_n^j| = \sqrt{|s_m^i s|^2 + |sr_n^j|^2} < \sqrt{|s_m^i s|^2 + |sp_j|^2} = |s_m^i p_j| = |p_0 p_j| = 1\,.$$

Finally, note that any two points r_i^j and s_i^j together with $\{p_1, \ldots, p_d\} \setminus \{p_j\}$ span a regular unit d-simplex, for all $i \in \{1, \ldots, k-1\}$ and $j \in \{1, \ldots, d\}$, which gives $d(k-1)+1$ simplices in total (we count also the initial simplex $p_0 p_1 \ldots p_d$).

4 Proof of Theorem 2

We start with some lemmas. It is shown in [17] that any two triangles in a graph of diameters in $\mathbb{R}^3$ must share a vertex. We extend this result to k-th largest triangles for $k \geq 2$.

Lemma 4.1. *There is a large enough constant c such that no matter how we choose at least k^{ck} triangles in a graph of k-th largest distances in $\mathbb{R}^3$, there are always two triangles that share a vertex.*

Proof. We can assume that $k \geq 2$. Suppose to the contrary we have $m \geq k^{ck}$ triangles no two of which share a vertex, for a large constant c. Let $a_1 b_1 c_1$ be one of these triangles. Then each of the remaining $m-1$ triangles has a vertex on one of the $3(k-1)$ spheres with centers a_1, b_1, c_1 and radii $d_1, \ldots, d_{k-1}$ (here we used the result from [17] mentioned above). Hence, we can find at least $\frac{m-1}{3(k-1)}$ triangles that have at least one vertex on one fixed sphere S_1. Now pick one of these triangles, say, $a_2 b_2 c_2$. Again, each of the remaining $\frac{m-1}{3(k-1)} - 1$ triangles has a vertex on one of the $3(k-1)$ spheres with centers a_2, b_2, c_2 and radii $d_1, \ldots, d_{k-1}$, and, just like before, at least $\frac{m}{3^2(k-1)^2} - \frac{1}{3^2(k-1)^2} - \frac{1}{3(k-1)}$ triangles have at least one vertex on one fixed sphere S_2. Note that all these triangles also have at least one vertex on S_1, and $S_2 \neq S_1$, since they have different centers. Proceeding in this manner, after t steps (we'll specify t shortly) we find at least

$$\frac{m}{3^t(k-1)^t} - \frac{1}{3^t(k-1)^t} - \cdots - \frac{1}{3(k-1)}$$

triangles that have at least one vertex on each of the t distinct spheres $S_1, \ldots, S_t$ of radii from $\{d_1, \ldots, d_{k-1}\}$. We take $t = 3 \cdot 2(k-1) + 1$ and claim that

$$\frac{m}{3^t(k-1)^t} - \frac{1}{3^t(k-1)^t} - \cdots - \frac{1}{3(k-1)} \leq 2\binom{t}{2(k-1)+1}. \tag{1}$$

Indeed, each triangle has one vertex on each of the t spheres, and, therefore, each triangle has a vertex that lies in the intersection of at least $2(k-1)+1$ spheres. However, any intersection of $2(k-1)+1$ spheres consists of at most two points. Thus, each triangle has a vertex that lies in a set of at most $2\binom{t}{2(k-1)+1}$ points. However, two different triangles never share a vertex, hence the inequality (1). On the other hand, we have that

$$\frac{m}{3^t(k-1)^t} - \frac{1}{3^t(k-1)^t} - \cdots - \frac{1}{3(k-1)} \geq k^{ck/2} \tag{2}$$

for large enough c. From (1) and (2) we have that

$$k^{ck/2} \leq 2\binom{6k-5}{2k-1},$$

which does not hold for large enough c. □

Next we count the number of k-th largest distances on a sphere in $\mathbb{R}^3$ under the condition that the radius of the sphere is large enough compared to the distances.

Lemma 4.2. *Among n points on a 2-sphere there can be at most $2kn$ pairs at distance d_k, provided that d_k is the radius of the sphere as well as the k-th largest distance.*

Proof. Denote the sphere by S. We consider the graph of k-th largest distances on the n given points. If all the vertices have degree at most $4k$, we are done, so we can assume that there is a vertex v of degree at least $4k+1$. Also, if there is a vertex of degree at most $2k$, we are done, since we can delete that vertex and proceed by induction. Therefore, we assume that all vertices have degree at least $2k+1$.

Let $u_1, \ldots, u_{4k+1}$ be neighbors of v. Then $|vu_1| = \cdots = |vu_{4k+1}| = d_k$ (Figure 4). The points $u_1, \ldots, u_{4k+1}$ are cocyclic and lie on one hemisphere with point v (this follows from the fact that the radius of S is d_k). Let C be the circle that contains $u_1, \ldots, u_{4k+1}$ and let p be its center. It is an easy exercise to show that there exists i such that the diameter of C that contains u_i divides the rest of the points into two parts of $2k$ points. From the assumption we know that u_i has a neighbor $w \neq v$. Now we want to locate the projection w' of point w to the plane of C. In the plane of C consider the line ℓ that is perpendicular to u_ip and passes through p. We claim that w' lies on the same side of ℓ as point u_i. This follows from the fact that w lies on the circle $S(u_i, d_k) \cap S$, which in turn lies on the same side of the plane determined by v and ℓ as point u_i (here $S(u_i, d_k)$ denotes the sphere with center u_i and radius d_k). Without loss of generality let the points lying on one

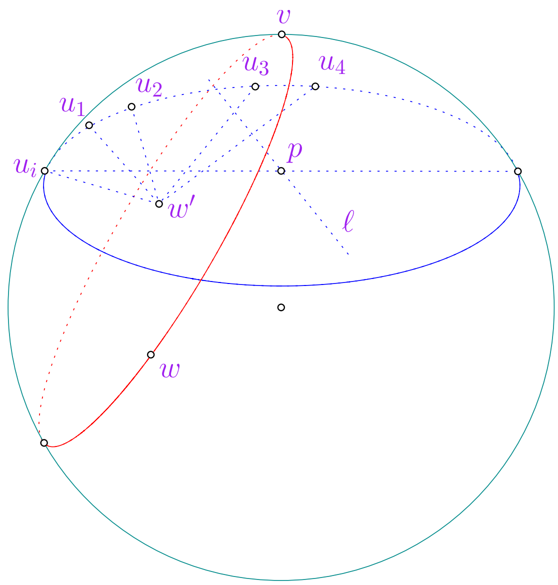

Figure 4: Proof of Lemma 4.2

side of the diameter of C through u_i be $u_1, \dots, u_{2k}$ and let w' lies on the other side of the diameter (or, possibly, on the diameter). Then $|w'u_j| > |w'u_i|$, for all $j = 1, \dots, 2k$. By the Pythagorean theorem it follows that $|wu_j| > |wu_i| = d_k$ for all j, and, hence, $|wu_j| \in \{d_1, \dots, d_{k-1}\}$. However, each of the $k-1$ possible values can be taken by $|wu_j|$ for at most two different j's. This is a contradiction. $\square$

Now the proof of Theorem 2 is not difficult.

Proof of Theorem 2. Consider the graph of k-th largest distances on the given set P. Take a maximal set of triangles in which no two share a vertex. Denote by m the number of triangles in this set and by M the total number of triangles. Then $m = k^{O(k)}$, by Lemma 4.1. Each of the remaining $M - m$ triangles shares a vertex with one of the m triangles. Thus, we can find at least $\frac{M-m}{3m}$ triangles that share a vertex v. The sphere with center v and radius d_k contains at most $n-1$ points of P that generated at least $\frac{M-m}{3m}$ distances d_k. Now Lemma 4.2 gives us $\frac{M-m}{3m} \le 2k(n-1)$, which implies $M \le k^{O(k)}n$. $\square$

5 Proof of Theorem 3

Here we discuss a method based on an interesting graph-theoretic lemma, from which Theorem 3 follows immediately. We formulated the lemma recently with Alexey Glazyrin and the following proof was given on MathOverflow by Timothy Gowers, Sergey Norin and Fedor Petrov [14] (we slightly modify their proof by choosing the probability more carefully, which gives a better bound).

Lemma 5.1. *Given a graph $G(V, E)$ whose edges are colored in two colors, red and blue. Suppose there are constants $c, c' > 0$ such that the following two conditions hold:*

(1) for any $S \subseteq V$, there are at most $c|S|$ red edges in $G[S]$;

(2) for any $S \subseteq V$, if $G[S]$ contains no red edges, then it contains at most $c'|S|$ blue edges (resp., triangles).

Then the total number of blue edges (resp., triangles) is at most $e(4c+1)c'|V|$ (resp., $e^2(3c+1)^2c'|V|$).

Proof. We will give a detailed proof just for the case of edges. The proof for triangles can be obtained with minor changes, which we explain at the end.

First we show that we can label the vertices of G by $v_1, v_2, \ldots, v_n$ such that v_{i+1} has at most $2c$ neighbors in $\{v_1, \ldots, v_i\}$ to which it is connected by a red edge, for all $i = 1, \ldots, n-1$. Indeed, let v_n be a vertex with the smallest red-degree in G (the red-degree is the number of red edges incident to a vertex). Then $\deg_{red}(v_n) \le 2c$. We proceed by induction: supposing that the vertices $v_n, v_{n-1}, \ldots, v_{k+1}$ are already chosen, we pick a vertex $v_k \in V \setminus \{v_n, v_{n-1}, \ldots, v_{k+1}\}$ that has the smallest red-degree in $G[V \setminus \{v_n, v_{n-1}, \ldots, v_{k+1}\}]$. By the assumption, the red-degree of v_k in the restricted graph will not be larger than $2c$, as required.

Now we define a random subset S of V recursively, as follows: if $S \cap \{v_1, \ldots, v_i\}$ has already been chosen, we put v_{i+1} in S with probability p (to be specified later) if it is not joined by a red edge to any of the vertices already in S, otherwise we do not put it in S. Thus, we obtain a random *red-independent* set S of vertices (where by *red-independent* we mean that $G[S]$ contains no red edges).

The next step is to give a lower bound for the probability that a fixed blue edge is chosen. Let x and y be two vertices connected by a blue edge. We have that $\Pr[x \in S \;\&\; y \in S]$ is at least the probability that none of the neighbors of x and y is chosen multiplied by the probability that x and y are chosen, i.e.,

$$\Pr[x \in S \;\&\; y \in S] \ge (1-p)^{4c} \cdot p^2 \,,$$

since both x and y have at most $2c$ vertices preceding them, to which they are connected by a red edge. Note that we did not use that x and y are connected by a blue edge, we used just that xy is not a red edge. Finally, for the expected number of blue edges in S we have

$$\mathrm{E}[\#\text{blue edges in } S] \ge (1-p)^{4c} \cdot p^2 \times \#\text{blue edges}$$

on one hand, and

$$\mathrm{E}[\#\text{blue edges in } S] \le c' \,\mathrm{E}[|S|] \le c'p|V|$$

on the other hand, since each vertex of G lies in S with probability at most p. By combining the two inequalities and setting $p = \frac{1}{4c+1}$, we get

$$\#\text{blue edges} \le c'p^{-1}(1-p)^{-4c}|V| \le e(4c+1)c'|V|,$$

as claimed (we used the well known fact that $(1+1/x)^x < e$ for all $x \ge 1$).

To get the bound for the number of blue triangles, we proceed in the same manner, except that we use the estimate

$$\Pr[x \in S \;\&\; y \in S \;\&\; z \in S] \ge (1-p)^{6c} \cdot p^3 \,,$$

for any blue triangle xyz, and we consider the expected number of blue triangles in S, instead of edges. In this case we take $p = \frac{1}{3c+1}$. □

Now Theorem 3 follows with no difficulty.

Proof of Theorem 3. Consider a graph on n points in $\mathbb{R}^3$, whose red and blue edges are, respectively, diameters and second largest distances. Then the conditions of the edge version of Lemma 5.1 are satisfied with $c = c' = 2$ (since the maximum number of diameters is $2n-2$, see [21]). Hence, we conclude that the number of second largest distances is at most $18en < 50n$. Applying the lemma repeatedly $k-1$ times finishes the proof. □

The constant c_k we get is tower exponential in k, while we expect a polynomial dependence on k. This might possibly be achieved by improving the dependence on c and c' of the final constant in Lemma 5.1. So far the above graph-theoretic approach is the only way how we can derive a linear upper bound for k-th largest distance in $\mathbb{R}^3$. Note that the triangle version of Lemma 5.1 (with $c = 2$ and $c' = 1$, by Theorem B) also provides an instant proof of Theorem 2, although with a much weaker constant.

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