

Thermodynamic Consistency of a Pseudoincompressible Approximation for General Equations of State

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ABSTRACT

In soundproof model equations for geophysical fluid dynamics, the momentum and mechanical energy budgets decouple from the thermodynamics for adiabatic flows. In applying such models to nonadiabatic flows of fluids with general equations of state, thermodynamic consistency of the soundproof approximations needs to be ensured. Specifically, a physically meaningful total energy conservation law should arise as an integral of adiabatic dynamics, while for diabatic flows the effective energy source terms should be related through thermodynamic relationships to the rates of change of entropy and other pertinent internal degrees of freedom. Complementing earlier work by one of the authors on the Lipps and Hemler-type anelastic approximation, this paper discusses the thermodynamic consistency of an extension of Durran's pseudoincompressible model to moist atmospheric motions allowing for a general equation of state.

1. Introduction

Geophysical flows involve a large range of motions operating on multiple spatial and temporal scales, including propagating waves of various types, buoyancy-driven convective overturning, and geostrophic flow constrained by rotation. All these flows can be represented based on the compressible Navier–Stokes equations. However, because the compressible Navier–Stokes equations account for all types of fluid motions, they are often not needed to study specific aspects of the circulation. The details of sound wave propagation, in particular, are generally considered to be irrelevant for flows on scales of, say, 100 km and less, and this motivates the derivation and study of “soundproof” flow equations. These capture the net effect of sound waves (i.e., the rapid local equilibration of the thermodynamic pressure) while suppressing the need to numerically represent the propagation of sound waves.

Over the years, several mathematical approximations of the compressible Navier–Stokes equations have been developed with the specific goal of filtering out specific types of motions. The Boussinesq approximation (Boussinesq 1903) is probably the best known of these. It filters out sound waves by replacing the continuity equation by an incompressibility condition. The anelastic and pseudoincompressible approximations have further expanded the mathematical framework to account for large vertical variations of pressure and density and are the basis for a wide range of numerical models (Ogura and Phillips 1962; Dutton and Fichtl 1969; Lipps and Hemler 1982, 1985; Durran 1989; Bannon 1995, 1996; Prusa et al. 2008; Klein et al. 2010). See also Davies et al. (2003), Klein et al. (2010), and Achatz et al. (2010) for discussions of the regime of validity of soundproof approximations.

The Boussinesq and anelastic approximations rely on partial linearizations of the pressure and gravity terms in the momentum equation to filter out sound waves based on the assumptions of a low Mach number. The pseudoincompressible approximation does not invoke such linearizations in the case of an ideal gas equation of state and in the formulation using the Exner pressure as proposed originally in Durran (1989, 2008). Yet, our

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thermodynamically consistent extension to general equations of state will require an appropriate linearization [see (10) below]. In all the soundproof models, these approximations raise the issue of how they impact energy conservation and the underlying thermodynamics. Indeed, the fundamental laws of thermodynamics imply a nonlinear relationship among variations of enthalpy, pressure, and entropy. This problem is made worse when studying moist convection, as condensation itself implies a strong nonlinearity that results in a discontinuity in the partial derivatives of the equation of state.

This issue has been recognized for some time. In their original derivation of the anelastic approximation, Ogura and Phillips (1962) note that their derivation could not be extended to a moist atmosphere. The version obtained by Lipps and Hemler (1982) does not conserve energy. It is only recently that Pauluis (2008) showed that it is possible to obtain a thermodynamically consistent version of the anelastic approximation for a moist atmosphere if one accepts a restriction to nearly moist adiabatic background states. By thermodynamic consistency here we mean that the approximation

- (i) conserves energy, with an expression for the internal energy that is consistent with the nonlinear equation of state;
- (ii) maintains the same conversion between internal and mechanical energy as found in a fully compressible fluid for reversible adiabatic flows; and
- (iii) ensures that diabatic processes have the same impacts on the thermodynamic state variable as for the compressible Navier–Stokes equations.

These requirements ensure not only that the approximation conserves the total energy, but also that the conversion rate between internal and mechanical energy is fully consistent with the thermodynamic constraints. See also recent work by Tailleux (2011, manuscript submitted to *J. Fluid Mech.*) in this context.

The purpose of this article is to expand the results of Pauluis (2008) to obtain a thermodynamically consistent version of the pseudoincompressible approximation for a moist atmosphere. Whereas the anelastic approximation's primary simplifying assumption is that of a time-independent spatial density distribution, the pseudoincompressible approximation relies instead on small pressure variations (Durran 1989, 2008). As shown recently by Klein et al. (2010) and Achatz et al. (2010) through formal asymptotic arguments, the anelastic and pseudoincompressible models share a broad regime of validity involving low Mach number atmospheric flows of an ideal gas with constant specific heat capacities. Almgren et al. (2006) developed a set of low-Mach number model equations for astrophysical applications

that reduces to Durran's pseudoincompressible model in the case of an ideal gas but is applicable to more general fluids. This work did not address the issue of thermodynamic consistency, however, and this motivates the present paper.

There are two main difficulties when dealing with the thermodynamic relationship within the pseudoincompressible equation. First, the derivation of the pseudoincompressible model has been historically tied to the ideal gas law by introducing the Exner pressure and the dry potential temperature as primary prognostic variables. The associated particular transformations do not easily transfer to general equations of state, and this renders adaption of the pressure-based modeling approach to an arbitrary fluid rather difficult. Second, in contrast with the anelastic approximation, the pseudoincompressible approximation keeps a contribution from the pressure perturbation in the buoyancy, which must be accounted for in the conservation of total energy conservation. We manage to overcome these difficulties in this paper.

Our key result is a thermodynamically consistent pseudoincompressible model for fluids with general equation of state that allows for quite general background stratifications. Its derivation closely maintains the spirit of Durran's original arguments in Durran (1989). An important difference between our model on the one hand and the pseudoincompressible equations of Durran (1989) and Almgren et al. (2006) is that the velocity divergence constraint cannot generally be cast in the usual form $\nabla \cdot (\beta \mathbf{v}) = 0$ for a prescribed vertically stratified $\beta(z)$ but must rather be deduced from mass continuity given the equation of state, background pressure distribution, and advection equations for additional scalars representing the internal degrees of freedom of the fluid under consideration.

2. Pseudoincompressible approximation for fluids with a general equation of state

The starting point for our developments is the compressible Euler equations:

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (1a)$$

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{1}{\rho} \nabla p = -g \mathbf{k}, \quad (1b)$$

$$S_t + \mathbf{v} \cdot \nabla S = \dot{S}, \quad (1c)$$

$$q_t + \mathbf{v} \cdot \nabla q = \dot{q}. \quad (1d)$$

Here (p, S, q) are the pressure, entropy, and an additional scalar that represents any pertinent additional

internal degree of freedom characterizing the fluid microstate. For example, q could be the total water content in a model for moist atmospheric flows. These quantities will serve as our primary thermodynamic variables in what follows. Furthermore, \mathbf{v} is the velocity, g the acceleration of gravity, and \mathbf{k} the vertical unit vector. Through an equation of state for the density

$$\rho = \rho(p, S, q), \tag{2}$$

the mass balance (1a) yields effectively an evolution equation for p .

On the right-hand sides of (1c) and (1d), \dot{S} corresponds to the rate of change of entropy due to both external diabatic heating and internal production by irreversible processes and \dot{q} is an externally imposed rate of change of the additional internal degree of freedom. For moist atmospheric flow, \dot{q} would include the total moisture fluxes due to precipitation.

The equations in (1) can be combined to describe the balance of total energy

$$(\rho E_T)_t + \nabla \cdot (\rho H_T \mathbf{v}) = \rho(T\dot{S} + \mu\dot{q}), \tag{3}$$

where the total specific energy and enthalpy are

$$E_T = H_T - p/\rho \quad \text{and} \quad H_T = H + \frac{\mathbf{v}^2}{2} + gz, \tag{4}$$

respectively. Note that the total energy and enthalpy include mechanical energy $(\mathbf{v}^2/2) + gz$. Importantly, changes in total energy are generally associated with sources either of entropy or of any additional internal degrees of freedom. The total energy changes are related to these sources through the thermodynamic potentials, temperature T for entropy, and the Gibbs free energy μ for any other internal degree of freedom. Equation (3) ensures that the total energy is conserved for reversible adiabatic flow (i.e., for vanishing \dot{S} and \dot{q}). For more general problems, energy conservation requires that the right-hand side of (3) can be expressed as a convergence of an energy flux \mathbf{F} —that is,

$$\rho(T\dot{S} + \mu\dot{q}) = -\nabla \cdot \mathbf{F}. \tag{5}$$

The dependence of the specific enthalpy H on the primary thermodynamic variables [i.e., $H = H(p, S, q)$] defines the equation of state of the fluid. In fact, the inverse density, temperature, and chemical potential $\alpha \equiv 1/\rho$, T , and μ , respectively, satisfy the thermodynamic differential relations

$$\alpha = \frac{1}{\rho} = \frac{\partial H}{\partial p}, \quad T = \frac{\partial H}{\partial S}, \quad \mu = \frac{\partial H}{\partial q}. \tag{6}$$

The enthalpy equation of state also determines that for density referred to in (2). For moist atmospheric flow, $\mu = \mu_v - \mu_d$, where μ_v and μ_d denote the chemical potentials for water vapor and dry air, respectively.

The central approximation in Durran’s pseudoincompressible model (Durran 1989) is that deviations of pressure from a given background hydrostatic distribution are small. Along with the original derivation of the model, Durran demonstrated that the equations satisfy a total energy conservation law, where the pseudoincompressible total energy was defined as the sum of internal, kinetic, and gravitational potential energy with the pressure constrained to equal its background hydrostatic value at the given vertical level. Durran’s original derivation of total energy conservation was based on an ideal gas equation of state with constant specific heat capacities and to the adiabatic case, and this allowed for a few simplifications. Here we generalize the model to account for general equations of state and we demonstrate thermodynamic consistency between entropy production, changes of internal degrees of freedom of the fluid, and the total energy balance given a suitable definition of “pseudo-incompressible thermodynamic potentials.” Our derivation borrows its key ideas from the analogous derivations for an anelastic model by Pauluis (2008).

The pseudoincompressible model addresses flows as deviations from a hydrostatic background state at rest so that there is a background pressure distribution $\bar{p}(z)$ satisfying

$$\frac{d\bar{p}}{dz} = -\bar{\rho}g, \quad \bar{p}(0) = p_0, \tag{7}$$

where $\bar{\rho}(z)$ is a suitable horizontally averaged density distribution and g is the acceleration of gravity. In the following no further assumptions are made regarding \bar{p} other than that it is positive. Of course, in any practical application one will choose a background density that matches, for example, any soundings relevant to the considered atmospheric flow situation.

Given this background state, our proposed extension of the pseudoincompressible model for general equations of state involves the compressible Euler equations in the form presented in (1) with the following modifications. First, we replace

$$\rho \rightarrow \rho^* \equiv \rho(\bar{p}, S, q) \tag{8}$$

in the mass balance; second, we introduce the pressure perturbation

$$p \rightarrow \bar{p} + \delta p; \tag{9}$$

and third, we expand the inverse density multiplying the pressure gradient in the momentum equation as

$$\frac{1}{\rho} \rightarrow \frac{1}{\rho^*} - \frac{1}{\rho^{*2}} \frac{\partial \rho^*}{\partial \bar{p}} \delta p. \quad (10)$$

Replacing the density by the ‘‘pseudodensity’’ (Durrant 1989, 2008) amounts to using its leading-order approximation under the perturbation expansion for the pressure in (9). The expansion in (10) ensures that we carry consistently all terms in the momentum equation up to first order in δp . In fact, the leading-order terms in the momentum equation from (1b) are those constituting the hydrostatic balance in (7). The actual acceleration, $(\partial_t + \mathbf{v} \cdot \nabla)\mathbf{v}$, therefore balances with the first- and higher-order perturbations of the pressure gradient term. Keeping all terms up to first order in the pressure perturbation δp , we find the pseudo-incompressible model

$$\rho_t^* + \nabla \cdot (\rho^* \mathbf{v}) = 0, \quad (11a)$$

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{1}{\rho^*} \nabla \delta p = -g\mathbf{k} - \left(\frac{1}{\rho^*} - \frac{1}{\rho^{*2}} \frac{\partial \rho^*}{\partial \bar{p}} \delta p \right) \frac{d\bar{p}}{dz} \mathbf{k}, \quad (11b)$$

$$S_t + \mathbf{v} \cdot \nabla S = \dot{S}, \quad (11c)$$

$$q_t + \mathbf{v} \cdot \nabla q = \dot{q}. \quad (11d)$$

The particular form of the right-hand side of (11b) will be convenient in subsequent calculations, but notice that it may be recast in the form of an effective buoyancy term:

$$\begin{aligned} & -g\mathbf{k} - \left(\frac{1}{\rho^*} - \frac{1}{\rho^{*2}} \frac{\partial \rho^*}{\partial \bar{p}} \delta p \right) \frac{d\bar{p}}{dz} \mathbf{k} \\ & = g\mathbf{k} \left(\frac{\bar{p} - \rho^*}{\rho^*} - \frac{\bar{p}}{\rho^{*2}} \frac{\partial \rho^*}{\partial \bar{p}} \delta p \right). \end{aligned} \quad (12)$$

The key difference between this system and the compressible system from (1) is that the pressure components \bar{p} and δp are not determined by the mass balance through the equation of state. Rather, \bar{p} is defined by the background hydrostatic balance in (7), whereas δp is a perturbation pressure that adjusts to guarantee compliance of the velocity field with a divergence constraint. To reveal this constraint, we carry out the differentiations in (11a), insert (8), (11c), and (11d), and find

$$\nabla \cdot \mathbf{v} = -\frac{\partial \ln \rho^*}{\partial \bar{p}} w \frac{d\bar{p}}{dz} - \frac{\partial \ln \rho^*}{\partial S} \dot{S} - \frac{\partial \ln \rho^*}{\partial q} \dot{q}. \quad (13)$$

In the appendix we demonstrate that δp satisfies a second-order elliptic pressure equation for externally prescribed diabatic source terms, \dot{S} and \dot{q} , by taking the time derivative of (13) and comparing it with the divergence of the momentum equation in (11b). The effective equation for the perturbation pressure may be considerably more complex than that for more general forms of the diabatic source terms. A detailed discussion of this issue is beyond the scope of this paper, but it needs to be worked out prior to computational implementations of the model equations.

We are now ready to check the consequences of the pseudoincompressible approximation for an associated total energy budget. Letting

$$\begin{aligned} H^* &= H(\bar{p}, S, q), \quad \alpha^* = \frac{1}{\rho^*} = \frac{\partial H^*}{\partial \bar{p}}, \\ T^* &= \frac{\partial H^*}{\partial S}, \quad \mu^* = \frac{\partial H^*}{\partial q}, \end{aligned} \quad (14)$$

we find

$$(\partial_t + \mathbf{v} \cdot \nabla)H^* = \frac{1}{\rho^*} w \frac{d\bar{p}}{dz} + T^* \dot{S} + \mu^* \dot{q}. \quad (15)$$

After multiplication by ρ^* , using the pseudoincompressible mass balance from (11a), and considering that $\rho^* H^* = \rho^* E^* + \bar{p}$ with time-independent \bar{p} , one finds

$$(\rho^* E^*)_t + \nabla \cdot (\rho^* H^* \mathbf{v}) = w \frac{d\bar{p}}{dz} + \rho^* (T^* \dot{S} + \mu^* \dot{q}). \quad (16)$$

Next we derive an equation for the kinetic energy by multiplying (11b) by \mathbf{v} and collecting terms:

$$\begin{aligned} & \rho^* (\partial_t + \mathbf{v} \cdot \nabla) \frac{\mathbf{v}^2}{2} + \mathbf{v} \cdot \nabla \delta p \\ & = -w \frac{d\bar{p}}{dz} - \rho^* g w + \delta p \frac{1}{\rho^*} \frac{\partial \rho^*}{\partial \bar{p}} w \frac{d\bar{p}}{dz}. \end{aligned} \quad (17)$$

Using the fact that $(\partial_t + \mathbf{v} \cdot \nabla)z \equiv w$, employing again the pseudoincompressible mass balance from (11a) to recast terms in conservation form, and letting $\mathbf{v} \cdot \nabla \delta p = \nabla \cdot (\delta p \mathbf{v}) - \delta p \nabla \cdot \mathbf{v}$, we have

$$\begin{aligned} & \left[\rho^* \left(\frac{\mathbf{v}^2}{2} + gz \right) \right]_t + \nabla \cdot \left[\rho^* \left(\frac{\mathbf{v}^2}{2} + gz \right) \mathbf{v} \right] + \nabla \cdot (\delta p \mathbf{v}) \\ & = -w \frac{d\bar{p}}{dz} + \delta p \left(\nabla \cdot \mathbf{v} + \frac{\partial \ln \rho^*}{\partial \bar{p}} w \frac{d\bar{p}}{dz} \right). \end{aligned} \quad (18)$$

With the divergence constraint in (13) and the thermodynamic derivatives in (6) we re-express the last bracket in (18) as

$$\nabla \cdot \mathbf{v} + \frac{\partial \ln \rho^*}{\partial \bar{p}} w \frac{d\bar{p}}{dz} = \rho^* \left(\frac{\partial T^*}{\partial \bar{p}} \dot{S} + \frac{\partial \mu^*}{\partial \bar{p}} \dot{q} \right). \quad (19)$$

Then, combining (18) and (19) with the internal energy equation in (16) we obtain our main result:

$$(\rho^* E_T^*)_t + \nabla \cdot [(\rho^* H_T^* + \delta p) \mathbf{v}] = \rho^* (T^{**} \dot{S} + \mu^{**} \dot{q}), \quad (20)$$

where

$$H_T^* = H^* + \frac{v^2}{2} + gz \quad \text{and} \quad E_T^* = H_T^* - \bar{p}/\rho^*, \quad (21)$$

and

$$T^{**} = T^* + \delta p \frac{\partial T^*}{\partial \bar{p}} \quad \text{and} \quad \mu^{**} = \mu^* + \delta p \frac{\partial \mu^*}{\partial \bar{p}} \quad (22)$$

are thermodynamic potentials accurate up to and including first order in δp . From a physical point of view, the additional corrections to the temperature and Gibbs free energy associated with the dynamic pressure perturbation account for the small changes in these thermodynamics potentials that result from adiabatic fluctuations of the pressure field. For example, if the dynamical pressure perturbation is positive, it implies that the parcel is slightly more compressed than the background reference state, and thus has a slightly higher temperature.

These corrected potentials are also the ones that should be used in the physical parameterizations used to compute the tendencies \dot{S} and \dot{q} to ensure thermodynamic consistency. More specifically, let us assume that energy conservation is established in the form of (5), with a decomposition of the heat flux into contribution due to sensible heat and latent fluxes \mathbf{F}_T and $h_v \mathbf{F}_q$:

$$\mathbf{F} = \mathbf{F}_T + h_v \mathbf{F}_q. \quad (23)$$

Here h_v is the specific enthalpy of the water vapor. The diabatic source terms \dot{q} and \dot{S} must be defined as

$$\rho^* \dot{q} = -\nabla \cdot \mathbf{F}_q \quad \text{and} \quad \rho^* \dot{S} = -\nabla \cdot \mathbf{F}_S + \dot{S}_{\text{irr}}. \quad (24)$$

Here, the entropy flux given by

$$\mathbf{F}_S = s_v \mathbf{F}_q + \frac{\mathbf{F}_T}{T^{**}}, \quad (25)$$

with s_v being the specific entropy of water vapor. For consistency, we consider here that the specific enthalpy of water vapor can be related to its entropy and free

energy by $h_v = \mu^{**} + T^* s_v$. This allows us to write the irreversible entropy production as

$$\dot{S}_{\text{irr}} = \mathbf{F}_T \cdot \nabla \left(\frac{1}{T^{**}} \right) - \frac{1}{T^{**}} \mathbf{F}_q \cdot (s_v \nabla T^{**} + \nabla \mu^{**}). \quad (26)$$

The first term on the left-hand side of (26) corresponds to the internal entropy production due to diffusion of sensible heat along a temperature gradient, and the second term corresponds to the internal entropy production due to diffusion of water vapor. Note that if one treats water vapor as an ideal gas, this term can be rewritten as $(-1/T) \mathbf{F}_q \cdot (s_v \nabla T + \nabla \mu) = -R_v \mathbf{F}_q \cdot \nabla \ln e$, with R_v being the specific gas constant and e the partial pressure of water vapor. According to the second law of thermodynamics, the internal entropy production must always be positive, $\dot{S}_{\text{irr}} \geq 0$, which sets a key constraint on the physical parameterization for \mathbf{F}_q and \mathbf{F}_T . Other processes, such as radiation or frictional dissipation, can be treated similarly by following the classic approach for a compressible fluid but using the value of T^{**} and μ^{**} for the corresponding thermodynamic potentials.

We have seen that a thermodynamically consistent representation of temperature as the thermodynamic potential in a pseudoincompressible model may be achieved by linearization of the temperature equation of state with respect to the pressure variable, namely

$$T_{\text{psinc}} = T^{**} = T(\bar{p}, S, q) + \delta p \frac{\partial T}{\partial \bar{p}}(\bar{p}, S, q). \quad (27)$$

In particular, taking into account just the leading-order term $T^* = T(\bar{p}, S, q)$ will lead to thermodynamic or energetic inconsistencies.

It is well known that deviations of the pressure from a hydrostatic background distribution $\bar{p}(z)$ in small to mesoscale atmospheric flows scale with some positive power of the flow Mach number [i.e., $\delta p \sim M^\alpha$ with $0 < \alpha \leq 2$; see Klein et al. (2010) for a detailed recent discussion]. Thus, for realistic values $M < 0.1$, first, the linearization in (27) will be a quite accurate approximation to the real temperature; and second, with $|\delta p/\bar{p}| \ll 1$ the approximated temperature T^{**} will generally remain positive and thus represent a meaningful thermodynamic potential. In turn, the pseudoincompressible approximation loses its validity when the perturbation term in (27) is no longer small compared with the leading approximation [i.e., when $|(\partial T/\partial p)(\delta p/T^*)| \rightarrow 1$].

Interestingly, the thermodynamically consistent temperature approximation T^{**} is obtained by combining the result of a leading-order evolution equation for T^* ,

which can be derived straightforwardly from (11), with the result of the determining, in many cases elliptic, equation for δp (see the appendix).

3. Conclusions

A thermodynamically consistent pseudoincompressible model according to the three criteria listed in the introduction has been achieved: As regards the first criterion, the internal energy E_T^* is directly related to the internal energy of the original compressible system in that it is obtained by replacing p with \bar{p} in the equation of state—that is, $E_T^*(z, S, q) \equiv E_T[\bar{p}(z), S, q]$. As regards the second criterion, the conversion between internal and mechanical energies in the case of an adiabatic system is mediated by the term pressure term $\alpha \mathbf{v} \cdot \nabla p$. In the pseudoincompressible case, the pressure field is the sum of the reference pressure and the dynamical pressure perturbation, and the specific volume α includes the leading- and first-order expansion of the density to the pressure perturbation (9). As regards the third criterion, a diabatic process associated with a net heating is related to the change of entropy and water content by $\delta Q = T\dot{S} + \mu\dot{q}$ in the compressible system. In the pseudoincompressible framework, a similar relationship applies, with $\delta Q = T^{**}\dot{S} + \mu^{**}\dot{q}$ in (20), in which the temperature and Gibbs free energy each include a term depending on the pressure perturbation corresponding to first-order accurate approximations to the proper thermodynamic potentials. In doing so, one consistently accounts for how the dynamic pressure affects locally the temperature and Gibbs free energy, and for how this changes the relationship among variations in energy, entropy, and water content. While the discussion here was mostly concerned with the dynamics of a moist atmosphere, the framework proposed here is applicable to any equation of state and can thus be applied to a wide range of fluids.

Pauluis (2008) derives a total energy budget analogous to our (20) for the anelastic system. He argues that this equation may be discretized directly and in conservation form instead of the entropy transport equation so as to guarantee total energy conservation. The situation is somewhat more subtle here as the effective thermodynamic potentials T^{**} and μ^{**} involve the (elliptic) pressure perturbation δp . If one follows the line of development in Klein (2009), however, one may consider the total energy equation as providing the divergence constraint while the mass conservation equation effectively describes entropy transport. Then, in a semi-implicit discretization, the terms involving δp on the right-hand side of (20) will naturally appear as additional diagonal entries in the elliptic pressure equation. We leave

the detailed development of a related numerical method for future work.

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APPENDIX

The Elliptic Pressure Equation

In section 2, in the context of (11), we have claimed the pressure perturbation field δp to satisfy a second-order elliptic equation as a result of the divergence constraint in (13) for externally prescribed source terms \dot{S}, \dot{q} . Here we provide a derivation of this equation.

Taking the time derivative of the divergence constraint in (13) we find

$$\begin{aligned} \frac{\partial}{\partial t}(\nabla \cdot \mathbf{v}) &= (\mathbf{v} \cdot \nabla S - \dot{S}) \left(\frac{\partial^2 \rho^*}{\partial S^2} \dot{S} + \frac{\partial^2 \rho^*}{\partial S \partial q} \dot{q} + \frac{\partial^2 \rho^*}{\partial S \partial \bar{p}} w \frac{d\bar{p}}{dz} \right) \\ &+ (\mathbf{v} \cdot \nabla q - \dot{q}) \left(\frac{\partial^2 \rho^*}{\partial q \partial S} \dot{S} + \frac{\partial^2 \rho^*}{\partial q^2} \dot{q} + \frac{\partial^2 \rho^*}{\partial q \partial \bar{p}} w \frac{d\bar{p}}{dz} \right) \\ &- \frac{\partial \ln \rho^*}{\partial \bar{p}} \frac{d\bar{p}}{dz} \frac{\partial w}{\partial t} - \frac{\partial \ln \rho^*}{\partial S} \frac{\partial \dot{S}}{\partial t} - \frac{\partial \ln \rho^*}{\partial q} \frac{\partial \dot{q}}{\partial t}. \end{aligned} \quad (\text{A1})$$

In the last line on the right-hand side we use the vertical component of the momentum equation in (11b) to replace

$$\frac{\partial w}{\partial t} = -\mathbf{v} \cdot \nabla w - \frac{1}{\rho^*} \frac{\partial \delta p}{\partial z} - g - \frac{1}{\rho^*} \left(1 - \frac{\partial \ln \rho^*}{\partial \bar{p}} \delta p \right) \frac{d\bar{p}}{dz}. \quad (\text{A2})$$

We obtain another expression for $(\nabla \cdot \mathbf{v})_t$ by taking the divergence of (11b), namely

$$\begin{aligned} \frac{\partial}{\partial t}(\nabla \cdot \mathbf{v}) &= -\nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) - \nabla \cdot \left(\frac{1}{\rho^*} \nabla \delta p \right) \\ &- \frac{\partial}{\partial z} \left[\frac{1}{\rho^*} \left(1 - \frac{\partial \ln \rho^*}{\partial \bar{p}} \delta p \right) \frac{d\bar{p}}{dz} \right]. \end{aligned} \quad (\text{A3})$$

Comparison of (A1) with (A2) on the one hand and (A3) on the other yields

$$\begin{aligned}
 \nabla \cdot \left(\frac{1}{\rho^*} \nabla \delta p \right) - \frac{\partial}{\partial z} \left[\frac{1}{\rho^*} \left(\frac{\partial \ln \rho^*}{\partial \bar{p}} \right) \frac{d\bar{p}}{dz} \right] \delta p &= -\nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) - \frac{\partial}{\partial z} \left(\frac{1}{\rho^*} \frac{d\bar{p}}{dz} \right) + \left[\frac{1}{\rho^*} \left(\frac{\partial \ln \rho^*}{\partial \bar{p}} \right) \frac{d\bar{p}}{dz} \right] \frac{\partial \delta p}{\partial z} \\
 &- (\mathbf{v} \cdot \nabla S - \dot{S}) \left(\frac{\partial^2 \rho^*}{\partial S^2} \dot{S} + \frac{\partial^2 \rho^*}{\partial S \partial q} \dot{q} + \frac{\partial^2 \rho^*}{\partial S \partial \bar{p}} w \frac{d\bar{p}}{dz} \right) \\
 &- (\mathbf{v} \cdot \nabla q - \dot{q}) \left(\frac{\partial^2 \rho^*}{\partial q \partial S} \dot{S} + \frac{\partial^2 \rho^*}{\partial q^2} \dot{q} + \frac{\partial^2 \rho^*}{\partial q \partial \bar{p}} w \frac{d\bar{p}}{dz} \right) + \frac{\partial \ln \rho^*}{\partial S} \frac{\partial \dot{S}}{\partial t} + \frac{\partial \ln \rho^*}{\partial q} \frac{\partial \dot{q}}{\partial t} \\
 &- \left(\frac{\partial \ln \rho^*}{\partial \bar{p}} \frac{d\bar{p}}{dz} \right) \left[\mathbf{v} \cdot \nabla w + \frac{1}{\rho^*} \frac{\partial \delta p}{\partial z} + g + \left(\frac{1}{\rho^*} - \frac{1}{\rho^{*2}} \frac{\partial \rho^*}{\partial \bar{p}} \delta p \right) \frac{d\bar{p}}{dz} \right]. \tag{A4}
 \end{aligned}$$

After cancellation of matching terms and collection of all terms involving δp on the left-hand side we obtain the Helmholtz-type elliptic equation:

$$\begin{aligned}
 \nabla \cdot \left(\frac{1}{\rho^*} \nabla \delta p \right) - \left[\frac{\partial}{\partial z} \left(\frac{1}{\rho^*} \frac{\partial \ln \rho^*}{\partial \bar{p}} \frac{d\bar{p}}{dz} \right) + \frac{1}{\rho^*} \left(\frac{\partial \ln \rho^*}{\partial \bar{p}} \frac{d\bar{p}}{dz} \right)^2 \right] \delta p &= -\nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) - \left(\frac{\partial \ln \rho^*}{\partial \bar{p}} \frac{d\bar{p}}{dz} \right) \mathbf{v} \cdot \nabla w \\
 &- (\mathbf{v} \cdot \nabla S - \dot{S}) \left(\frac{\partial^2 \rho^*}{\partial S^2} \dot{S} + \frac{\partial^2 \rho^*}{\partial S \partial q} \dot{q} + \frac{\partial^2 \rho^*}{\partial S \partial \bar{p}} w \frac{d\bar{p}}{dz} \right) \\
 &- (\mathbf{v} \cdot \nabla q - \dot{q}) \left(\frac{\partial^2 \rho^*}{\partial q \partial S} \dot{S} + \frac{\partial^2 \rho^*}{\partial q^2} \dot{q} + \frac{\partial^2 \rho^*}{\partial q \partial \bar{p}} w \frac{d\bar{p}}{dz} \right) \\
 &+ \frac{\partial \ln \rho^*}{\partial S} \frac{\partial \dot{S}}{\partial t} + \frac{\partial \ln \rho^*}{\partial q} \frac{\partial \dot{q}}{\partial t} - \left(\frac{\partial \ln \rho^*}{\partial \bar{p}} \frac{d\bar{p}}{dz} \right) \left(g + \frac{1}{\rho^*} \frac{d\bar{p}}{dz} \right) \\
 &- \frac{\partial}{\partial z} \left(\frac{1}{\rho^*} \frac{d\bar{p}}{dz} \right). \tag{A5}
 \end{aligned}$$

This equation may be simplified somewhat by introducing

$$\frac{d\bar{p}}{dz} = -\bar{\rho}g, \quad \frac{\partial \ln \rho^*}{\partial \bar{p}} = \frac{1}{\rho^* c^{*2}}, \quad c^{*2} = gh^*. \tag{A6}$$

We find

$$\begin{aligned}
 \nabla \cdot \left(\frac{1}{\rho^*} \nabla \delta p \right) + \left\{ \frac{\partial}{\partial z} \left[\frac{g\bar{\rho}}{(\rho^* c^{*2})} \right] - \frac{1}{\rho^*} \left(\frac{g\bar{\rho}}{\rho^* c^{*2}} \right)^2 \right\} \delta p &= -\nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) + \frac{g\bar{\rho}}{\rho^* c^{*2}} \mathbf{v} \cdot \nabla w \\
 &- (\mathbf{v} \cdot \nabla S - \dot{S}) \left(\frac{\partial^2 \rho^*}{\partial S^2} \dot{S} + \frac{\partial^2 \rho^*}{\partial S \partial q} \dot{q} + \frac{\partial^2 \rho^*}{\partial S \partial \bar{p}} w \frac{d\bar{p}}{dz} \right) \\
 &- (\mathbf{v} \cdot \nabla q - \dot{q}) \left(\frac{\partial^2 \rho^*}{\partial q \partial S} \dot{S} + \frac{\partial^2 \rho^*}{\partial q^2} \dot{q} + \frac{\partial^2 \rho^*}{\partial q \partial \bar{p}} w \frac{d\bar{p}}{dz} \right) \\
 &+ \frac{\partial \ln \rho^*}{\partial S} \frac{\partial \dot{S}}{\partial t} + \frac{\partial \ln \rho^*}{\partial q} \frac{\partial \dot{q}}{\partial t} + \frac{g^2}{c^{*2}} \left[\left(\frac{\bar{\rho}}{\rho^*} \right) \left(1 - \frac{\bar{\rho}}{\rho^*} \right) + h^* \frac{\partial}{\partial z} \left(\frac{\bar{\rho}}{\rho^*} \right) \right]. \tag{A7}
 \end{aligned}$$

The right-hand side of this equation involves the flow state variables ρ^* , \mathbf{v} ; the background state pressure and density \bar{p} , $\bar{\rho}$; the diabatic source terms \dot{S} , \dot{q} ; and gradients of these quantities. There are time derivatives only of \dot{S} , \dot{q} but not of ρ^* , \mathbf{v} , or \bar{p} , $\bar{\rho}$. Thus, if \dot{S} and \dot{q} are known prescribed functions of space and time, then (A7) is a second-order elliptic pressure perturbation equation

whose source term is determined given the current flow state and these source term functions.

Note that the perturbation pressure equation may become more complex when molecular or turbulent diffusion are included and when \dot{S} and \dot{q} depend explicitly on the flow variables and their derivatives, especially on the pressure perturbation. Thus it is known,

for instance, that determination of the perturbation pressure field for the incompressible Navier–Stokes equations requires the solution of several separate Poisson equations for some auxiliary quantities and an additional nonstandard elliptic equation that captures boundary effects (see, e.g., Waters et al. 2004). A detailed discussion of this issue is beyond the scope of the present paper.

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