Sources and Sinks of Available Potential Energy in a Moist Atmosphere

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ABSTRACT

Available potential energy (APE) is defined as the difference between the total static energy of the atmosphere and that of a reference state that minimizes the total static energy after a sequence of reversible adiabatic transformations. Determining the rate at which APE is generated in the atmosphere allows one to estimate the amount of kinetic energy that can be generated by atmosphere flows. Previous expressions for the sources and sinks of APE rely on a dry framework and are limited by the fact that they require prior knowledge of the distribution of latent heat release by atmospheric motion. In contrast, this paper uses a moist APE framework to derive a general formula for the sources and sinks of APE that can be equally applied to dry and moist circulations.

Two key problems are addressed here. First, it is shown that any reorganization of the reference state due to diabatic heating or addition of water does not change its total static energy. This property makes it possible to determine the rate of change in APE even in the absence of an analytic formula for the reference state, as is the case in a moist atmosphere. Second, the effects of changing the total water content of an air parcel are also considered in order to evaluate the changes of APE due to precipitation, evaporation, and diffusion of water vapor. Based on these new findings, one can obtain the rate of change of APE from that of atmospheric entropy, water content, and pressure.

This result is used to determine the sources and sinks of APE due to different processes such as external energy sources, frictional dissipation, diffusion of sensible heat and water vapor, surface evaporation, precipitation, and reevaporation. These sources and sinks are then discussed in the context of an idealized atmosphere in radiative–convective equilibrium. For a moist atmosphere, the production of APE by the surface energy flux is much larger than any observational or theoretical estimates of frictional dissipation, and, as is argued here, must be balanced by a comparable sink of APE due to the diffusion of water vapor from unstable to stable air parcels.

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1. Introduction

The maintenance of the general circulation requires that kinetic energy be continuously generated to balance frictional dissipation. To do so, the atmosphere acts as a heat engine that converts internal energy into kinetic energy through a combination of warm, moist air expansion and cold, dry air compression. As with any heat engine, the ability of the atmosphere to produce kinetic energy is closely tied to an energy transport from a warm source, corresponding to the energy flux at the earth's surface, to a cold sink due to the radiative imbalance of the troposphere. Quantitative estimates of the kinetic energy generated by the atmo-

spheric circulation can be obtained by analyzing the entropy budget of the atmosphere. The differential heating of the atmosphere corresponds to a net sink of entropy, which must be balanced by the entropy production due to various irreversible processes taking place in the atmosphere. By identifying the contribution of frictional dissipation to the total irreversible entropy production, one can determine the kinetic energy generated and dissipated by the atmospheric circulation. Rennó and Ingersoll (1996) and Emanuel and Bister (1996) argue that if frictional dissipation is the sole irreversible source in the atmosphere, the entropy budget analysis yields a kinetic energy production and dissipation that is similar to the work produced by a Carnot cycle. However, Pauluis and Held (2002a,b) show that in a moist atmosphere, diffusion of water vapor and irreversible phase transitions account for a large portion of the total entropy production—roughly two-thirds in the simulations discussed in Pauluis and Held

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(2002a). The irreversible entropy source associated with these moist processes reduces the amount of entropy that can be generated by frictional dissipation. As a result, the amount of kinetic energy produced in a moist atmosphere is much lower than what would be expected from a Carnot cycle with the same energy transport.

The results of Pauluis and Held (2002a,b) indicate that the amount of kinetic energy produced by the atmospheric circulation is directly influenced by the intensity of the hydrological cycle and the atmospheric transport of water vapor. The present paper is motivated by the need to address how moist processes affect the generation of available potential energy (APE). In the atmospheric sciences, early discussions of the maintenance of the general circulation were based on the APE budget rather than on the entropy budget. Lorenz (1955) defines the APE as the difference in total static energy (internal plus geopotential) between the current state of the atmosphere and that of an idealized reference state, defined as the state that minimizes the static energy of the atmosphere after a sequence of reversible adiabatic transformations. A reversible transformation here corresponds to a process where there is no irreversible entropy production, and an adiabatic transformation is one that does not involve any external energy source or sink. Hence, the entropy of an air parcel is conserved by reversible adiabatic transformations. By construction, the sum of APE and kinetic energy is conserved under reversible adiabatic redistribution of the air parcels, and the APE itself can be interpreted as an energy reservoir that can be converted into kinetic energy by atmospheric motions. The small-amplitude formulation of the APE derived in Lorenz (1955, 1967) yields a quadratic expression that is commonly used in various discussions of the energetics of atmospheric circulation.

Lorenz (1955, 1967) and Dutton and Johnson (1967) show that external energy sources can act as net sources of APE. In a statistically steady atmosphere, this production of APE is converted into kinetic energy by the circulation, and is then dissipated by friction. At first glance, the descriptions of the maintenance of the atmospheric circulation in terms of the entropy and APE budgets are identical. However, a key problem arises from the fact that the original concept of APE and the corresponding discussion of the maintenance of the atmospheric circulation have been focused on dry circulations.[1] Moisture, if it is taken into consideration at all,

only enters as a prescribed external energy source, accounting for the "latent heat release" associated with phase transitions of water. From a conceptual point of view, latent heat release is not an external process, but corresponds to an internal energy conversion. Thermodynamically, the external energy source corresponds to the evaporation of water at the earth's surface, and not to the condensation of water vapor in the troposphere. In practice, the main limitation of the dry framework lies in that it requires knowledge of the distribution of latent heat release, which itself depends on the circulation, in order to determine the source of APE. Furthermore, in light of the new understanding of the role of moist processes in the entropy budget that has emerged in recent years (see Goody 2000, 2003; Pauluis and Held 2002a,b; Pauluis et al. 2000), a dry framework seems inadequate to properly address the maintenance of the general circulation in the earth's atmosphere.

Lorenz (1978) and Randall and Wang (1992) show that the concept of APE can be applied to a moist atmosphere as long as the transformations leading to the reference state include reversible adiabatic phase transitions and conserve the total water content of each individual parcel. A fundamental difficulty here is that the reference state cannot be derived analytically, but must be determined from an iterative minimization procedure—a graphical one in Lorenz (1978), and a computational algorithm in Randall and Wang (1992). This raises a second difficulty: if the reference state can only be obtained iteratively, how does one determine its evolution, which is necessary to compute the rate of change of APE? Lorenz (1978) discusses this issue, but still relies on a graphical technique to estimate the change of APE. More recently, Bannon (2004, 2005) derives the effects of moisture in the context of various formulations for available energy, and obtains a general formula for the sources and sinks of available energy. However, the reference states used in both papers differ from that used in the Lorenz APE; in Bannon (2004) the reference state corresponds to one where parcels are brought back adiabatically to their initial pressure, while Bannon (2005) uses an arbitrary state of constant temperature and composition. By using these alternative forms for the available energy, Bannon (2004, 2005) is able to avoid the problems associated with changes in the reference state pressure, but the results in Bannon (2004, 2005) cannot be directly applied to the Lorenz APE.

The primary purpose of the present paper is to derive an analytic expression for the rate of change of the Lorenz APE in a moist atmosphere. In section 2a, it is shown that changes in the static energy of the reference

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[1] It should be stressed here that, while several textbooks discuss the general circulation in terms of APE, none provides an expression for the sources and sinks of APE in a moist atmosphere. They rely instead on a formulation that is only valid in the context of a dry atmosphere.

state can be related to a change of the parcel entropy, of total humidity, and of its pressure in the reference state. The central result here is that any reorganization of the reference state that results from either changes in entropy or total water content of air parcels does not affect the reference state enthalpy or the APE. Boer (1976) derives a similar result in the dry framework, but this derivation makes use of the analytic expression for the reference state in a dry atmosphere and cannot be straightforwardly extended to the moist case. Section 2b discusses the APE changes resulting from the addition and removal of water to the atmosphere, which occurs, for example, when water evaporates at the earth's surface or when condensed water precipitates. It is shown here that, in a hydrostatic atmosphere, the change in static energy due to a source or sink of water includes two contributions: a first term given by the enthalpy of the added water enthalpy, and a second term due to the compression of the air below the added mass. This second term integrated over the atmospheric column is equal to the geopotential energy of the water. These new findings make it possible to obtain an expression for the sources and sinks of APE even in the absence of an analytic formula for pressure in the reference state. This expression is equally valid for dry and moist atmospheres and does not require the knowledge of the evolution of the reference state.

In section 3, the sources and sinks of APE are evaluated for various atmospheric processes such as external energy sources, frictional dissipation, surface evaporation, diffusion of sensible heat, diffusion of water vapor, precipitation, and reevaporation. External energy sources generate an amount of APE given by their net energy input multiplied by an efficiency factor equal to the Carnot efficiency of a perfect heat engine acting between the parcel's temperature and its temperature in the reference state. The efficiency factor for evaporation is slightly lower, mainly due to a water loading contribution. For earthlike conditions, the efficiency factor can be as high as one-third. For a surface energy flux of 100 W m^{-2}, this would correspond to a generation of APE of more than 30 W m^{-2}. Such a large APE source is incompatible with either observed or theoretical estimates of the rate at which kinetic energy is dissipated in the atmosphere. It is argued here that such a large source of APE by the surface flux must be balanced by a comparable loss due to the diffusion of water vapor from unstable to stable air masses.

2. Time tendency of the available potential energy

The total energy E in the atmosphere is given by the integral

$$E = \int \left(u + gZ + \frac{1}{2}|\mathbf{V}|^2 \right) dm, \tag{1}$$

where u is the internal energy per unit of mass, g is the gravitational acceleration, Z is the geopotential height, and $\mathbf{V}$ is the three-dimensional velocity. The integral in (1) is taken over all the mass of the atmosphere, with $dm = (1 + r_T)dm_d$ as the total mass of an air parcel, dm_d is the mass of dry air in the air parcel, and r_T is the total water mixing ratio defined as the mass of water in all phases per unit mass of dry air.

The first law of thermodynamics implies that in the absence of external energy sources and exchanges with the lower boundary, the total energy is conserved. If all atmospheric transformations are reversible and adiabatic processes, then the net amount of kinetic energy generated by successive transformations is equal to the net loss in static energy:

$$\Delta \int \frac{1}{2}|\mathbf{V}|^2 \, dm = -\Delta \int (u + gz) \, dm. \tag{2}$$

Since the total amount of static energy in the atmosphere is much larger than the kinetic energy, one would want to determine an upper bound on the amount of static energy that can be converted into kinetic energy. This can be done by defining a reference state as the state that minimizes the total static energy after a set of reversible adiabatic transformations. Because of (2), the reference state also maximizes the kinetic energy. The APE is defined as the difference in total static energy between the atmosphere in its current state and in the reference state:

$$\text{APE} = \int \left[u - u_{\text{ref}} + g(z - z_{\text{ref}}) \right] dm, \tag{3}$$

where u_{ref} and z_{ref} denote the internal energy and geopotential height of a parcel in its reference state. The entropy per unit mass of dry air s and total water mixing ratio r_T are conserved in reversible adiabatic transformations, and the reference state itself is a function of the joint distribution of entropy s and total water mixing ratio r_T. A more detailed description of the properties of the reference state for a moist atmosphere can be found in Lorenz (1978) and Randall and Wang (1992).

A convenient simplification arises when the atmosphere is in hydrostatic balance, the lower boundary is flat, and the upper boundary is at zero pressure. In this case the vertical integral of the static energy is equal to the vertical integral of the enthalpy:

$$\int (u + gz) \, dm = \int h \, dm_d. \tag{4}$$

Here, h is the enthalpy *per unit mass of dry air*, defined as $h = (1 + r_T)(u + \alpha p)$, with α as the specific volume,

and p as the pressure. Lorenz (1955) shows that the reference state is always hydrostatic; thus, the APE of a hydrostatic atmosphere over a flat surface is given by the difference in the total enthalpy:

$$\mathrm{APE} = \int (h - h_{\mathrm{ref}}) \, dm_d, \qquad (5)$$

where h and h_{ref} are the enthalpy per unit mass of dry air in the current and reference states. This paper is focused primarily on the impacts of moisture, and will therefore be limited for simplicity to the treatment of a hydrostatic atmosphere over a flat surface, in which (5) is exactly equal to the Lorenz APE. For further information, the reader can consult Shepherd (1993) and Bannon (2005) who discuss the available energy in a nonhydrostatic atmosphere, and Boer (1989) who provides the expression of the APE in the presence of topography.

The rate of change of APE is obtained by taking the time derivative of (5), and is given by the difference between the rate of change of the total enthalpy of the atmosphere and the rate of change of the total enthalpy of the reference state:

$$\frac{d}{dt} \mathrm{APE} = \int \frac{dh}{dt} \, dm_d - \int \frac{dh_{\mathrm{ref}}}{dt} \, dm_d. \qquad (6)$$

On the right-hand side of (6), (d/dt) refers to the Lagrangian derivative following a (dry air) parcel's trajectory. The problem is thus to find an expression for the changes in the enthalpy of the current and reference states. In a multiple-component system like moist air, it is useful to distinguish between *open* and *closed* transformations, depending on whether a given parcel exchanges mass with its surroundings. Diffusion of heat, radiative heating, evaporation of cloud water, and expansion and compression are examples of closed transformations where the total amount of water is conserved. In contrast, evaporation at the earth's surface, diffusion of water vapor, and precipitation are open transformations, in which the amount of water in an air parcel varies. Closed transformations are discussed in section 2a, while open transformations are discussed in section 2b.

a. Closed transformations

Consider first the rate of change of APE for closed transformations only. The fundamental thermodynamics relationship between enthalpy, pressure, entropy, and chemical composition indicates that the rate of change of enthalpy can be expressed as

$$\frac{dh}{dt} = T \frac{ds}{dt} + \Delta\mu \frac{dr}{dt} + \alpha_d \frac{dp}{dt}. \qquad (7)$$

Here, r is the mixing ratio (defined as the mass of water vapor per unit mass of dry air), $\Delta\mu = \mu_v - \mu_l = R_v T \ln H$ is the difference between the chemical potential of water vapor μ_v and the chemical potential of liquid water μ_l, with H as the relative humidity and R_v as the gas constant of water vapor, and α_d as the specific volume of dry air. As only closed transformations are considered in this section, the mixing ratio for total water r_T is constant.

Similarly, the enthalpy change in the reference state is given by

$$\frac{dh_{\mathrm{ref}}}{dt}(t_0) = T_{\mathrm{ref}} \frac{ds}{dt} + \Delta\mu_{\mathrm{ref}} \frac{dr_{\mathrm{ref}}}{dt} + \frac{d}{dt} \left(\int_{p_{\mathrm{ref}}(t_0)}^{p_{\mathrm{ref}}(t_0+t)} \alpha_d \, dp \right). \qquad (8)$$

As before, the subscript "ref" indicates a quantity evaluated in the reference state. The third term on the right-hand side corresponds to the rate at which the parcel enthalpy would change when moved from its reference pressure at time t to its reference pressure at time $t + dt$. If p_{ref} is a smooth function of time, the third term is equal to $\alpha_{d,\mathrm{ref}}(dp_{\mathrm{ref}}/dt)$. However, p_{ref} can be a discontinuous function of time, and the integral notation is kept here as a reminder that from a mathematical point of view, the time derivatives in (8) must be taken in a weak sense. Discontinuity in the reference pressure occurs for example when a stable parcel with a reference level near the surface becomes conditionally unstable with a reference level at a much higher altitude. In such situations, both the enthalpy h_{ref} and reference pressure p_{ref} are discontinuous.

A first simplification arises in (8) from the fact that in the reference state a parcel is either unsaturated with no condensed water present and $r_{\mathrm{ref}} = r_T$, or it is saturated with $\Delta\mu_{\mathrm{ref}} = 0$. Hence, for a closed transformation, the second term on the right-hand side of (8) always vanishes:

$$\Delta\mu_{\mathrm{ref}} \frac{dr_{\mathrm{ref}}}{dt} = 0.$$

Combining (7) and (8) into (6) yields

$$\frac{d}{dt} \mathrm{APE} = \int \left[(T - T_{\mathrm{ref}}) \frac{ds}{dt} \right] dm_d + \int \left[\Delta\mu \frac{dr}{dt} \right] dm_d$$

$$+ \int \left(\alpha_d \frac{dp}{dt} \right) dm_d$$

$$- \int \left[\frac{d}{dt} \left(\int_{p_{\mathrm{ref}}(t_0)}^{p_{\mathrm{ref}}(t_0+t)} \alpha_d \, dp \right) \right] dm_d. \qquad (9)$$

The first term on the right-hand side includes the generation of APE by external energy sources, while the second term accounts for a loss of APE resulting from irreversible phase transitions. These will be discussed in greater detail in section 3. The third term on the right-hand side can be written as

$$\int \left(\alpha_d \frac{dp}{dt}\right) dm_d = \int \frac{\partial p}{\partial t} dx\, dy\, dz + \int (\mathbf{V} \cdot \mathbf{\nabla} p)\, dx\, dy\, dz$$

$$= \int \left[z \frac{\partial p}{\partial t}\right]_{z=0}^{z=\infty} dx\, dy - \int z \frac{\partial^2 p}{\partial t \partial z} dx\, dy\, dz$$

$$+ \int [\mathbf{\nabla} \cdot (p\mathbf{V})]\, dx\, dy\, dz$$

$$- \int (p\mathbf{\nabla} \cdot \mathbf{V})\, dx\, dy\, dz$$

$$= \frac{d}{dt} \int (\rho g z)\, dx\, dy\, dz - \int (p\mathbf{\nabla} \cdot \mathbf{V})\, dx\, dy\, dz$$

$$= \frac{d\Phi}{dt} - W = -C_{\mathrm{APE} \rightarrow \mathrm{KE}} \qquad (10)$$

after replacing the integral over the mass by a volume integral, with $\alpha_d dm_d = dx\,dy\,dz$. Equation (10) indicates that the contribution of the total pressure change is equal to the difference between the rate of change in geopotential energy $(d\Phi/dt) = (d/dt)(\int \rho g z\ dx\ dy\ dz)$ and the work W performed by the atmosphere on itself, and thus accounts for the conversion of APE into kinetic energy $C_{\mathrm{APE} \rightarrow \mathrm{KE}}$.

The main difficulty here lies with the fourth term on the right-hand side of (9):

$$- \int \left[\frac{d}{dt} \left(\int_{p_{\mathrm{ref}(t_0)}}^{p_{\mathrm{ref}(t_0+t)}} \alpha_d\, dp \right) \right] dm_d.$$

This term corresponds to the change in APE associated with a vertical reordering of the parcels in the reference state. From a physical point of view, this term can be interpreted as the amount of mechanical work necessary for reorganize the parcels in the reference state. Boer (1976; see also Egger 1976) shows that this term vanishes in the dry framework. However, Boer's proof uses the analytic expression of the reference state in a dry atmosphere and cannot be straightforwardly extended to the moist case. An additional difficulty arises from the possibility that in a moist atmosphere the reference pressure can jump from one level to another, as discussed above. However, it will now be shown that

the fact that the reference state is minimizes the total static energy guarantees that the change in APE associated with a reordering of the reference state always vanishes, making the computation of (dp_{ref}/dt) unnecessary.

Consider two atmospheric reference states A and C. For any parcel, the enthalpy difference between the two states is given by

$$h_C - h_A = \int_A^C T\, ds + \int_A^C \Delta\mu\, dr + \int_A^C \alpha_d\, dp$$

$$= \int_A^C T\, ds + \int_A^C \alpha_d\, dp, \qquad (11)$$

with $\int_A^C$ indicating an integral following a thermodynamic path between the two reference states. The integral of $\Delta\mu dr$ vanishes since phase transitions are reversible in the reference state. The transformation between A and C can follow an infinite number of paths. One possibility is to define an intermediary state B that has the same pressure distribution as A, but the same entropy as C. The transformation between A and C can thus be subdivided into two transformations: the first one is a change of entropy at constant pressure (stage A–B), and the second one is a reversible adiabatic pressure adjustment (stage B–C). In this case, we have

$$\int (h_B - h_A)\, dm_d = \int \left(\int_A^B T\, ds \right) dm_d \quad \text{and}$$

$$(12)$$

$$\int (h_C - h_B)\, dm_d = \int \left(\int_B^C \alpha_d\, dp \right) dm_d. \qquad (13)$$

By definition, C is the reference state with the same entropy and total water content as B. Therefore, the total static energy is smaller in state C than in state B, and the integral

$$\int \left(\int_B^C \alpha_d\, dp \right) dm_d \leq 0 \qquad (14)$$

is nonpositive. Indeed, if it were positive, B would have a lower total enthalpy than C. Similarly, state D is constructed as to have the same pressure distribution as C and the same entropy distribution as A. Hence, we have

$$\int (h_D - h_C)\, dm_d = \int \left(\int_C^D T\, ds \right) dm_d \quad \text{and}$$
(15)

$$\int (h_A - h_D)\, dm_d = \int \left(\int_D^A \alpha_d\, dp \right) dm_d.$$
(16)

As A is the reference state with the same entropy and total water content as in state D, we have

$$\int \left(\int_D^A \alpha_d\, dp \right) dm_d \le 0.$$
(17)

Adding (12)–(13) and (15)–(16) yields

$$\int \left(\int_B^C \alpha_d\, dp + \right) dm_d + \int \left(\int_D^A \alpha_d\, dp \right) dm_d$$

$$= -\int \left(\int_A^B T\, ds + \int_C^D T\, ds \right) dm_d.$$
(18)

The constraints (14) and (17) imply that the left-hand side must be nonpositive.

Instead of the cycle A–B–C–D, one can subdivide the path between A and C into $N + 1$ intervals by selecting a set of intermediary reference states $A_1, A_2, \ldots A_n$. The state B_i has with the same pressure distribution as A_i and the same entropy as in A_{i+1}. Similarly, the state D_i is constructed with the same pressure distribution as A_{i+1}, and the same entropy distribution as A_i. We can now construct the cycle $A = A_0, B_0, A_1, \ldots B_N, C = A_{N+1}, D_N, A_N, \ldots D_0, A = A_0$. Figure 1 shows a schematic representation for this cycle with two intervals. The equivalent of Eq. (18) includes now the integrals taken over $2N + 2$ segments:

$$\sum_{i=0}^{N} \left[\int \left(\int_{B_i}^{A_{i+1}} \alpha_d\, dp \right) dm_d \right] + \sum_{i=0}^{N} \left[\int \left(\int_{D_i}^{A_i} \alpha_d\, dp \right) dm_d \right]$$

$$= -\int \left[\sum_{i=0}^{N} \left(\int_{A_i}^{B_i} T\, ds \right) + \sum_{i=0}^{N} \left(\int_{A_{i+1}}^{D_i} T\, ds \right) \right] dm_d.$$
(19)

By increasing the number of subintervals N, the terms on the right-hand side converge toward the integral following a path of reference states:
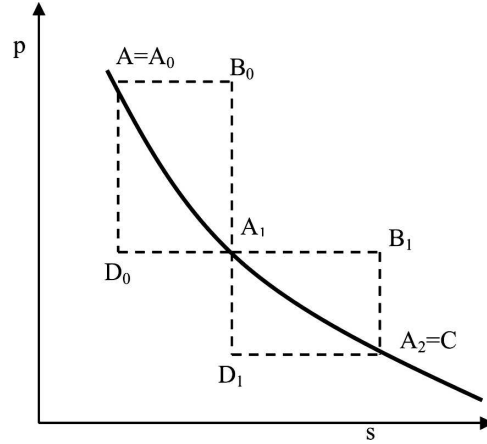

FIG. 1. Schematic diagram for the transformations discussed in section 2a. The figure illustrates a two-step transformation for a single parcel between the initial reference state A and the final reference state C. The horizontal axis is entropy, and the vertical axis is pressure. The thick curve corresponds to parcel properties in the different reference states between A and C (see section 2a for details).

$$\sum_{i=0}^{\infty} \left(\int_{A_i}^{B_i} T\, ds \right) = \sum_{i=0}^{\infty} \left(\int_{D_i}^{A_{i+1}} T\, ds \right) = \int_A^C T_{\text{ref}}\, ds,$$
(20)

where the ref subscript is used here to emphasize the fact that an integral is taken along a path of reference states. Equation (20) implies that in the limit of number of intervals between A and C going to infinity, $N \to \infty$, the right-hand side of (19) vanishes:

$$\sum_{i=0}^{\infty} \left[\int \left(\int_{B_i}^{A_{i+1}} \alpha_d\, dp \right) dm_d \right]$$

$$+ \sum_{i=0}^{\infty} \left[\int \left(\int_{D_i}^{A_i} \alpha_d\, dp \right) dm_d \right] = 0.$$
(21)

Since all states A_i are reference states, each term in the sum is nonpositive:

$$\int \left(\int_{B_i}^{A_{i+1}} \alpha_d\, dp \right) dm_d \le 0 \quad \text{and}$$

$$\int \left(\int_{D_i}^{A_i} \alpha_d\, dp \right) dm_d \le 0.$$
(22)

The infinite sum vanishes if and only if each of these integrals vanishes individually. In particular, we have

$$\sum_{i=0}^{\infty} \left[\int \left(\int_{B_i}^{A_{i+1}} \alpha_d \, dp \right) dm \right]$$

$$= \sum_{i=0}^{\infty} \left[\int \left(\int_{D_i}^{A_i} \alpha_d \, dp \right) dm_d \right] = 0. \qquad (23)$$

As long as we follow a path of reference states, the total enthalpy change due to the reordering of the reference state is identically zero. This is also true if the states A and C correspond to the reference states at times t_0 and $t_0 + dt$. In particular, the contribution of the pressure readjustment of the reference state to the APE vanishes:

$$\int \left[\frac{d}{dt} \right] \left(\int_{p_{\mathrm{ref}}(t_0)}^{p_{\mathrm{ref}}(t_0+t)} \alpha_d \, dp \right) dm_d = 0. \qquad (24)$$

It should be stressed here that, while (24) guarantees that the integral of the pressure term over the whole atmosphere vanishes, the contribution of the pressure term for a specific parcel can be nonzero.

From a mathematical point of view, this result can be seen as a direct consequence of the fact that the reference state minimizes the total static energy of the atmosphere for reversible adiabatic transformations. The cancellation of the pressure term in (24) is very similar to a property of constrained extremum. Given two smooth functions $F(\mathbf{X})$ and $G(\mathbf{X})$, with $\mathbf{X}$ a N-dimensional vector, so that F has an extremum at X_0 under the constraint that $G(\mathbf{X}) = 0$, then the directional derivative of F in any direction allowed by the constraint $G(\mathbf{X}) = 0$ must vanish:

$$\sum_i \frac{\partial F}{\partial x_i}(X_0) dx_i = 0 \quad \text{for} \quad \forall dx \in \mathfrak{R}^N \quad \text{such that}$$

$$\sum_i \frac{\partial G_k}{\partial x_i}(X_0) dx_i = 0.$$

This property is commonly used to validate the Lagrange method in calculus (see, e.g., Courant 1934, chapter 3). In the context of the previous discussion, the function F can be interpreted as the total enthalpy of the atmosphere, while the set of constraints G_k corresponds to the requirement that the entropy and total water of each parcel be conserved, and that the atmosphere is in hydrostatic balance. The fact that F has a local minimum under the constraint G implies that the total enthalpy F remains unchanged for any infinitesimal redistribution of the parcels near a reference state X_0 that minimizes F. The derivation of Eq. (24) can be viewed as a specialized proof of this property of con-

strained extremum for a case with an infinite number of degrees of freedom and with the various derivatives expressed in terms of thermodynamic variables.

From a physical point of view, Eq. (24) can be interpreted as stating that reorganization of the parcels between different reference states does not produce or consume any kinetic energy. Indeed, consider the evolution of an atmosphere that remains at all times in the reference state corresponding to its current entropy and humidity distribution. In this case, the pressure term in (24) is exactly to the amount of enthalpy that is converted into kinetic energy. However, as by definition the reference state minimizes the total static energy, the air parcels must be reorganized as soon as an infinitesimally small amount of kinetic energy could be produced through such reorganization. It is thus impossible to convert enthalpy into kinetic energy in an atmosphere that remains at all times in its reference state. Similarly, kinetic energy cannot be converted into enthalpy, as it would imply that the reverse transformation would be able to extract kinetic energy.

The cancellation of the contribution of the pressure term (24) can be used with Eqs. (9) and (10) to yield the expression for the rate of change of APE under closed transformations:

$$\frac{d}{dt} \mathrm{APE} = \int \left[(T - T_{\mathrm{ref}}) \frac{ds}{dt} \right] dm_d + \int \left(\Delta\mu \frac{dr}{dt} \right) dm_d$$

$$- C_{\mathrm{APE \to KE}}. \qquad (25)$$

b. Open transformations

In open transformations, water is either added or removed. The rate of change of enthalpy per unit mass of dry air is given by

$$\frac{dh}{dt} = T \frac{ds}{dt} + \mu_v \frac{dr}{dt} + \mu_l \frac{dr_l}{dt} + \alpha_d \frac{dp}{dt}, \qquad (26)$$

where $r_l = r_T - r$ is the liquid water mixing ratio, $\mu_v = h_v - Ts_v = C_l[T - T_0 - \ln(T/T_0)] + R_v \ln H$ and $\mu_l = h_l - Ts_l = C_l[T - T_0 \ln(T/T_0)]$ are the chemical potentials of water vapor and liquid water, $h_v = C_l(T - T_0) + L_v$ and $h_l = C_l(T - T_0)$ are the specific enthalpies of water vapor and liquid water, and $s_v = C_l \ln(T/T_0) + (L_v/T) - R_v \ln H$ and $s_l = C_l \ln(T/T_0)$ are the specific entropies of water vapor and liquid water. Here, T_0 is an arbitrary reference temperature, and C_l is the specific heat of liquid water, assumed here to be independent of temperature. For simplicity, the discussion here assumes that the temperature of the liquid water is the same as that of the surrounding air, and the treatment is limited here to water in either liquid or gas phase. The inclusion of an ice phase would require the addi-

tion of an additional variable for the ice mixing ratio but is otherwise straightforward. Note that for a closed transformation, the rate of change of the liquid water and water vapor are related by $(dr_l/dt) = -(dr/dt)$, and Eq. (26) reduces to Eq. (7).

When computing the rate of change in the reference state, a convenient simplification arises from the fact that the amount of liquid water in a parcel can only change when the parcel is saturated in the reference state. As at saturation, the chemical potentials of liquid water and water vapor are equal, the contribution of the chemical potential in (26) can be rewritten as

$$\mu_{v,\text{ref}} \frac{dr_{\text{ref}}}{dt} + \mu_{l,\text{ref}} \frac{dr_{l,\text{ref}}}{dt} = \mu_{v,\text{ref}} \frac{dr_T}{dt}. \quad (27)$$

The terms involving the changes in entropy and both mixing ratios can be computed directly, but changes in pressure must be treated carefully. Consider the case where water is added or removed in some parcel while the atmosphere remains hydrostatic at all times. In this case, the rate of change of pressure in a parcel is given by the weight of the total mass added above the parcel, divided by the area A of the parcel:

$$\frac{dp(m_d)}{dt} = \frac{1}{A} \int\limits_{p(m'_d) < p(m_d)} \left(g \frac{dr_T(m'_d)}{dt} \right) dm'_d. \quad (28)$$

The integral here is taken on all the parcels m'_d located directly above the parcel m_d.

Integrating the rate of change for enthalpy $\alpha_d(dp/dt)$ over the whole atmosphere yields

$$\int \left(\alpha_d(m_d) \frac{dp(m_d)}{dt} \right) dm_d$$

$$= \int \left[\alpha_d(m_d) A^{-1} \int\limits_{p(m'_d) < p(m_d)} \left(g \frac{dr_T(m'_d)}{dt} \right) dm'_d \right] dm_d$$

$$= \int \left[g \frac{dr_T(m'_d)}{dt} \int\limits_{p(m_d) > p(m'_d)} A^{-1}(\alpha_d(m_d)) \, dm_d \right] dm'_d$$

$$= \int \left[\frac{dr_T(m'_d)}{dt} gz \right] dm'_d. \quad (29)$$

The rate of change of enthalpy integrated over the entire atmosphere is exactly equal to the rate of change of the geopotential energy due to the addition of water.

The impacts of the reorganization of the reference state can be treated in a similar fashion as in the case of the closed transformations in section 2a. Changes in the total enthalpy between two reference states A and C can be split into two segments by introducing an inter-

mediate state B. The state B is in hydrostatic balance, and the parcels in state B occupy the same relative position as in the initial state A, but their entropy and total mixing ratio are the same as in the final state C. In the first stage of the transformation between A and B, the entropy and total humidity of the parcels are changed while keeping the relative position of the parcels unchanged (and maintaining the hydrostatic balance). The change in enthalpy is obtained by combining (26) with (28) and (29):

$$\int (h_B - h_A) \, dm_d$$

$$= \int \left[\int_A^B T \, ds_d + \int_A^B (\mu_v + gz) \, dr_T \right] dm_d. \quad (30)$$

This is the equivalent of Eq. (12) in section 2a. The term involving the change in the total mixing ratio accounts both for the effects of the chemical potential and for the changes in the hydrostatic pressure field resulting from the addition of mass [Eq. (28)].

In the second part of the transformation, the parcels are redistributed by reversible adiabatic transformations to their new reference level:

$$\int (h_C - h_B) \, dm_d = \int \left(\int_B^C \alpha_d \, dp \right) dm_d. \quad (31)$$

As in the proof given in section 2a, the path between A and C can be subdivided into a sequence of smaller transitions by passing through a set of intermediate reference states A_i. The intermediary states B_i and D_i are constructed in a similar way to those in section 2a. The state B_i has the same entropy and total water distribution as A_{i+1}, but the parcels are located at the same relative position as in A_i. The state D_i has the same entropy and total water distribution as A_i, but the parcels are located at the same relative position as in A_{i+1}. One can then show that because the reference state minimizes the total enthalpy, the change in the total enthalpy that results from the reorganization of the reference state must vanish. In essence, the steps from (12) to (24) can be reproduced for open transformations. The sole difference is that in addition to the contribution of entropy changes Tds, there is also a contribution due to changes in the total water mixing ratio $(\mu_v + gz)dr_T$ in Eqs. (12), (15), and (18)–(20).

The changes in enthalpy in the reference state is thus given by taking the integral (29) along a path of the reference state, and the rate of change in APE is given by

TABLE 1. Sources and sinks of APE in an idealized radiative–convective equilibrium (see the text for details).

Process	APE contribution
Surface sensible heat flux	$(d/dt)\mathrm{APE}|_{\mathrm{sen}} \approx 3.3 \ \mathrm{W \ m^{-2}}$
Surface evaporation	$(d/dt)\mathrm{APE}|_{E} \approx 26.8 \ \mathrm{W \ m^{-2}}$
Radiative cooling	$(d/dt)\mathrm{APE}|_{\mathrm{rad}} \approx 0.5 \ \mathrm{W \ m^{-2}}$
Precipitation	$(d/dt)\mathrm{APE}|_{\mathrm{prec}} \approx 3.6 \ \mathrm{W \ m^{-2}}$
Reevaporation	$-3 \ \mathrm{W \ m^{-2}} < (d/dt)\mathrm{APE}|_{\mathrm{re\text{-}ev}} < 3 \ \mathrm{W \ m^{-2}}$
Diffusion	$(d/dt)\mathrm{APE}|_{\mathrm{diff}} \approx -\varepsilon[(d/dt)\mathrm{APE}|_{\mathrm{sen}} + (d/dt)\mathrm{APE}|_{E}]$

$$
\frac{d}{dt}\mathrm{APE} = \int \left((T - T_{\mathrm{ref}}) \frac{ds}{dt} \right) dm_d
$$

$$
+ \int \left(\mu_v \frac{dr}{dt} + \mu_l \frac{dr_l}{dt} - \mu_{v,\mathrm{ref}} \frac{dr_T}{dt} \right) dm_d
$$

$$
+ \int \left(g(z - z_{\mathrm{ref}}) \frac{dr_T}{dt} \right) dm_d - C_{\mathrm{APE \to KE}}.
$$

$$
(32)
$$

For a closed transformation, the total mixing ration is constant, with $(dr_t/dt) = 0$ and $(dr_l/dt) = -(dr/dt)$, and Eq. (32) reduces to Eq. (25). The addition of water to an air parcel affects the APE in three distinct ways. First, the change of entropy (ds/dt) must include the added water entropy. Second, the term proportional to the chemical potential of the water accounts for the different impacts of the addition of water vapor to a saturated or unsaturated air parcel. Finally, the third integral on the right-hand side of (32) represents the changes in the hydrostatic pressure in the parcels below the added water, and its total impact is equal to the geopotential energy of the added water.

3. Sources and sinks of APE

Equation (32) can be used to compute the rate of change of the APE for various atmospheric phenomena. While the APE itself is a property of the distribution of entropy and water vapor of the whole atmosphere, the production of APE can be computed for changes in individual parcels, independent of the transformations taking place in the rest of the atmosphere. This section discusses the APE changes associated with seven atmospheric processes: external energy source, frictional dissipation, surface evaporation, diffusion of heat and of water vapor, precipitation, and reevaporation of condensed water. The APE sources and sinks are also estimated for an idealized radiative–convective equilibrium. The estimated sources and sinks of APE

for different processes are shown in Table 1. The discussion here is meant to be illustrative, and the complete computation of the APE sources and sinks are currently under way.

a. External energy sources

An external energy source Q, corresponding to a convergence of a sensible heat flux from the surface or of a convergence of the radiation flux, is associated with a rate of change of entropy $(ds/dt) = (Q/T)$, while the water content remains unchanged $(dr_T/dt) = 0$. The rate of change of the APE obtained from (32) is equal to

$$
\frac{d}{dt}\mathrm{APE}\bigg|_{\mathrm{ext}} = \int Q \frac{T - T_{\mathrm{ref}}}{T} dm_d, \qquad (33)
$$

with Q expressed here as the heating rate per unit mass of dry air. The rate of change of APE is equal to the heating rate multiplied by an efficiency factor. This efficiency factor is the same as that of a perfect heat engine acting between the parcel's temperature and its reference state. In the "dry" framework, Lorenz (1967) and Dutton and Johnson (1967) show that the efficiency factor is $(\pi - \pi_{\mathrm{ref}}/\pi)$ with $\pi = (p/p_0)^{(R/C_p)}$ and $\pi_{\mathrm{ref}} = (p_{\mathrm{ref}}/p_0)^{(R/C_p)}$ being the values of the Exner function for the parcel in its current state and in the reference state. In a dry atmosphere, the temperature and Exner function are related by $T = \pi\theta$ and $T_{\mathrm{ref}} = \pi_{\mathrm{ref}}\theta$, where θ is the potential temperature of the parcel. For a dry atmosphere, the efficiency factor is the same whether it is computed in terms of the Exner function or temperature. For a moist atmosphere however, a parcel's potential temperature is not necessarily the same in the current and reference states, and the efficiency factor must be computed using the temperature as in (33).

For an energy source at the surface, the efficiency factor can be quite large, up to 1/3 for an air parcel at $T \sim 300$ K and with a reference temperature $T_{\mathrm{ref}} \sim 200$ K. The efficiency factor is, however, highly sensitive to the parcel's properties. In particular, heating in a parcel whose reference level remains near the surface would generate little or no APE. The efficiency factor is negative when the reference temperature is higher than the current temperature. In such a case, heating would reduce the APE, while cooling would increase it. In particular, tropospheric cooling acts as a source of APE when the cooling occurs in parcels whose reference level is lower than their current locations.

b. Frictional dissipation

Frictional dissipation is associated with a net irreversible entropy production equal to $(ds/dt) = (D/T)$, with

D the rate at which kinetic energy is dissipated. The rate of change of APE is

$$\frac{d}{dt}\mathrm{APE}\bigg|_{\mathrm{diss}} = \int D\,\frac{T - T_{\mathrm{ref}}}{T}\,dm_d. \qquad (34)$$

The frictional heating resulting from frictional dissipation can be associated with a source of APE. Since the efficiency factor is always less than 1, frictional heating always produces less APE than the amount of kinetic energy dissipated, and the sum of the APE and total kinetic energy decreases. Rennó and Ingersoll (1996) provide a more detailed discussion of this partial recycling of the frictional heating and of its impacts on the atmospheric energy cycle.

c. Surface evaporation

The rate of change of entropy due to the addition of water vapor at a rate (dr_T/dt) is given by $(ds/dt) = s_v(dr_T/dt) = [C_l \ln(T/T_0) + (L_v/T) - R_v \ln H](dr_T/dt)$. After expanding the expression for the chemical potential, Eq. (32) becomes

$$\frac{d}{dt}\mathrm{APE}\bigg|_E = \int L_v(T)\,\frac{T - T_{\mathrm{ref}}}{T}\,\frac{dr_T}{dt}\,dm_d$$

$$+ \int C_l\left(T - T_{\mathrm{ref}} - T_{\mathrm{ref}}\ln\frac{T}{T_{\mathrm{ref}}}\right)\frac{dr_T}{dt}\,dm_d$$

$$+ \int R_v T_{\mathrm{ref}}(\ln H - \ln H_{\mathrm{ref}})\,\frac{dr_T}{dt}\,dm_d$$

$$+ \int g(z - z_{\mathrm{ref}})\,\frac{dr_T}{dt}\,dm_d. \qquad (35)$$

The first term on the right-hand side is similar to the contribution of an external energy source in (33), and is proportional to the latent heat source. The second term on the right-hand side of (35) accounts for the increased heat capacity of the air parcel due to an increase in water content. A Taylor expansion of $\ln(T/T_{\mathrm{ref}})$ yields

$$C_l\left(T - T_{\mathrm{ref}} - T_{\mathrm{ref}}\ln\frac{T}{T_{\mathrm{ref}}}\right) \approx \frac{1}{2}\,C_l\,\frac{(T - T_{\mathrm{ref}})^2}{T_{\mathrm{ref}}}. \qquad (36)$$

The second term on the right-hand side of (35) is thus approximately equal to the rate of work performed by a perfect heat engine transporting energy at a rate given by $\frac{1}{2}C_l(T - T_{\mathrm{ref}})(dr_T/dt)dm_d$ over a temperature difference $(T - T_{\mathrm{ref}})$.

The third term accounts for the variations of the chemical potential for water vapor. If the parcel is saturated in the reference state (i.e., the relative humidity in the reference state is $H_{\mathrm{ref}} = 1$) then the loss of APE is proportional to the irreversible entropy production due to evaporation:

$$R_v T_{\mathrm{ref}}(\ln H - \ln H_{\mathrm{ref}})\,\frac{dr_T}{dt} = -T_{\mathrm{ref}}\Delta S_{\mathrm{irr}} \qquad (37)$$

with ΔS_{irr} as the irreversible entropy production rate associated with the evaporation of water in unsaturated air. The fourth term corresponds to the work required to lift the additional water from the surface to its reference level.

For a quantitative example, consider evaporation in an air parcel at the surface, with $T = 300$ K, $H = 0.8$, $Z = 0$, and with a reference state in the upper troposphere corresponding to $T_{\mathrm{ref}} = 200$ K, $H_{\mathrm{ref}} = 1$, $Z_{\mathrm{ref}} = 15\,000$ m. The contribution of the latent heating is $(T - T_{\mathrm{ref}}/T)L_v \approx 810$ kJ kg^{-1}, the impact of the irreversible entropy production is $R_v T_{\mathrm{ref}}\ln H \approx -20$ kJ kg^{-1}, the contribution of the specific heat of liquid water amounts to $C_l[T - T_{\mathrm{ref}} - T_{\mathrm{ref}}\ln(T/T_{\mathrm{ref}})] \approx 80$ kJ kg^{-1}, and the water loading is $g(Z - Z_{\mathrm{ref}}) \approx -147$ kJ kg^{-1}. The latent heat term dominates (35), though the combined contribution of the other terms cannot be entirely neglected. The overall effect is a change of APE on the order of 720 kJ kg^{-1} of water vapor added to the atmosphere. The efficiency factor of the latent heat flux, defined as the ratio of the APE production to the surface flux of latent heat, is slightly smaller than that for a sensible heat flux, primarily due to the water loading term. As for an external energy source, the net effect of the evaporation is highly sensitive to a parcel reference state. In particular, evaporation into a "stable" parcel with a reference state at the earth surface would not change the APE.

d. Diffusion of heat

A molecular flux of sensible heat acts as an energy sink for the parcels where the flux is divergent, and an energy source for parcels where the flux is convergent. If diffusion extracts energy at a rate Q from the parcel A and transport it to parcel B, the rate of change of the APE can be obtained (32):

$$\frac{d}{dt}\mathrm{APE}\bigg|_{\mathrm{diff,sen}} = \left(\frac{T_{\mathrm{ref},A}}{T_A} - \frac{T_{\mathrm{ref},B}}{T_B}\right)Q\,dm_d$$

$$\approx \frac{T_{\mathrm{ref},A} - T_{\mathrm{ref},B}}{T}\,Q\,dm_d, \qquad (38)$$

where it has been assumed that the parcel temperatures are similar $T_A \approx T_B \approx T$.

In a dry atmosphere, the reference temperature of an air parcel is determined by its potential temperature. If diffusion takes place between two parcels at the same

pressure, the term $(T_{\mathrm{ref},A} - T_{\mathrm{ref},B}/T)$ is proportional to the buoyancy difference between the air parcels. This implies that, outside very limited circumstances, diffusion of heat will be limited to exchange between air parcels with similar reference temperatures.

In a moist atmosphere, the reference temperature is a function of both the entropy and humidity of a parcel. Two parcels can have the same specific volume at a given pressure and still have very different reference temperatures. For example, in the boundary layer, an unstable air parcel, with a reference level in the upper troposphere, can have the same specific volume as a stable parcel with a reference level near the earth's surface. These two parcels would have the same buoyancy, and can remain close to each other for a long period of time. This makes it possible to have significant diffusion of heat between parcels of different reference temperatures. In these circumstances, Eq. (36) indicates that diffusion of heat could have a fairly large impact on APE.

e. Diffusion of water vapor

A diffusive flux of water vapor corresponds to a sink of water vapor in the parcels where the flux is divergent, and a source of water vapor in the parcels where the flux is convergent. If molecular diffusion transports water vapor at a rate (dr_T/dt) from a parcel A to a parcel B, the APE changes can be obtained by applying Eq. (35). As discussed in section 3c, the contribution of the latent heat in (35) dominates, and the rate of change of the APE can be approximated by

$$
\begin{aligned}
\frac{d}{dt}\mathrm{APE}\bigg|_{\mathrm{diff},v} &\approx \left(\frac{T_{\mathrm{ref},A}}{T_A} - \frac{T_{\mathrm{ref},B}}{T_B}\right) L_v \frac{dr_T}{dt} dm_d \\
&\approx \frac{T_{\mathrm{ref},A} - T_{\mathrm{ref},B}}{T} L_v \frac{dr_T}{dt} dm_d. \qquad (39)
\end{aligned}
$$

The dominant contribution of the diffusion of water vapor is the same as the APE change due to a sensible heat flux of the same amplitude as the latent heat transport. Since the diffusive flux of latent heat is usually an order of magnitude larger than that of sensible heat, diffusion of water vapor has the potential to be the largest sink of APE in the atmosphere. For example, consider an ascending updraft where diffusion results in a net loss of water vapor of $dr_T \approx 0.001$ kg kg^{-1}. If the reference temperature of the ascending air is $T_{\mathrm{ref},A} \approx 200$ K and the water vapor is diffused in a stable environment with $T \approx T_{\mathrm{ref},B} \approx 270$ K, then the loss of APE from (37) would be approximately 650 J kg^{-1} of dry air in the updraft. For comparison, a typical value of the

convective available potential energy (CAPE) in the Tropics is on the order of 1–2 kJ kg^{-1}, which can be destroyed by a loss of water vapor in the ascending air on the order of $dr_T \approx -0.003$ kg kg^{-1}.

f. Precipitation

Falling precipitation can be treated as a transfer of the precipitation water from one air parcel to another, until it reaches the ground. In this case, the rate of change of the total water mixing ratio would be given by $(dr_T/dt) = \alpha_d[\partial(\rho_p v_T)/\partial z]$, where ρ_p is the mass of precipitation per unit volume, and v_T is the terminal velocity of the precipitation. A simpler method used here, albeit more approximate, is to remove the condensed water when the precipitation forms. The rate of change of entropy is $(ds/dt) = s_l(dr_T/dt)$. The rate of change of APE is obtained from Eq. (32):

$$
\begin{aligned}
\frac{d}{dt}\mathrm{APE}\bigg|_{\mathrm{Prec}} &= \int C_l\left(T - T_{\mathrm{ref}} - T_{\mathrm{ref}} \ln \frac{T}{T_{\mathrm{ref}}}\right) \frac{dr_T}{dt} dm_d \\
&+ \int g(z - z_{\mathrm{ref}}) \frac{dr_T}{dt} dm_d. \qquad (40)
\end{aligned}
$$

Precipitation affects the APE both through its effect on the heat capacity [the first term on the right-hand side of (40)] and geopotential (the second term). For precipitation falling out of a convective updraft, one can assume that the reference temperature of the parcel is lower than the current temperature $T > T_{\mathrm{ref}}$ while its reference height is higher than the current location $Z < Z_{\mathrm{ref}}$. In this case, the reduction in heat capacity reduces the APE, while the reduction in water loading increases the APE. The positive contribution of the water loading term can be seen as the result of the increase in buoyancy due to a reduction of water loading. Its effect is equal to the difference in geopotential energy between the removed water's current and reference states. A quantitative comparison between the two terms in (40) indicates that the water loading is the dominant effect.

Equation (40) indicates that the removal of the precipitating water results in a net increase of the APE. This increase in APE is not balanced by any equivalent reduction of the kinetic energy, and corresponds to an increase in the sum of the APE and the kinetic energy. In contrast, the sum of APE and kinetic energy in a dry atmosphere can only increase in the presence of external energy sources. Hence, the fact that precipitation can increase the APE is an important way in which dry and moist atmospheres differ.

The effect of precipitation on APE can be better

understood by looking at the impact of adding water to a parcel at the earth's surface, lifting the parcel to a higher level, and removing the water. If the ascent is reversible, it does not change the parcel reference level. The net effect of water loading is a reduction of APE: $d\text{APE} = g(z_{\text{in}} - z_{\text{out}})dr_T dm_d$. This is the amount of work required to lift the water to the level at which it is removed. If the precipitation falls through the air at rest, this is also the amount of kinetic energy dissipated in the microscopic shear zones surrounding the falling precipitation, as discussed in Pauluis et al. (2000).

g. Reevaporation

Consider that due to the reevaporation of precipitation, the mixing ratio of a parcel increases at a rate (dr_T/dt). For an adiabatic reevaporation, the parcel entropy increases at a rate given by $(ds/dt) = (s_l - R_v \ln H)(dr_T/dt)$. Here, the first term corresponds to the entropy increase due to the addition of liquid water, while the second term is the entropy increase due to the irreversible adiabtic evaporation. The change in APE can be obtained by applying (32):

$$\frac{d}{dt}\text{APE}\bigg|_{\text{RE}} = \int R_v T_{\text{ref}}(\ln H - \ln H_{\text{ref}})\frac{dr_T}{dt}dm_d$$
$$+ \int C_l\left(T - T_{\text{ref}} - T_{\text{ref}}\ln\frac{T}{T_{\text{ref}}}\right)\frac{r_T}{dt}dm_d$$
$$+ \int g(z - z_{\text{ref}})\frac{dr_T}{dt}dm_d. \qquad (41)$$

The rate of change of APE due to reevaporation differs from the APE source due to surface evaporation in that it does not include the first term of (35) corresponding to the latent heat source.

By assuming that the air parcel is also unsaturated in its reference state $r = r_{\text{ref}} = r_T$, and approximating the partial pressure of water vapor by $e \approx (R_v/R_d)R_T p$, one can express the relative humidity term in (41) as

$$R_v T_{\text{ref}}(\ln H - \ln H_{\text{ref}}) \approx R_v T_{\text{ref}}\left(\ln\frac{p}{p_{\text{ref}}}\right) - R_v T_{\text{ref}}\left(\ln\frac{e_s}{e_{s,\text{ref}}}\right)$$
$$\approx -\frac{R_v}{R_d}g(z - z_{\text{ref}}) + L_v\frac{T - T_{\text{ref}}}{T}. \qquad (42)$$

The last approximation takes advantage of the facts that the hydrostatic balance can be written as $(\partial \ln p/\partial z) \approx (-g/R_d\overline{T})$, and that the Clausius–Clapeyron relationship can be approximated by $\ln[e_s(T_1)/e_s(T_2)] \approx (L_v/R_v)[(1/T_2) - (1/T_1)]$. Substituting (42) into (41), and neglecting the second integral yields

$$\frac{d\text{APE}}{dt}\bigg|_{\text{RE}} \approx \left[L_v\frac{T - T_{\text{ref}}}{T} - g\frac{R_v}{R_d}(z - z_{\text{ref}})\right.$$
$$\left. + g(z - z_{\text{ref}})\right]\frac{dr_T}{dt}dm_d. \qquad (43)$$

The first term is the work that would be produced by a Carnot cycle transporting the latent heat from the reference temperature to its current temperature. The second term on the right-hand side of (43) corresponds to the work required to compress the newly evaporated water from its current level to its reference level. The third term is the net gain of geopotential energy resulting from the addition of water at a higher level than its reference state. As $R_v > R_d$, the contribution of the second term dominates the third term.

For an air parcel with a reference level lower than the current level, reevaporation increases the APE. Conversely, if the reference level is higher than the current level, for example in the case of reevaporation in the subcloud layer, then reevaporation of precipitation decreases the APE. It should be stressed here that, similar to precipitation in section 3f, reevaporation in an unsaturated downdraft can result in a net increase of not only the APE but also of the sum of the APE and kinetic energy, without any external source of energy. This is impossible in a dry atmosphere in which the sum of the APE and kinetic energy can only be increased by external energy source.

It is useful here to compare the effects of reevaporation and diffusion of water vapor on the APE. Consider a rising updraft in an unsaturated environment. Some condensed water can fall from the updraft into the environment and reevaporate. A similar effect can be achieved if water vapor diffuses directly from the updraft to the environment. From a thermodynamic point of view, reevaporation and diffusion are irreversible and are associated with exactly the same entropy production. Yet, from the point of view of APE, diffusion of water vapor would be associated with a large destruction of APE, as discussed in section 3e, while the reevaporation can potentially increase APE. The difference arises from the fact that diffusion reduces the upward energy transport by the updraft, while reevaporation can increase the net upward energy transport by generating a cold downdraft. This example indicates that the production or destruction of APE is not directly related to the dissipative nature of the process.

h. Radiative–convective equilibrium

To provide a quantitative example, the sources and sinks of APE for idealized radiative–convective equilibrium are considered here. Consider a quasi-steady

atmosphere, heated at the surface with a surface latent heat flux of 90 W m^{-2}, and a sensible heat flux of 10 W m^{-2}. This is balanced by a net tropospheric cooling of 100 W m^{-2}. The surface temperature is 300 K, and the tropopause temperature is 200 K. These numbers are illustrative of tropical conditions. The reference state can be constructed by lifting the "unstable" parcels from the boundary layer to the tropopause, while the rest of the atmosphere—the stable parcels—is compressed by the ascent of the unstable parcels. The portion of the atmosphere that rises to the upper troposphere cannot be determined a priori, but it is reasonable to assume here that it corresponds to the mass of the subcloud layer, approximately 50 mb.

For the surface flux, one can assume that the surface energy fluxes are transferred across a thin surface layer to unstable air parcels. In this case, the effective efficiency for the sensible heat flux is 1/3, and the source of APE is $(d/dt)\mathrm{APE}|_{\mathrm{sen}} = 3.3$ W m^{-2}. Using the same value as the example in section 3b for the latent heat flux, the source of APE associated with the latent heat flux is $(d/dt)\mathrm{APE}|_E = 26.8$ W m^{-2}. As noted in section 3c, the efficiency factor is slightly lower for the latent heat flux than for the sensible heat flux. The effect of the radiative cooling depends on its vertical structure and on the portion of the atmosphere that is unstable. If the mass of unstable parcels corresponds to an atmospheric layer of 50 mb, the reference temperature of a stable parcel is higher by approximately 5 K than its current temperature. For a uniformly distributed radiative cooling, the energy loss in the stable parcels is 95 W m^{-2}, which translates into a source of APE of approximately 2 W m^{-2}. The rest of the radiative cooling, 5 W m^{-2} acts on unstable parcels, with $T \sim 300$ K and $T_{\mathrm{ref}} \sim 200$ K, and removes about 1.7 W m^{-2} of APE. The net contribution of the radiative cooling is small, less than $(d/dt)\mathrm{APE}|_{\mathrm{rad}} < 0.5$ W m^{-2}. There is a near cancellation between the cooling of the stable and unstable parcels, which results from the choice of a uniform cooling rate and the fact that the mean temperature of the reference state is typically close to the mean atmospheric temperature.

The contribution of internal processes requires further assumptions, and should thus be viewed as tentative until further study. While precipitation is a dissipative mechanism, it is shown in section 3e that it acts as a source of APE when precipitation forms in parcels that are lower than their reference level. In our example, assuming that the precipitation falls on average from 5000 m yields a source of APE given by $(d/dt)\mathrm{APE}|_{\mathrm{prec}} \sim Pg(z_{\mathrm{ref}} - z) \sim 3.6$ W m^{-2}, with P the precipitation rate. Reevaporation can act either as a source or as a sink of APE depending on where it oc-

curs. Taking $\ln H \sim \ln H_{\mathrm{ref}} \sim 1$ in (42) yields a scale for the rate of change of the APE due to reevaporation $(d/dt)\mathrm{APE}|_{\mathrm{re-ev}} \sim PR_vT \sim 3$ W m^{-2}. The sign of this contribution is unclear, and this value most likely overestimates the net effect of reevaporation.

The impact of diffusion can be obtained if one considers that a fraction ε of the surface fluxes is diffused from unstable parcels to stable parcels. In this case, the APE destroyed by diffusion of heat and water vapor would be given by

$$\frac{d}{dt}\mathrm{APE}\Big|_{\mathrm{diff}} \approx -\varepsilon \frac{d}{dt}\mathrm{APE}\Big|_{\mathrm{sfc}},$$

with $(d/dt)\mathrm{APE}|_{\mathrm{sfc}} = (d/dt)\mathrm{APE}|_{\mathrm{sen}} + (d/dt)\mathrm{APE}|_E$ being the contribution of the surface fluxes.

The APE production due to the external energy sources and sinks is close to $(d/dt)\mathrm{APE}|_{\mathrm{sfc}} \approx 30$ W m^{-2}. This is extremely large in comparison to both the observed atmospheric dissipation of 2–4 W m^{-2} (see, e.g., Peixoto and Oort 1993) and the results from numerical simulations of radiative–convective equilibrium such as Pauluis and Held (2002a) where the kinetic energy production and dissipation in convective motions is about 1 W m^{-2}. This difference cannot be explained by the contribution of reevaporation or precipitation (the latter is a source of APE in any case). The only significant sink of APE is the diffusion of sensible heat and water vapor. The net APE source due to the surface heat flux and diffusion is approximately

$$\frac{d}{dt}\mathrm{APE}\Big|_{\mathrm{sfc}} + \frac{d}{dt}\mathrm{APE}\Big|_{\mathrm{diff}} \approx Q_{\mathrm{surf}}\frac{T_{\mathrm{surf}} - T_{\mathrm{trop}}}{T_{\mathrm{surf}}}(1 - \varepsilon),$$

$$(44)$$

where Q_{surf} is the total surface energy flux, and T_{surf} and T_{trop} are the surface and tropopause temperatures, respectively. (The contribution of evaporation has been approximated to that of an external energy source.) The difference between the large APE source due to surface flux and the weak observed dissipation can only be reconciled if, after diffusion is taken into account, only a small fraction of the surface fluxes is used to increase the energy of stable parcels, which implies $\varepsilon \sim 0.8 \ldots 1$.

4. Discussion

In this paper, an analytic formula for the sources and sinks of APE has been derived. A key element of this derivation lies in proving that the contribution due to the reorganization of air parcels in the reference state vanishes. This is a direct consequence of the choice of a reference state that minimizes the total static energy of the atmosphere, and is therefore valid only in the con-

text of the Lorenz APE framework. Despite the widespread use of APE to discuss the maintenance of the atmospheric circulation, this is the first time that an analytic expression for the production and dissipation of APE in a moist atmosphere has been explicitly derived.

The expression for the APE production is such that the contribution of individual processes can be identified, independently of the transformations taking place in other parts of the atmosphere. This has been used here to discuss the contribution of external energy sources, frictional dissipation, diffusion of sensible heat, surface evaporation, precipitation, diffusion of water vapor, and reevaporation of water.

The production of APE by an external energy source is equal to the net heat source multiplied by an efficiency factor. The efficiency factor is equal to the efficiency of a Carnot cycle with an energy source at the current parcel temperature and an energy sink at the parcel temperature in its reference state. For an energy source at the surface, the same energy flux will generate more APE if it occurs in a parcel with low reference temperature than in an air parcel with high reference temperature. This indicates that the generation of APE is maximized for heating in "unstable" air parcels (i.e., parcels whose reference level is situated in the upper troposphere), while heating "stable" parcels (i.e., whose reference level is situated near the surface) has little impact on the APE.

A comparison between "dry" and "moist" processes indicates that the differences are related to the effects of the water loading, the heat capacity of liquid water, and to the chemical potential difference between water vapor and liquid water. These contributions are in general small when an energy source or transport is involved. For example, the efficiency factor associated with a latent heat flux is found to be only slightly smaller than that of a sensible heat flux. The primary difference between dry and moist atmospheres lies not in the expression for the APE sources and sinks, but rather in the relative importance of the different processes taking place.

Indeed, it is argued here that diffusion can potentially play a dominant role in a moist atmosphere. Diffusion of either sensible or latent heat between two parcels at the same level corresponds to a sink of APE that is proportional to the difference between the parcels' reference temperature. For a dry atmosphere, the direct relationship between potential temperature, density, and reference temperature should prevent diffusion between parcels with large differences in their reference temperature. In contrast, in a moist atmosphere, parcels with the same specific volume and pressure can

have a very different entropy and thus reference temperature. For this reason, one expects mixing and diffusion to play a major role in the APE budget of a moist atmosphere.

Estimates of the APE production for an idealized atmosphere in radiative–convective equilibrium have been considered. For typical conditions, the surface energy flux produces a very large source of APE if all the heating occurs in unstable air parcels, close to 30 W m^{-2} for a surface flux of 100 W m^{-2}. This APE source is much larger than typical estimates of the work produced and dissipated by atmospheric motions. It is argued here that a large fraction of the surface energy flux must be diffused into stable parcels. If only a small fraction $1 - \varepsilon$ of the surface flux heats up unstable parcels, the generation of APE can be approximated by

$$\frac{d}{dt}\text{APE} \approx Q_{\text{surf}} \frac{T_{\text{surf}} - T_{\text{trop}}}{T_{\text{surf}}} (1 - \varepsilon). \qquad (45)$$

The net generation of APE due to the surface energy flux and diffusion is strongly controlled by how much mixing takes place between stable and unstable air in the boundary layer. The same surface flux would produce very little APE in subsidence regions, where there is little to no unstable air and ε is close to 1, but can produce tremendous amounts of APE in convectively active regions such as hurricane eyewalls.

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