Thermodynamic Consistency of the Anelastic Approximation for a Moist Atmosphere

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ABSTRACT

The primary goal of this paper is to validate the use of the anelastic approximation for fluids with a complex equation of state such as moist air or seawater. The anelastic approximation is based on a leading-order expansion of the equations of motion for a compressible fluid in terms of density. Its application to atmospheric flows has been based on a dry framework that treats phase transitions as an external energy source. However, cloudy air is more accurately described as a two-phase fluid in which condensed water and water vapor are in thermodynamic equilibrium. Thermodynamic equilibrium reduces to three the number of state variables necessary to describe the thermodynamic state of moist air, and leads to a discontinuity in the partial derivatives of the equation of state at the saturation point. A version of the anelastic approximation for a moist atmosphere is derived here by considering the atmospheric density as a small perturbation from a moist-adiabatic reference profile, and using moist entropy and total water content as prognostic variables, with buoyancy determined from the complete nonlinear equation of state.

The key finding of this paper is that this implementation of the anelastic approximation conserves energy. The total energy is equal to the sum of the kinetic energy and the thermodynamic energy. The latter is found to be equal to the sum of the enthalpy and geopotential energy of the parcel. Furthermore, the state relationships between this thermodynamic energy, entropy, and other state variables are the same as those for moist air after replacing the total pressure with the reference state pressure. This guarantees that, as long as the pressure perturbation remains small, the thermodynamic behavior of a fluid under the anelastic approximation is fully consistent with both the first and second laws of thermodynamics.

Two implications of this finding are also discussed. First, it is shown that the first and second laws of thermodynamics constrain the vertically integrated buoyancy flux. This is equivalent to deriving the total work performed in a compressible atmosphere from its entropy and energy budgets. Second, it is argued that an anelastic model can be built with temperature or enthalpy as a prognostic variable instead of entropy. The rate of change for this new state variable can be obtained from energy conservation, so that such a model explicitly obeys the first law of thermodynamics. The entropy in this model is equal to the entropy of the parcel evaluated at the reference pressure, and its evolution obeys the second law of thermodynamics.

1. Introduction

Within the earth's atmosphere, water can be found in gaseous, liquid, and solid states. Water molecules undergo multiple phase transitions as they are transported by atmospheric flows. These phase transitions are associated with significant release or absorption of latent heat, and play an important role in the atmospheric circulation. For example, evaporation of water accounts for approximately 80% of the energy exchange between the atmosphere and oceans (Peixoto and Oort 1992). Phase transitions are at the core of the complex interplay between thermodynamics and fluid dynamics that is an intrinsic aspect of many weather phenomena ranging from convection to hurricanes and midlatitude storms.

Moist air can be treated as an ideal mixture of dry air, water vapor, and condensed water. The state of moist air can be uniquely described by the combination of entropy S, total pressure p, specific humidity q_v, and liquid water content q_l. However, not all possible combinations of these four state variables can be realized within the atmosphere. Indeed, air parcels within the lower 100 km of the atmosphere are considered to be in local thermodynamic equilibrium. This assumption implies that, when water vapor and condensed water are

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both present, the partial pressure of the water vapor in the parcel is equal to the saturation vapor pressure. Consistently with Gibbs's phase rule, the requirement that the different water phases be in thermodynamic equilibrium reduces by one the number of variables necessary to describe the state of moist air. Here, we introduce the total water content $q_T = q_v + q_l$ as the mass of water in all phases per unit mass of moist air, so that the state of a moist air parcel is uniquely determined by its entropy S, pressure p, and total water content q_T. In particular, the partitioning of the total water content q_T between the water vapor q_v and liquid water q_l can be obtained once the three state variables, S, p, and q_T, are known. For example, at constant pressure and total water content, higher entropy corresponds to higher temperature and water vapor content but lower condensed water content. At sufficiently large entropy, low pressure, or low total water content, all of the water will be in the vapor phase, with no condensed water present. Air parcels with only water vapor and no condensed phase are referred to as unsaturated parcels, while parcels with both phases present are referred to as saturated parcels.

A key feature of moist convection arises from the fact that parcels behave differently depending on whether they are saturated or not. This fact can be illustrated by one of its commonly observed repercussions in tropical meteorology. The lapse rate is defined as minus the derivative of the temperature with height: $\Gamma = -(\partial T/\partial z)$. For a typical tropical sounding, it is roughly 0.01 K m^{-1} near the surface but decreases abruptly to a value on the order of 0.004 K m^{-1} above the cloud base. This abrupt change in the lapse rate is a reflection of the difference in behavior between unsaturated and saturated air parcels. The temperature profile in regions of active convection in the tropics follows closely that of a parcel raised adiabatically from the surface. For such an adiabatic ascent, the temperature T is a function of the entropy S, total water content q_T, and pressure p:

$$T = T(S, q_T, p).$$

Below the cloud base, the lapse rate corresponds to the adiabatic expansion of an unsaturated parcel with

$$\Gamma_u = -\left(\frac{\partial T}{\partial z}\right)_{S,q_T} = \rho g \left(\frac{\partial T}{\partial p}\right)_{S,q_T} = \frac{g}{C_p}, \quad (1)$$

with ρ the mass of air per unit volume, g the gravitational acceleration, and C_p the heat capacity at constant pressure. The second equality arises from the hydrostatic balance: $(\partial p/\partial z) = -\rho g$. Above the cloud base, the lapse rate corresponds to the adiabatic expansion of

the same parcel that is now saturated. Emanuel (1994) and others have computed the lapse rate for a saturated ascent:

$$\Gamma_m = -\left(\frac{\partial T}{\partial z}\right)_{S,q_T} \approx \frac{g}{C_p} \frac{1 + \dfrac{L_v q_T}{R_d T}}{1 + \dfrac{L_v^2 q_T (1 + q_T/\epsilon)}{C_p R_v T^2}}, \quad (2)$$

where L_v is the latent heat of vaporization; R_d and R_v are the ideal gas constants of dry air and water vapor, respectively; and $\epsilon = (R_d/R_v)$ is their ratio. The approximation lies in neglecting the contribution of the condensed water and replacing the mixing ratio by the specific humidity. The saturated lapse rate is always lower than the unsaturated lapse rate because the cooling due to the parcel's expansion is partially compensated for by the latent heat release due to condensation. Observations in the tropics reflect the fact that, for a parcel raised adiabatically, the lapse rate is given by the unsaturated value Γ_u below the cloud base, and abruptly drops to its saturated value Γ_m above it.

The abrupt change in the adiabatic lapse rate is one example of the fact that multiphase fluids often exhibit a discontinuity in the partial derivative in the state law at saturation. Consider a generic state function $F(S, p, q_v, q_l)$ with q_v the specific humidity and q_l the liquid water content. The state property F is a smooth function and can be determined for any combination of S, p, q_v, and q_l. However, as discussed above, not all of these combinations can be realized due to the requirement that moist air be in local thermodynamic equilibrium. For an unsaturated air parcel, only water vapor is present, $q_v = q_T$ and $q_l = 0$. Partial derivatives must be taken assuming that $dq_l = 0$ and $dq_v = dq_T$; for example,

$$\left(\frac{\partial F}{\partial p}\right)_{S,q_T,\text{unsaturated}} = \left(\frac{\partial F}{\partial p}\right)_{S,q_v = q_T, q_l = 0}.$$

In contrast, for a saturated parcel with both water vapor and liquid water present, the thermodynamic equilibrium between the two phases is expressed by requiring the chemical potential of both water vapor μ_v and liquid water μ_l to be equal: $\mu_v = \mu_l$. In this case, partial derivatives are taken assuming both $dq_v + dq_l = dq_T$ and $d\mu_v = d\mu_l$; for example,

$$\left(\frac{\partial F}{\partial p}\right)_{S,q_T,\text{saturated}} = \left(\frac{\partial F}{\partial p}\right)_{S,q_v + q_l = q_T, \mu_v - \mu_l = 0}.$$

Local thermodynamic equilibrium leads to different constraints on saturated and unsaturated parcels. Par-

tial derivatives are taken along different directions in four-dimensional space (S, q_v, q_l, p) depending on whether the air is saturated or unsaturated. As a result, the partial derivatives of the state equation are discontinuous at the saturation point.

The Boussinesq approximation (Oberbeck 1879; Boussinesq 1903; Spiegel and Veronis 1960; Mihaljan 1962) offers a practical simplification of the Navier–Stokes equations for a compressible fluid by replacing the continuity equation with a nondivergence constraint on the mass flux. This not only simplifies the analytic treatment of the equations, but also filters out fast-propagating sound waves. The use of the Boussinesq approximation to study atmospheric motion is limited by the fact that it requires the vertical extent of the domain to be small in comparison to the density-scale height. This problem is addressed in the anelastic approximation by allowing for a height-dependent reference density profile (Batchelor 1953; Ogura and Phillips 1962; Dutton and Fitchl 1969; Lipps and Hemler 1982; Durran 1989; Vallis 2006). The original derivations of the Boussinesq and anelastic approximations require the atmosphere to be close to a dry isentropic reference state in hydrostatic balance, and an equation of state based on the ideal gas law, with density proportional to the potential temperature.

The primary motivation of this work is to ensure that the use of the anelastic approximation is fully consistent even for a fluid exhibiting a highly nonlinear thermodynamic state law such as moist air. Section 2 discusses an implementation anelastic approximation for a moist atmosphere. It assumes that the atmospheric density distribution is a small perturbation around a hydrostatic reference profile with constant entropy and total water content. The impacts of density and pressure on the momentum equation are linearized, so that density fluctuations enter the momentum equations only as a vertical buoyancy acceleration. A central feature of this version of the anelastic approximation is that it keeps the full nonlinear dependency of the buoyancy on the entropy and total water content.

The central result presented in section 3 can be seen as an extension of Ingersoll's (2005) work on nonlinear equations of state. It is shown that the anelastic approximation conserves the sum of the kinetic and thermodynamic energy. The thermodynamic energy here is equal to the difference between the enthalpy of the parcel estimated at the reference pressure and the enthalpy of the reference state at the same level. The temperature and chemical potential are given by the temperature and chemical potential of a moist air parcel with the same entropy and total water content but

they are estimated at the reference pressure. Furthermore, the thermodynamic relations between these various quantities under the anelastic approximation are the same as these for moist air except that the pressure has been replaced by the reference state pressure. As the anelastic approximation discussed here uses entropy as a prognostic variable, this results implies that this anelastic model simultaneously obey the first and second laws of thermodynamics.

Sections 4 and 5 address some implications of this result. For example, analysis of the entropy and energy budgets such as that in Pauluis and Held (2002) can provide an estimate of the work produced by the atmosphere. In section 4, it is shown that, under the anelastic approximation, the same considerations yield a constraint on the vertically integrated buoyancy flux. This is illustrated here with a simple example that compares a direct calculation of the buoyancy flux with the predictions from the thermodynamic constraints. Section 5 discusses the possibility of using enthalpy or temperature as a prognostic variable instead of entropy, with the evolution equation obtained from the thermodynamic energy equation. The entropy in this model would be given by the entropy of the parcel evaluated at the reference state pressure. Such a model, while explicitly obeying the first law of thermodynamics, would also implicitly obey the second law.

2. The anelastic approximation

We are interested here in deriving the anelastic approximation for a fluid in which the specific volume $\alpha(S, q_T, p)$ is an arbitrary function of three state variables: here, the entropy S, total water content q_T and pressure p. As discussed in the introduction, moist air can be treated as an ideal mixture of dry air, water vapor, and condensed water, with the water phases in thermodynamic equilibrium.[1] This later requirement implies that the partitioning of the total water between its gas and condensed phases can be uniquely determined from the three state variables S, p, and q_T, consistently with Gibb's phase rule. The state law for moist

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[1] Precipitation is made of large water droplets and ice crystals that are not in thermodynamic equilibrium with the surrounding air. Condensed water in the anelastic model discussed here is limited to small cloud droplets and crystals for which the thermodynamic equilibrium assumption is appropriate. A very rough treatment of rain and snow within this framework is to altogether remove the condensed water when it starts falling. A more comprehensive approach requires the addition of at least one state variable for precipitating water, but does not introduce any particular difficulty as far as the anelastic approximation is concerned.

air exhibits a strong nonlinear behavior in that its partial derivatives are discontinuous at the saturation point. In this section, we derive a version of the anelastic approximation that maintains the nonlinear dependency of the specific volume on both the entropy and total water content.

In the anelastic approximation, the atmospheric state is considered to be a small perturbation from an atmospheric reference state. Here, the original approach to the anelastic approximation (Ogura and Phillips 1962) is followed by requiring a reference state with uniform entropy S_0 and total water content q_{T0}. The entropy here includes contributions from the water vapor and condensed water. The reference state pressure $p_0(z)$ is in hydrostatic balance:

$$\alpha_0 \frac{\partial p_0}{\partial z} = -g, \quad (3)$$

with $\alpha_0(z) = \alpha[S_0, q_{T0}, p_0(z)]$ the specific volume in the reference state. The total pressure is then given by $p(x, y, z, t) = p_0(z) + \delta p(x, y, z, t)$, and the specific volume is expanded as

$$\alpha(S, q_T, p) = \alpha_0(z) + \delta\alpha(S, q_T, p_0)$$
$$+ \left(\frac{\partial \alpha}{\partial p}\right)_{S,q_T} \delta p + 0(\delta p^2). \quad (4)$$

The contributions of the pressure and specific volume terms in the momentum equations are linearized:

$$\frac{D\mathbf{u}}{Dt} = -\alpha\boldsymbol{\nabla}p - g\mathbf{k}$$
$$= -\alpha_0\boldsymbol{\nabla}\delta p + g\mathbf{k}\alpha_0^{-1}\delta\alpha(S, q_T, p_0)$$
$$+ g\mathbf{k}\alpha_0^{-1}\left(\frac{\partial \alpha}{\partial p}\right)_{S,q_T} \delta p + 0(\delta\alpha^2, \delta p^2). \quad (5)$$

Note that in (5), the dependency of the specific volume on pressure has been linearized but the full nonlinear dependency of the specific volume on the two other states variables, S and q_T, has been kept. This has some specific advantages over versions based on a fully linearized equation of state. First, while the pressure and specific volume perturbations must be small for the expansion of the momentum equation to remain accurate, a nonlinear equation of state allows for large fluctuations of the entropy or the total water content as long as their combined impacts on the specific volume is small. Second, this makes it possible to handle discontinuous partial derivatives and other complex aspects of the equation of state. Finally, we will establish in section 3 that preserving the nonlinearity dependency of the spe-

cific volume on entropy and the total water content also preserves some key thermodynamics relationships.

To obtain an energy-conserving form for the anelastic approximation, it is necessary to approximate the partial derivative of the specific volume with respect to the pressure by that of the reference state:

$$\left(\frac{\partial \alpha}{\partial p}\right)_{S,q_T}(S, q_T, p_0) = \left(\frac{\partial \alpha}{\partial p}\right)_{S,q_T}(S_0, q_{T0}, p_0). \quad (6)$$

In this case, the two terms involving the pressure perturbation can be combined into a single-gradient term:

$$-\alpha_0\boldsymbol{\nabla}\delta p + g\mathbf{k}\alpha_0^{-1}\left(\frac{\partial \alpha}{\partial p}\right)_{S,q_T}\delta p = -\alpha_0\boldsymbol{\nabla}\delta p - g\mathbf{k}\left(\frac{\partial \alpha_0}{\partial z}\right)$$
$$= -\alpha_0\boldsymbol{\nabla}\delta p - \delta p\boldsymbol{\nabla}\alpha_0$$
$$= -\boldsymbol{\nabla}(\alpha_0\delta p). \quad (7)$$

It must be stressed here that for a moist atmosphere, the partial derivatives are discontinuous between saturated and unsaturated regions. The difference between the saturated and unsaturated values depends on the specific humidity, and can be up to 30% of the unsaturated value for earth-like conditions. The error introduced by the assumption (6) is of the same magnitude as the error resulting from the inclusion of the reference state potential temperature in the pressure gradient term in Lipps and Hemler (1982). Physically, this corresponds to neglecting the difference in sound speed between the saturated and unsaturated environments. The recombination (7) of the two pressure terms into a single gradient term cannot be motivated on dimensional grounds alone and should, thus, be viewed as an heuristic treatment. It is nevertheless a key step in ensuring an energy-conserving version of the anelastic approximation.

The anelastic momentum equation can thus be written as

$$\frac{D\mathbf{u}}{Dt} = -\boldsymbol{\nabla}(\alpha_0\delta p) + \mathbf{k}B, \quad (8)$$

where the buoyancy B is given by

$$B[S, q_T, p_0(z)] = B(S, q_T, z)$$
$$= g\frac{\alpha[S, q_T, p_0(z)] - \alpha_0(z)}{\alpha_0(z)}. \quad (9)$$

The linearization of the momentum equation is based on the Taylor expansion of the specific volume α and pressure terms p, and is only accurate for small fluctuations in these quantities.

In the anelastic approximation, the continuity equa-

tion is replaced by the requirement that the reference state mass flux be nondivergent:

$$\boldsymbol{\nabla} \cdot (\rho_0 \mathbf{u}) = 0. \tag{10}$$

For an adiabatic flow, the terms neglected in the continuity equation can be shown to scale as the square of the Mach number Mach $= (U/c_s)$, with U a velocity scale for the flow and c_s the speed of the sound. Hence, only flows with low Mach numbers (Mach $\ll 1$) can be accurately represented under the anelastic approximation. This requirement is separate from the previous requirement that the density fluctuations be small.

In addition to the momentum and continuity equations, one requires two prognostic equations for the entropy and the water content:

$$\frac{DS}{Dt} = \dot{S} \quad \text{and} \tag{11}$$

$$\frac{Dq_T}{Dt} = \dot{q}_T. \tag{12}$$

While the exact form of the entropy Eq. (11) has been deliberately left open, it is assumed here that it should be determined by the second law of thermodynamics. The validity of the anelastic approximation requires that the rate of change of both entropy and the total water content must induce a small rate of change in the density (in comparison to the change implied by the advective tendency in the continuity equation). The anelastic system thus includes the three-dimensional momentum Eq. (8), a state law for buoyancy (9), the reduced continuity Eq. (10), and the thermodynamic equations for entropy (11) and total water (12).

The kinetic energy equation is obtained by multiplying the momentum equation by $\rho_0 \mathbf{u}$:

$$\frac{\partial \left(\rho_0 \frac{|\mathbf{u}|^2}{2} \right)}{\partial t} + \boldsymbol{\nabla} \cdot \left(\rho_0 \frac{|\mathbf{u}|^2}{2} \mathbf{u} \right) = -\boldsymbol{\nabla} \cdot (\mathbf{u} \delta p) + \rho_0 B w. \tag{13}$$

The equation differs from the kinetic energy for a compressible fluid in that it uses the density of the reference state rather than the actual density of the fluid, and in that the net production of kinetic energy is given by the buoyancy flux $\rho_0 B w$.

3. Thermodynamic energy in the anelastic approximation

Given the form of the kinetic energy equation in the anelastic approximation (13), one would like to obtain a formulation for the thermodynamic energy that ensures conservation of the sum of the kinetic and thermodynamic energy. As the pressure term $-\boldsymbol{\nabla} \cdot (\mathbf{u} \delta p)$ is the divergence of the flux, it integrates to zero over a closed domain and does not affect the energy conservation. The objective here is to find a function of entropy and total water content (and possibly height and pressure) whose tendency cancels out the buoyancy term when added to the kinetic energy equation. Following Ingersoll (2005) and Vallis (2006, chapter 2.5), we define the thermodynamic energy as

$$\mathcal{E}(S, q_T, z) = \Phi_0(S, q_T) - \int_{z_0}^{z} B(S, q_T, z') \, dz'. \tag{14}$$

Here, the function Φ_0 can be any arbitrary function of the entropy and humidity.

The Lagrangian derivative of $\mathcal{E}$ is obtained by differentiating (14):

$$\frac{D\mathcal{E}}{Dt} = -Bw + \mathcal{T}\dot{S} + \mathcal{M}\dot{q}_T, \tag{15}$$

where $\mathcal{T}$ and $\mathcal{M}$ are the partial derivatives of the thermodynamic energy, respectively, to the entropy and total water content:

$$\mathcal{T}(S, q_T, z) = \left(\frac{\partial \mathcal{E}}{\partial S} \right)_{(q_T, z)} = \frac{\partial \Phi_0}{\partial S} - \int_{z_0}^{z} \frac{\partial B}{\partial S} \, dz' \quad \text{and} \tag{16}$$

$$\mathcal{M}(S, q_T, z) = \left(\frac{\partial \mathcal{E}}{\partial q_T} \right)_{(S, z)} = \frac{\partial \Phi_0}{\partial q_T} - \int_{z_0}^{z} \frac{\partial B}{\partial q_T} \, dz'. \tag{17}$$

Multiplying the thermodynamic energy Eq. (15) by ρ_0 and adding the kinetic energy equations, yields the following equation for the conservation of the perturbation energy:

$$\frac{\partial \left[\rho_0 \left(\frac{|\mathbf{u}|^2}{2} + \mathcal{E} \right) \right]}{\partial t} + \boldsymbol{\nabla} \cdot \left[\rho_0 \left(\frac{|\mathbf{u}|^2}{2} + \mathcal{E} \right) \mathbf{u} \right]$$

$$= -\boldsymbol{\nabla} \cdot (\mathbf{u} \delta p) + \rho_0 \mathcal{T}\dot{S} + \rho_0 \mathcal{M}\dot{q}_T. \tag{18}$$

Equation (14) defines the thermodynamic energy $\mathcal{E}$ up to an additive function of entropy and water content, and the same holds for its derivatives $\mathcal{T}$ and $\mathcal{M}$. However, their dependency on height is uniquely determined by the state equation for buoyancy. The integrals in (16) and (17) can be further simplified by taking advantage of the Maxwell relationships:

$$\left(\frac{\partial\alpha}{\partial S}\right)_{p,q_T}(S,q_T,p) = \frac{\partial^2 H}{\partial S\partial p}(S,q_T,p)$$

$$= \left(\frac{\partial T}{\partial p}\right)_{S,q_T}(S,q_T,p) \quad \text{and} \qquad (19)$$

$$\left(\frac{\partial\alpha}{\partial q_T}\right)_{p,S}(S,q_T,p) = \frac{\partial^2 H}{\partial q_T\partial p}(S,q_T,p)$$

$$= \left[\frac{\partial(\mu_v - \mu_d)}{\partial p}\right]_{S,q_T}(S,q_T,p). \qquad (20)$$

Here, μ_v and μ_d are the chemical potentials for water vapor and dry air. The Maxwell relationships are obtained by comparing the second-order derivatives of enthalpy $H(S,q_T,p)$. These relationships make it possible to rewrite the partial derivative of the buoyancy with respect to entropy:

$$\left(\frac{\partial B}{\partial q_T}\right)_{q_T,z}(S,q_T,z) = \frac{g}{\alpha_0}\left(\frac{\partial\alpha}{\partial S}\right)_{q_T,p}[S,q_T,p_0(z)]$$

$$= \frac{g}{\alpha_0}\left(\frac{\partial T}{\partial p}\right)_{S,q_T}[S,q_T,p_0(z)]$$

$$= -\left(\frac{\partial T}{\partial z}\right)_{S,q_T}(S,q_T,z)$$

$$= \Gamma(S,q_T,z), \qquad (21)$$

where $\Gamma = -(\partial T/\partial z)_{S,q_T}(S,q_T,z)$ is the adiabatic lapse rate, which is a function of the entropy S, the total water content q_T, and height z. Similarly, the partial derivative the of buoyancy, respectively, to the total water mixing ratio is

$$\left(\frac{\partial B}{\partial q_T}\right)_{S,p}(S,q_T,z) = \frac{g}{\alpha_0}\left(\frac{\partial\alpha}{\partial q_T}\right)_{q_T,p}[S,q_T,p_0(z)]$$

$$= \frac{g}{\alpha_0}\left[\frac{\partial(\mu_v - \mu_d)}{\partial p}\right]_{S,q_T}[S,q_T,p_0(z)]$$

$$= -\left[\frac{\partial(\mu_v - \mu_d)}{\partial z}\right]_{S,q_T}(S,q_T,z). \qquad (22)$$

The vertical dependencies of $\mathcal{T}$ and $\mathcal{M}$ are directly related to the variations of the temperature and chemical potential:

$$\mathcal{T}(S,q_T,z) = \frac{\partial\Phi_0}{\partial S} + T[S,q_T,p_0(z)]$$

$$- T[S,q_T,p_0(z_0)] \quad \text{and} \qquad (23)$$

$$\mathcal{M}(S,q_T,z) = -\frac{\partial\Phi_0}{\partial q_T} + \mu_v[S,q_T,p_0(z)]$$

$$- \mu_d[S,q_T,p_0(z)] - \mu_v[S,q_T,p_0(z_0)]$$

$$- \mu_d[S,q_T,p_0(z_0)]. \qquad (24)$$

If one chooses the function Φ_0 to be given by,

$$\Phi_0(S,q_t) = H[S,q_T,p_0(z_0)] - H[S_0,q_{T0},p_0(z_0)], \qquad (25)$$

then its partial derivatives are

$$\frac{\partial\Phi_0}{\partial S} = \left(\frac{\partial H}{\partial S}\right)_{q_T,p} = T[S,q_T,p_0(z)] \qquad (26a)$$

$$\frac{\partial\Phi_0}{\partial q_T} = \left(\frac{\partial H}{\partial q_T}\right)_{S,p} = (\mu_v - \mu_d)[S,q_T,p_0(z)]. \qquad (26b)$$

Using these expressions in (23) and (24) results in the functions $\mathcal{T}$ and $\mathcal{M}$ being, respectively, equal to the temperature of the parcel at the reference pressure and to the difference between the chemical potential of the water vapor and dry air both evaluated at the reference pressure:

$$\mathcal{T}(S,q_T,z) = T[S,q_T,p_0(z)] \quad \text{and} \qquad (27)$$

$$\mathcal{M}(S,q_T,z) = \mu_v[S,q_T,p_0(z)] - \mu_d[S,q_T,p_0(z)]. \qquad (28)$$

The differential relationship (15) can now be written as

$$d\mathcal{E} = TdS - Bdz + (\mu_v - \mu_d)dq_T, \qquad (29)$$

where the temperature T and the chemical potential μ_v and μ_d are evaluated at the reference state pressure $p_0(z)$. This can be compared with the thermodynamic relationship between enthalpy, entropy, and water content for moist air:

$$dH = TdS + \alpha dp + (\mu_v - \mu_d)dq_T. \qquad (30)$$

The thermodynamic energy in the anelastic approximation $\mathcal{E}$ at a given height has the same dependency on entropy and total water content as the enthalpy H at the reference state pressure $p_0(z)$. By construction, the thermodynamic energy of the reference is $\mathcal{E}(S_0,q_T,z) = 0$. Hence, the thermodynamic energy is equal to the enthalpy perturbation defined as the enthalpy difference between the parcel and the reference state:

$$\mathcal{E}(S,q_T,z) = H[S,q_T,p_0(z)] - H[S_0,q_{T0},p_0(z)]. \qquad (31)$$

As the reference state has a constant entropy and total water content, the vertical derivative of the reference state enthalpy is

$$\frac{\partial H[S_0,q_{T0},p_0(z)]}{\partial z} = \alpha_0(z)\frac{\partial p_0}{\partial z} = -g. \qquad (32)$$

The total static energy $H + gz$ of the reference state is thus constant, and the thermodynamic energy (31) can then be written as

$$\mathcal{E}(S, q_T, z) = H[S, q_T, p_0(z)] + gz + C, \qquad (33)$$

where $C = -H[S_0, q_{T0}, p_0(z_0)] - gz_0$ can be treated as an arbitrary integration constant. The thermodynamic energy in the anelastic approximation can thus be interpreted either as the perturbation enthalpy or as the moist static energy (enthalpy plus geopotential energy).

The contribution of the change in height to the thermodynamic energy in (29) can be better understood after rewriting it in terms of the change in the reference pressure:

$$-B\, dz = -g\, \frac{\alpha - \alpha_0}{\alpha_0}\, dz = \alpha\, dp_0 - g\, dz. \qquad (34)$$

The first term on the right-hand side is equal to the change in enthalpy due to variations in the reference pressure; while the second term is equal to the change of the geopotential energy (which is also equal to the change of the enthalpy in the reference state). The differential relationship for the thermodynamic energy (29) is thus consistent with both interpretations of $\mathcal{E}$ either as the perturbation enthalpy or as the moist static energy of the parcel.

The energy conservation and the corresponding formulation for the thermodynamic energy are valid even for large fluctuations of the entropy and water content. Deriving (31) and (33) does not require us to make any approximation in the equation of state and, therefore, holds for complex fluid. The key difference between the compressible Navier–Stokes equations and the anelastic approximations lies in the fact that the latter uses the reference state pressure rather than the actual parcel's pressure when determining the different thermodynamic variables. For example, the temperature under the anelastic approximation corresponds to the temperature of an air parcel after it has been moved adiabatically to its reference pressure. If δT is the difference between these two temperatures, the relative temperature error scales as

$$\frac{\delta T}{T} \sim \frac{\delta p}{p_0} \approx \frac{U^2}{c_s^2} = \text{Mach}^2, \qquad (35)$$

where U is the velocity scale, c_s is the speed of sound, and Mach $= (u/c_s)$ is the Mach number. Hence, the temperature in the anelastic approximation is close to the actual temperature as long as the Mach number remains low. A similar scaling also applies for the error in the chemical potential and other thermodynamic variables. As a low Mach number is a necessary condition for the validity of the anelastic approximation, the thermodynamic consistency is guaranteed in the general conditions of use of the anelastic approximation.

The equivalence between perturbation energy $\mathcal{E}$ and the moist static energy only occurs for the very specific choice of $\Phi_0(S, q_T) = H[S, q_T, p_0(z)]$ in (14). Different choices for Φ_0 can be perfectly justified and would yield alternative forms of the perturbation energy that would differ from the enthalpy by a function of entropy and total water. These multiple forms of the perturbation energy can be thought of as capturing the thermodynamic behavior of other "fluids" for which the buoyancy would have the same dependency to the state variables S and q_T within the atmospheric domain, but with different state relationships outside the range of states considered in the atmospheric domain. The nonuniqueness of the perturbation energy is thus a reflection of the wide range of hypothetical fluids that would have the same behavior under the anelastic approximation.

4. Thermodynamic constraints on the buoyancy flux

In the previous section, it has been established that total energy is conserved under the anelastic approximation, with the thermodynamic energy given by the moist static energy evaluated at the reference state pressure $p_0(z)$. In this section, we show that the thermodynamics constrain the buoyancy flux in a manner similar to the way the entropy budget constrains the work performed in the atmosphere.

Comparing kinetic energy Eq. (13) and thermodynamic energy Eq. (15) indicates that the conversion rate between thermodynamic energy and kinetic energy is given by the buoyancy flux, with the total generation of the kinetic energy W given by the domain integral of the buoyancy flux:

$$W = \int (\rho_0 w B)\, dx\, dy\, dz. \qquad (36)$$

For an atmosphere in statistical equilibrium, the work W can be determined if the sources and sinks of entropy and water are known. Multiplying (15) by ρ_0 and integrating over the entire domain yields

$$W = \int_{z_l}^{z_u} \rho_0(\mathcal{T}\dot{S} + \mathcal{M}\dot{q}_T)\, dx\, dy\, dz, \qquad (37)$$

after taking advantage of the fact that the integral of $\rho_0(D\mathcal{E}/DT)$ over the atmospheric domain vanishes in statistical equilibrium. Equation (37) makes it possible to directly estimate the production of kinetic energy from the knowledge of the entropy and water sources. A similar expression can be obtained for a compressible atmosphere, and differs from (37) in that the actual density replaces the reference density and that the temperature and chemical potential are estimated at the

actual pressure. Hence, understanding how the production of kinetic energy is constrained by the thermodynamics under the anelastic approximation should translate directly to a fully compressible case.

For example, one can consider a moist analog to Rayleigh–Bénard convection forced by an energy source flux Q_{in} at the lower boundary at temperature T_{in} and an energy sink Q_{out} at the upper boundary at temperature T_{out}. There is no source or sink of water here. The main difference from the traditional Rayleigh–Bénard problem is that a portion of the atmospheric domain is saturated. To keep the discussion as simple as possible, it is also assumed that the work W is not dissipated but rather extracted from the system, which is equivalent to omitting the frictional heating and the corresponding irreversible entropy production. In this case, the only entropy sources and sinks in the system are the external energy sources and sinks. For the system to be in statistical equilibrium, the total entropy remains constant over time, which requires that $(Q_{in}/T_{in}) + (Q_{out}/T_{out}) = 0$. The work performed by the atmosphere is obtained by using (37):

$$\int (\rho_0 w B)\, dx\, dy\, dz = W = \frac{T_{in} - T_{out}}{T_{in}} Q_{in}. \quad (38)$$

This corresponds to the work performed by a Carnot cycle acting between the temperatures T_{in} and T_{out}. The same expression applies for the work performed by a compressible atmosphere.

This expression can be compared to a direct computation of the buoyancy flux. The matching entropy source and sink at the top and bottom must be compensated for by a constant upward entropy flux and not the net water flux:

$$\rho_0 \overline{w'S'} = \frac{Q_{in}}{T_{in}} \quad \text{and} \quad (39)$$

$$\rho_0 \overline{w'q_T'} = 0, \quad (40)$$

where the overbar indicates a horizontal average and the prime denotes a departure from the horizontal mean. If the entropy is close to the reference value, one can take advantage of the fact that the derivative of the buoyancy with respect to the entropy is given by the lapse rate (21):

$$\rho_0 \overline{w'B'} \approx \Gamma \rho_0 \overline{w'S'} = -\left(\frac{\partial T}{\partial z}\right)_{S=S_0, q_T=q_{T0}} \frac{Q_{in}}{T_{in}}. \quad (41)$$

Integrating the buoyancy flux over the depth of the domain yields the same expression as the work obtained from the thermodynamic constraint (38).

In deriving (41), the adiabatic lapse rate is approximated by that of the reference state. Given the fact that the lapse rate is discontinuous when a parcel goes through saturation, this is only accurate for levels at which the atmosphere is either entirely unsaturated or entirely saturated. This would be the case through most of the atmosphere for a stratus-topped boundary layer, but (41) cannot be applied to a problem such as cumulus convection where saturated and unsaturated air parcels coexist within a thick atmospheric layer. Nevertheless, the relation between the vertically integrated buoyancy flux and thermodynamic forcing (37) holds even in the case where the fluctuations of the entropy and total water content are large.

Equation (41) also indicates that the buoyancy flux in moist convection can be discontinuous. Indeed, below the cloud base where all air parcels are unsaturated, the buoyancy flux is equal to the entropy flux multiplied by the dry-adiabatic lapse rate; while above the cloud base, it is given by the moist-adiabatic lapse rate. The buoyancy flux exhibits a jump at the cloud base even when the flux of the entropy and water vapor are smooth. This is a consequence of the discontinuity in the partial derivatives of the state equation for moist air—in this case, the adiabatic lapse rate—as a parcel goes through saturation.

An alternative proof of the discontinuity of the buoyancy flux can also be obtained from the expression of the buoyancy flux in terms of the temperature, water vapor, and total water fluxes:

$$\rho_0 \overline{wB} \approx \rho_0 g \left(\frac{\overline{wT'}}{T_0} + \frac{R_v}{R_d} \overline{wq_v'} - \overline{w'q_T'} \right), \quad (42)$$

where q_v is the specific humidity, R_v is the ideal constant for the water vapor, and R_d is the ideal gas constant for dry air. Such an expression for the buoyancy flux can be obtained directly from the definition of the buoyancy (9) and the ideal gas law. Similarly, the entropy flux can be approximated by

$$\rho_0 \overline{w'S'} \approx \rho_0 \left(C_p \frac{\overline{w'T'}}{T_0} + L_v \frac{\overline{w'q_v'}}{T_0} \right). \quad (43)$$

In expression (42), a discontinuity in the buoyancy flux arises from the different state relationships for temperature, entropy, specific humidity, and total water content between the saturated and unsaturated regions of the atmosphere. For a level where all parcels are unsaturated, the specific humidity is equal to the total water content. Hence, in the absence of an upward water flux, the entropy flux is given by

$$\rho_0 \overline{w'S'} \approx \rho_0 C_p \frac{\overline{w'T'}}{T_0}, \quad (44)$$

with C_p the heat capacity of the moist air at constant pressure. Comparing this expression with (42) shows that the buoyancy flux here is equal to the entropy flux multiplied by the dry-adiabatic lapse rate $\Gamma_d = (g/C_p)$.

In contrast, in the saturated regions, the specific humidity fluctuations are related to the temperature fluctuation through the Clausius–Clapeyron relationship. For small-amplitude perturbations, this yields

$$q_v' \approx \frac{L_v}{R_v T_0^2} q_{s0} T', \tag{45}$$

with L_v the latent heat of the vaporization and q_{s0} the saturation specific humidity in the reference state. For a saturated atmospheric layer, the entropy flux becomes

$$\rho_0 \overline{w'S'} \approx \rho_0 \left(C_p + \frac{L_v^2 q_{s0}}{R_v T_0^2} \right) \frac{\overline{w'T'}}{T_0}, \tag{46}$$

while the buoyancy flux is

$$\rho_0 \overline{w'B'} \approx \rho_0 g \left(1 + \frac{L_v q_{s0}}{R_d T_0} \right) \frac{\overline{w'T'}}{T_0}. \tag{47}$$

The ratio between these two quantities is indeed the moist-adiabatic lapse rate (2).

5. Thermodynamic energy as a prognostic variable

The anelastic framework discussed in the previous sections uses entropy and total water content as prognostic variables to describe the thermodynamic properties of moist air. Other thermodynamic state variables such as temperature, enthalpy, chemical potential, or specific humidity can be obtained using state relationships for moist air. Relying on moist entropy as a prognostic variable is not a new proposition. Ogura and Phillips (1962) mention their unsuccessful attempt to base their derivation of the anelastic approximation based on moist entropy. Ingersoll (2005) also uses the entropy of salty water as a prognostic variable for the anelastic approximation. Nevertheless, many authors (Ogura and Phillips 1962; Lipps and Hemler 1982; Durran 1989; Bannon 1996) have derived the anelastic approximations for different combinations of thermodynamic variables, most commonly temperature or potential temperature. Here, it is shown that the thermodynamic energy Eq. (15) can be used to obtain an alternative prognostic equation for these other prognostic variables.

Equation (15) provides a prognostic equation for the perturbation energy $\mathcal{E}$. For the choice of $\mathcal{T} = T$ and $\mathcal{M} = \mu_v - \mu_d$, the last two terms on the right-hand side

of (15) can be combined into $T\dot{S} + (\mu_v - \mu_d)\dot{q}_T = Q$. Here, Q is the external heating rate, which includes contributions from radiative heating, frictional heating, and molecular diffusion but not latent heat release. Equation (15) thus yields a prognostic equation for the thermodynamic energy:

$$\rho_0 \frac{D\mathcal{E}}{Dt} + \rho_0 Bw = Q. \tag{48}$$

Given the definition of the thermodynamic energy (33), its rate of change is given by

$$\frac{D\mathcal{E}}{Dt} = C_p \frac{DT}{Dt} - \rho_0^{-1} L_v C + wg, \tag{49}$$

where C is the rate at which the water vapor condenses (evaporation corresponding to a negative value for C). The thermodynamic energy equation then becomes

$$\rho_0 C_p \frac{DT}{Dt} - \rho_0 \alpha \frac{Dp_0}{Dt} = Q + L_v C, \tag{50}$$

after using $\rho_0 w(g + B) = \rho_0 g(\alpha/\alpha_0)w = -\rho_0 \alpha(Dp_0/Dt)$. Equation (50) corresponds to the first law of thermodynamics in which the pressure has been replaced by the reference state pressure $p_0(z)$.

Equation (50) has been derived from the tendency equations for entropy and total water. Any two of these three equations are sufficient to determine the thermodynamic evolution of the fluid. On the one hand, we can use entropy and total water as prognostic variables as in section 2 and 3, with the anelastic system conserving energy and the first law of thermodynamic is being implicitly obeyed. On the other hand, we can also use temperature and total water content as prognostic variables, with the temperature tendency obtained from Eq. (50). In this case, energy conservation is explicitly enforced. In addition, the system would have an implicit entropy equation that is consistent with the second law of thermodynamics. The entropy in the anelastic approximation would be defined as the entropy of parcel of moist air at the same temperature and water content, but estimated at the reference state pressure: $S = S[T, q_T, p_0(z)]$.

Despite the fact that Eq. (50) might have been derived directly from first principles, several authors, including Ogura and Phillips (1962) and Lipps and Hemler (1982), have used incorrect formulations for the first law of thermodynamic. For instance, Lipps and Hemler (1982) use the following equation for potential temperature:

$$\rho_0 C_{pd} \frac{D\theta}{Dt} = \frac{Q + L_v C}{\pi}. \tag{51}$$

They define the potential temperature as $\theta = \pi^{-1}T$, where $\pi = (p/p_{\mathrm{ref}})^{(R_d/C_{pd})}$ is the Exner function *using the specific heat C_{pd} and gas constant R_d for dry air*. The quantity p_{ref} is an arbitrary pressure. While Eq. (51) is perfectly appropriate for a dry atmosphere, it fails to account for the effect of water vapor on both the heat capacity and gas constant, and as such is inconsistent with the first law of thermodynamics.

In the anelastic approximation, the pressure is replaced by the reference state pressure, so that the temperature is $T = (p_0/p_{\mathrm{ref}})^{(R_d/C_{pd})}\theta = \pi_0\theta$. In the case of Lipps and Hemler (1982), the potential temperature tendency is

$$\frac{D\theta}{Dt} = \frac{1}{\pi_0}\frac{DT}{Dt} + \frac{R_d\theta}{C_{pd}p_0}\frac{Dp_0}{Dt}. \qquad (52)$$

Using this expression in the potential temperature, Eq. (51) yields

$$\rho_0 C_{pd}\frac{DT}{Dt} + \rho_0\frac{R_d T}{p_0}\frac{Dp_0}{Dt} = Q + L_v C. \qquad (53)$$

This equation differs from the thermodynamic energy Eq. (48) in two respects. First, it uses the heat capacity of dry air C_{pd} rather than that of moist air C_p. Second, the term multiplying the rate of change of pressure $(R_d T/p_0) = \alpha(p_0, T, q_T = 0)$ is equal to the specific volume of a dry air parcel. This omits the impacts of water on the density of moist air. Lipps and Hemler (1982) indeed observe that their formulation of the anelastic approximation does not conserve energy. This fact however should be attributed to their use of an incorrect form of the potential temperature equation for a moist atmosphere, and is not indicative of an intrinsic limitation of the anelastic approximation. Indeed, an anelastic model based on Eq. (50) and the full nonlinear equation of state for buoyancy not only conserves energy but also obeys the second law of thermodynamics.

6. Conclusions

This paper has investigated the use of a complex equation of state in conjunction with the anelastic approximation. The version of the anelastic approximation discussed here differs slightly from the previous work of Ogura and Phillips (1962), Dutton and Fitchl (1969), Lipps and Hemler (1982), and Durran (1989) in that it uses moist entropy and total water content as prognostic variables and has a reference state with constant entropy and total water content. Buoyancy is computed with the full nonlinear equation of state for the specific volume. This anelastic approximation remains accurate even for large fluctuations of the entropy and total water as long as the perturbations in the specific volume and total pressure remain small. While this paper has mainly focused on the behavior of a moist atmosphere, the results reported here are valid for a wide range of fluids with complex equations of state. In particular, the use of the anelastic approximation is justified for studying deep convection in a moist atmosphere in which the equation of state is highly nonlinear due to the discontinuity of its partial derivatives at saturation, or in the ocean, in which case the nonlinearity results from thermobaric effects.

The key finding here is that the anelastic approximation conserves energy, with the thermodynamic energy given by the difference between the enthalpy of the parcel at the reference pressure and the enthalpy of the reference state at the same level. Furthermore, the thermodynamic relationships between the thermodynamic energy, entropy, and total water content for an anelastic fluid are the same as for a fully compressible fluid, with the exception that the reference state pressure must be used instead of the total pressure. The thermodynamic consistency of the anelastic approximation holds as long as the Mach number of the flow is small. This result indicates that, even for complex fluids, anelastic models can be constructed so that they simultaneously obey the first and second laws of thermodynamics.

The thermodynamic energy equation can also serve as an alternative prognostic equation without affecting the thermodynamic consistency. In an anelastic model, temperature or enthalpy can replace moist entropy as a prognostic variable, with the evolution equation derived from the thermodynamic energy equation. Such a model would explicitly follow the first law of thermodynamics, while implicitly obeying the second law of thermodynamics, with the entropy given by the moist entropy of the parcel at the reference state pressure.

These findings ensure that anelastic models offer an appropriate framework within which to study the thermodynamic behavior of the atmosphere. In a compressible atmosphere, the work produced by the circulation can be determined from the energy and entropy budgets (Pauluis and Held 2002). Here, the same approach has been applied to an anelastic model to determine the buoyancy flux from the thermodynamic constraints imposed by the first and second laws of thermodynamics. In numerical models, ensuring mass and energy conservation is usually straightforward under the anelastic approximation, but presents several difficult challenges under the compressible Navier–Stokes equations. Hence, while there has been a strong push toward the development and use of fully compressible atmospheric

codes, anelastic models still offer a very attractive platform, particularly for studying the maintenance of the atmospheric circulation over long time scales.

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