1 Riemannian metrics

We refer to Lecture 1 for basic intuition. The goal of this lecture is to formally introduce Riemannian manifolds and to study some of their basic properties.

Let $M$ be a smooth manifold. We write $TM$ for the tangent bundle, $T^*M$ for the cotangent bundle, and $T^*M \otimes T^*M$ for the vector bundle of $(0,2)$-tensors. We write $\Gamma$ for smooth sections; e.g. $X \in \Gamma(TM)$ means that $X$ is a smooth vector field.

**Definition 1.1 (Riemannian metric)**

A Riemannian metric is a smooth section of $T^*M \otimes T^*M$, that is symmetric and positive definite.

In other words we have an element $g \in \Gamma(T^*M \otimes T^*M)$ that satisfies $g_p(X,Y) = g_p(Y,X)$ and $g_p(X,X) \geq 0$ for all $p \in M$ and all $X,Y \in T_pM$, with equality only for $X = 0$. We sometimes write $\langle \cdot , \cdot \rangle_p$ instead of $g_p$, and we often suppress the point $p$ in the notation. In local coordinates $x^i$, with the corresponding coordinate vectors $\partial / \partial x^i$, the metric is represented by the symmetric positive definite matrix

$$g_{ij}(p) = g_p \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right). \quad (1.1)$$

We also sometimes use the notation $g = ds^2 = g_{ij} dx^i dx^j$.

**Theorem 1.2 (Existence of Riemannian metrics)**

Every smooth manifold admits a Riemannian metric.

For the proof of Theorem 1.2, we introduce the following notion.

**Definition 1.3 (Pullback metric)**

If $\phi : M \to (N,g_N)$ is a smooth embedding (or immersion), then the pullback metric $\phi^* g_N$ is defined by $(\phi^* g_N)(X,Y) = g_N(\phi_* X, \phi_* Y)$.

Note that $\phi^* g_N$ is indeed a Riemannian metric.

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1 According to our definition manifolds are Hausdorff and second countable.
Proof 1. By Whitney’s embedding theorem, for every smooth manifold $M$ there is a smooth embedding $\phi : M \to \mathbb{R}^N$ into some Euclidean space. Writing $g_{\text{eucl}}$ for the Euclidean metric on $\mathbb{R}^N$, we get a metric $g$ on $M$ via pullback, $g = \phi^*g_{\text{eucl}}$. \hfill \Box$

Proof 2. Let $M$ be a smooth manifold, and let $\phi_\alpha$ be a smooth partition of unity subordinate to a locally finite atlas $(U_\alpha, x_\alpha)$ of $M$. Let $g_\alpha$ be the pullback of the Euclidean metric to $U_\alpha$, and define

$$g = \sum_\alpha \phi_\alpha g_\alpha. \quad (1.2)$$

It is easy to check that $g$ has the desired properties. \hfill \Box

Definition 1.4 (Isometry)
An isometry $\phi : (M, g_M) \to (N, g_N)$ between Riemannian manifolds is a diffeomorphism such that $g_M = \phi^*g_N$.

Recalling Definition 1.3, the condition $g_M = \phi^*g_N$ means that

$$g_M(X, Y) = g_N(\phi_\ast X, \phi_\ast Y)$$

for all $X, Y \in \Gamma(TM)$.

Exercise 1 (Hyperbolic space $\mathbb{H}^2$). Find an isometry between the upper half plane model $(\mathbb{R} \times \mathbb{R}_+, \frac{dx^2 + dy^2}{y^2})$ and the Poincare disk model $(B_1(0), \frac{4(dx^2 + dy^2)}{(1-x^2-y^2)^2})$.

Definition 1.5 (Isometry group)
Let $(M, g)$ be a Riemannian manifold. The isometry group $\text{Isom}(M, g)$ is defined as the set of all isometries $\phi : (M, g) \to (M, g)$, with the group structure given by composition.

Note that the isometry group, is indeed a group.

Example 1.6 (Isometry groups)
$\text{Isom}(S^n) = O_{n+1}$, $\text{Isom}(\mathbb{R}^n) = \mathbb{R}^n \rtimes O_n$, etc.

Definition 1.7 (Riemannian immersions and submersions)
A Riemannian immersion (or isometric immersion) $\phi : (M, g_M) \to (N, g_N)$ is a smooth immersion such that $g_M = \phi^*g_N$. A Riemannian submersion $\phi : (M, g_M) \to (N, g_N)$ is a smooth submersion such that for each $p \in M$, $D\phi : \ker^\perp(D\phi) \to T_{\phi(p)}N$ is a linear isometry.

Example 1.8 (Riemannian immersions and submersions)
Inclusions $\mathbb{R}^n \to \mathbb{R}^N$ and projections $\mathbb{R}^N \to \mathbb{R}^n$ are trivial examples. Note however, that any curve $\gamma : \mathbb{R} \to \mathbb{R}^N$ parametrized by arclength is also an example of a Riemannian immersion.
Exercise 2 (Complex projective space). Define a Riemannian metric on $\mathbb{CP}^n$ such that the projection $(S^{2n+1}, g_{\text{can}}) \to \mathbb{CP}^n$ becomes a Riemannian submersion.

Exercise 3 (Flat torus). Consider torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the flat metric induced from $\mathbb{R}^2$. Find an isometry with $S^1 \times S^1$ equipped with the product metric. Find an isometric embedding into $\mathbb{R}^4$. Is there any isometric embedding into $\mathbb{R}^3$?

2 Explicit computations

Let us practice a bit to write down Riemannian metrics explicitly.

Example 2.1 (Flat $\mathbb{R}^2$)
In cartesian coordinates $(x, y)$ the flat metric on $\mathbb{R}^2$ is given by
$$g = dx^2 + dy^2,$$

i.e. $g_{11} = g_{22} = 1$ and $g_{12} = g_{21} = 0$. Switching to polar coordinates $(r, \theta)$ via $x = r \cos \theta$, $y = r \sin \theta$ the representation of the metric transforms into
$$g = dr^2 + r^2 d\theta^2,$$

i.e. $g_{rr} = g(\partial / \partial r, \partial / \partial r) = 1$, $g_{\theta\theta} = g(\partial / \partial \theta, \partial / \partial \theta) = r^2$, $g_{r\theta} = g_{\theta r} = 0$.

Example 2.2 ($S^2 \subset \mathbb{R}^3$)
We work in spherical coordinates $x = \sin \theta \cos \phi$, $y = \sin \theta \sin \phi$, $z = \cos \theta$. To find the formula for the metric, we first compute
$$\frac{\partial}{\partial \theta} = \begin{pmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ -\sin \theta \end{pmatrix}, \quad \frac{\partial}{\partial \phi} = \begin{pmatrix} -\sin \theta \sin \phi \\ \sin \theta \cos \phi \\ 0 \end{pmatrix} \quad (2.1)$$

Note that $|\frac{\partial}{\partial \theta}|^2 = 1$, $|\frac{\partial}{\partial \phi}|^2 = \sin^2 \theta$ and $\langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \rangle = 0$. Thus
$$g = d\theta^2 + \sin^2 \theta d\phi^2.$$

This can be used e.g. to find the formula for the area element (as in calculus),
$$d\mu = \sqrt{|\det g|} d\theta d\phi = \sin \theta d\theta d\phi.$$

Exercise 4 (Catenoid). The catenoid $C \subset \mathbb{R}^3$ is obtained by rotating the curve $x = \cosh z$ around the $z$-axis. Find a parametrization of $\phi$ of $C$ and compute the induced Riemannian metric $g_{ij} = \langle \partial_i \phi, \partial_j \phi \rangle$. Find the inverse metric $g^{ij}$ defined by the formula $g^{ij} g_{jk} = \delta^i_k$. Compute the normal vector $\nu$. Compute $g^{ij} \langle \nu, \partial_i \partial_j \phi \rangle$. 

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3 Length and volume

Given a Riemannian manifold $(M, g)$ we can compute length and volume.

**Definition 3.1 (Length)**

If $\gamma : [a, b] \rightarrow (M, g)$ is a piecewise smooth curve, then its **length** is

$$L(\gamma) = \int_a^b \sqrt{g(\frac{d\gamma}{dt}, \frac{d\gamma}{dt})} \, dt. \quad (3.1)$$

**Definition 3.2 (Volume measure)**

Let $(M^n, g)$ be a Riemannian manifold. The **volume measure** $d\mu_g = \sqrt{|\det g|} d^n x$ is the unique Borel measure on $M$ such that for any coordinate chart $(U, x)$ and any continuous function $f : U \rightarrow \mathbb{R}$ we have

$$\int_U f \, d\mu_g = \int_{x(U)} f \circ x^{-1} \sqrt{|\det g|} d^1 \cdots d^n. \quad (3.2)$$

For example, we can compute the volume (aka area) of the two-sphere,

$$\text{Vol}(S^2) = \int_0^\pi \int_0^{2\pi} \sin \theta \, d\theta \, d\phi = 4\pi.$$  

If we want to integrate a continuous function $f$ over a Borel set $\Omega \subseteq M$ that is not (almost) contained in a single chart, then we can reduce everything to computations in charts by means of a partition of unity,

$$\int_{\Omega} f \, d\mu_g = \sum_{\alpha} \phi_{\alpha} \int_{x_{\alpha}(\Omega \cap U_{\alpha})} f \circ x^{-1}_{\alpha} \sqrt{|\det g|} d^1 \cdots d^n. \quad (3.3)$$

Equation (3.3) is also useful for convincing oneself that there is indeed a unique Borel measure as claimed in Definition 3.2.