No.
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Differential Geometry II

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- Course site: cims.nyu.edu/~ryoung/courses/diffgeo1
- Office hours: Mon: 1-2.
- Announcements through Brightspace/email (collect emails)
- Texts: Milnor (excellent, somewhat concise)
  Lee (more traditional exposition)
  Cheeger-Ebin (comparison geometry).
  Warner (Lie groups, Hodge theory)

What did you learn last semester? Smooth manifolds, tangent bundle, vector fields, ~~Gauss curvature~~?, Jeff said he did some Riem geometry?

Plan: Introduce Riemannian geometry - try not to assume too much, but moving quickly. - curvature $\xrightarrow{\text{branch to various topic from here}}$ calculus of variations - Morse theory - hyperbolic geometry / nonpos. curvature. - Lie groups / symmetric spaces - cohomology / Hodge theory

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Riemannian geometry: Basic idea of Rien. geom: a topological/smooth manifold has no local structure - locally diffeo/homeo to $\mathbb{R}^n$. A Riemannian manifold has a metric (describing lengths), so it has local structure. How can we describe that local structure? How does the local structure affect the global structure?

Let's start with the most concrete case: Manifolds in $\mathbb{R}^n$:
Let M be a smooth embedded submanifold in $\mathbb{R}^n$. This is equipped with a natural metric, because we can measure the lengths of paths in M. ~~How can~~ Let's we do geometry on M? First: what are shortest paths on M?

Def: For $\gamma: [0,1] \to M$ a smooth curve, let $\ell(\gamma) = \int_0^1 \|\gamma'(t)\| dt$

Q: Minimize $\ell(\gamma)$ over curves s.t. $\gamma(0)=p$, $\gamma(1)=q$.

First, ℓ is a poor functional here - every reparam of a minimizer is a min.

Instead, let $E(\gamma) = E_0^1(\gamma) = \int_0^1 \|\gamma'(t)\|^2 dt$

Minimizing $E \Leftrightarrow$ minimizing ℓ: By C-S, $\int_a^b \|\gamma'(t)\| dt \leq$
$\sqrt{\int_a^b \|\gamma'(t)\|^2 dt}\sqrt{\int_a^b 1\, dt} = \sqrt{E_a^b(\gamma)}\sqrt{b-a}$, with equality

in our case,

iff $\|\gamma'(t)\|$ is constant — ~~so~~ $E(\gamma) \geq \ell(\gamma)^2$ $\Rightarrow$ γ is an E-minimizer $\Leftrightarrow$ γ is an ℓ-minimizer and $\|\gamma'(t)\|$ is const.

So let's minimize E. ~~If M is closed, connected, then a minimizer exists~~ ~~$\exists$~~ If M is closed, connected, then $\exists$ a ~~length-minimizing curve~~ energy ~~so~~ minimizer by Arzela–Ascoli. ~~what does this satisfy?~~ call it γ. What can we say about γ?

Calculus of Variations: since γ minimizes energy, any other curve has at least as much energy. Let $h_u : [0,1] \to M$, $u \in (-\varepsilon, \varepsilon)$ be a smoothly family of such curves (a variation of γ). Then

$$E(h_u) = \int_0^1 \left\|\frac{\partial h}{\partial t}\right\|^2 dt \quad \text{should be minimized at } u = 0, \Rightarrow$$

$$\frac{1}{2}\frac{d}{du}E(h_u) = \frac{1}{2}\int_0^1 \frac{\partial}{\partial u}\left\|\frac{\partial h}{\partial t}\right\|^2 dt = \frac{1}{2}\int_0^1 \frac{\partial}{\partial u}\left\langle \frac{\partial h}{\partial t} \,\middle|\, \frac{\partial h}{\partial t}\right\rangle dt$$

$$= \int_0^1 \left\langle \frac{\partial^2 h}{\partial u \partial t} \,\middle|\, \frac{\partial h}{\partial t}\right\rangle dt. \quad \text{But this looks like a derivative}$$

~~But~~ $\frac{\partial}{\partial t}\left\langle \frac{\partial h}{\partial u} \,\middle|\, \frac{\partial h}{\partial t}\right\rangle = \left\langle \frac{\partial^2 h}{\partial u \partial t} \,\middle|\, \frac{\partial h}{\partial t}\right\rangle + \left\langle \frac{\partial h}{\partial u} \,\middle|\, \frac{\partial^2 h}{\partial t^2}\right\rangle$, so we use IBP:

$$\frac{1}{2}\frac{dE}{du} = \int_0^1 \frac{\partial}{\partial t}\left\langle \frac{\partial h}{\partial u} \,\middle|\, \frac{\partial h}{\partial t}\right\rangle dt - \int_0^1 \left\langle \frac{\partial h}{\partial u} \,\middle|\, \frac{\partial^2 h}{\partial t^2}\right\rangle dt$$

$$= \left\langle \frac{\partial h}{\partial u} \,\middle|\, \frac{\partial h}{\partial t}\right\rangle\Big|_{t=0}^{1} - \int_0^1 \left\langle \frac{\partial h}{\partial u} \,\middle|\, \frac{\partial^2 h}{\partial t^2}\right\rangle dt$$

Let's call these: $U = \frac{\partial h}{\partial u}$ = variation field

$V = \frac{\partial h}{\partial t}$ = velocity

$A = \frac{\partial^2 h}{\partial t^2}$ = acceleration — then

$$\frac{1}{2}\frac{dE}{du} = \underbrace{\langle V(1) | U(1)\rangle - \langle V(0) | U(0)\rangle} - \int_0^1 \langle U | A \rangle\, dt.$$

boundary term — which way do endpoints move?

curvature term — toward/away from direct. of curv.

In particular, if γ is an E-minimizer & h fixes endpts ~~then~~ ($h_u(0) = p$, $h_u(1) = q$), then ~~$0 = \frac{1}{2}$~~ $0 = \frac{1}{2}\frac{dE}{du} = -\int_0^1 \langle U | A \rangle\, dt$ $\forall U$.

But $U(t)$ is any vector tangent to M at $\gamma(t)$.

~~So~~ $\Rightarrow$ A is ~~orthogonal to~~ normal to M.

Let TM be tangent space of M $\Rightarrow$ $\pi_{TM}(A) = 0$.

~~The So this generalizes straight lin~~

More generally, if $\pi_{TM}(\gamma'') = 0$, we call γ a geodesic (not all l-min ex: great circle but crit pts.) This generalizes "straight lines" in $\mathbb{R}^n$ — if $M = \mathbb{R}^n$, $TM = \mathbb{R}^n$, so γ a geodesic $\Leftrightarrow \gamma''(t) = 0 \Leftrightarrow \gamma$ is a straight line.

One note: This calc was entirely extrinsic & depends on embedding of M into $\mathbb{R}^n$. It should be intrinsic — depend only on lengths of paths: if ~~we embed M differently, w/ same lengths, should get same length-minimizers~~ $f: M \to N$ preserves lengths, ~~then~~ and γ is a geodesic in M, then $f \circ \gamma$ should be a geod. in N — geodesics are critical pts of length functional, so if length is preserved, so are geodesics.

In fact, something stronger: Prop: Suppose $f: M \to N$ is length-preserving. $\gamma: [0,1] \to M$
Then $f_*(\pi_{TM}(\gamma''(t))) = \pi_{TN}((f\circ\gamma)''(t))$ (Not true, not statable for γ'': sheet of paper, curve. But the tangential cpt is preserved.)

Pf: Let $h_u: [0,1] \to M$, $u \in (-\varepsilon, \varepsilon)$ be a variation of γ w/ variation field U. Then $\frac{1}{2}\frac{d}{du}E_0^b(h_u) = \frac{1}{2}\frac{d}{du}E_0^b(f\circ h_u)$.
$U(0) = U(1) = 0 \Rightarrow \int_0^b \langle \gamma'' | U \rangle dt = \int_0^b \langle (f\circ\gamma)'' | f_*(U) \rangle dt \quad \forall U$.

$\Rightarrow \forall X \in T_{\gamma(t)}M, \quad \langle \gamma''(t) | X \rangle = \langle (f\circ\gamma'')(t) | f_* X \rangle$
$\Rightarrow \langle \pi_{TM}(\gamma''(t)) | X \rangle = \langle \pi_{TN}((f\circ\gamma'')(t)) | f_*(X) \rangle$
Since f_* preserves inner prod., $\Rightarrow \langle f_*(\pi_{TM}(\gamma''(t))) | f_* X \rangle =$ "
True for all X, so $\Rightarrow f_*(\pi_{TM}(\gamma''(t))) = \pi_{TN}(f\circ\gamma'')(t)$ //

So $k = \pi_{TM}(\gamma'')$ (AKA geodesic curvature) is preserved by isoms! (overflow)

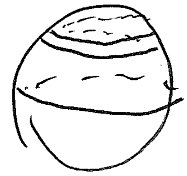
Particularly nice for surfaces in $\mathbb{R}^3$. (Do you know abt. Gauss. curv, Euler char, Gauss-Bonnet?) ~~Interpret in terms of this pic~~:

~~Gaussian~~ Geodesic curvature for surfaces:
Let n be the unit normal ~~s.t.~~ at $\frac{\pi}{2}$ ...
Ex: Let $\gamma: [0,1] \to \mathbb{R}^2$ a unit-speed curve. Then $\exists k$ $\gamma'' = kn$ measures turning speed. If γ ~~is a~~ simple closed curve, $\gamma'(0) = \gamma'(1)$, then $\int$ ~~$\gamma''(t)$~~ $k(t)\, dt = 2\pi$.

$-\gamma''$

~~Likewise~~ Generalize: ~~turning~~ ~~turning speed~~ ⇝ ~~geodesic curvature $\kappa = \pi_{TM}(\gamma'')$~~

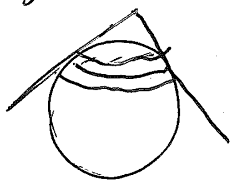
~~If~~ For $M \neq \mathbb{R}^2$, can likewise define unit normal n, $\pi_{TM}(\gamma''(t)) = \kappa n$, but $\int_0^1 \kappa$ need not equal 2π.

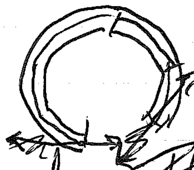


equator has $\pi_{TM}(\gamma''(t)) = 0$

What about in between? 1) Calculate.

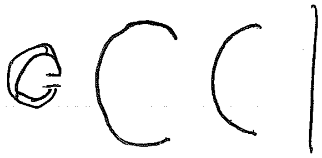
2) v. nice visualization. Imagine a thin strip around the curve. We know geod. curv is preserved by isometries, so if this were isometric to a strip in $\mathbb{R}^2$, we'd be done. It's not, but it's very close — if you've ever peeled an apple, you know that a strip of apple peel will lay essentially flat:





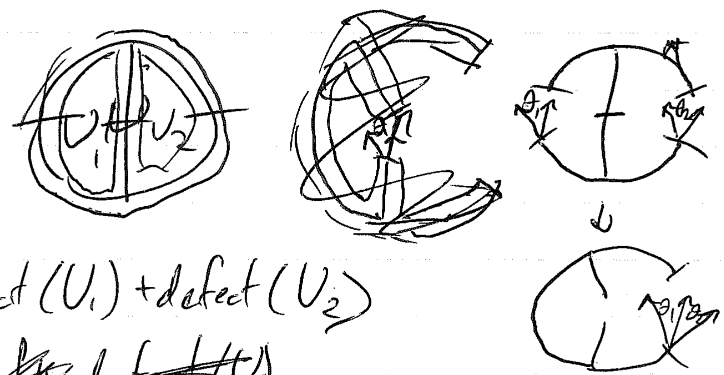
Rigorously, there's an embedding in $\mathbb{R}^2$ that preserves lengths on γ, preserves $\frac{dE}{du}$ for all variations of γ. Since geod curv. is determined by $\frac{dE}{du}$, curv is preserved, ~~then~~ and $\int \kappa \, dt = 2\pi + \angle(v_{start}, v_{end})$

Gaussian curv for surfaces: One thing you may notice: $2\pi - \int \kappa \, dt$ depends on the size of the curve:



We can make that rigorous:

Let $\text{defect}(U) = 2\pi - \int_{\partial U} \kappa \, dt$



$$\text{defect}(U_1 \cup U_2) = \text{defect}(U_1) + \text{defect}(U_2)$$

~~and we define $K = \frac{\text{defect}(U)}{\text{area}(U)}$~~

By Riesz representation, this is a measure — define $K(x) = \lim_{r \to 0} \frac{\text{defect}(B_r(x))}{\text{area}(B_r(x))}$.

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Smooth and Riemannian Manifolds

Last time: started talking about geometry of submanifolds of $\mathbb{R}^n$.

Today: back up a step — Recall smooth manifolds in general.

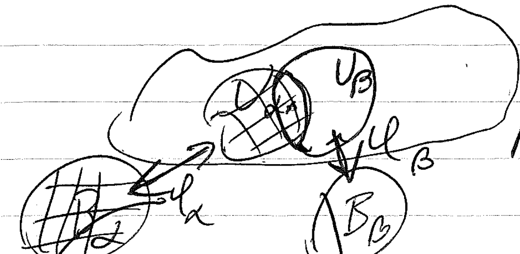
Def: A smooth manifold is a mfld M equipped w/ a smooth structure

— a collection of charts $\{(U_\alpha \cancel{\subset M}, B_\alpha \cancel{\subset \mathbb{R}^d}, \varphi_\alpha : U_\alpha \to B_\alpha)\}_{\alpha \in A}$ s.t.

- $\forall \alpha$, φ_α ~~is a homeomorphism~~ $U_\alpha \subset M$, $B_\alpha \subset M$ are open sets; $\varphi_\alpha : U_\alpha \to B_\alpha$ is a homeo.

— $M = \bigcup_\alpha U_\alpha$

- $\forall \alpha, \beta$ the transition map $\varphi_\beta^{-1} \circ \varphi_\alpha \big|_{\varphi_\alpha^{-1}(U_\alpha \cap U_\beta)}$ is smooth.



M (Two coord systems on $U_\alpha \cap U_\beta$ — want these to be smooth wrt one another)

Standard example: Let $f : \mathbb{R}^n \to \mathbb{R}$ a smooth function such that $\forall x \in \mathbb{R}^n$, if $f(x) = 0$, then $\nabla f(x) \neq 0$. Then $M = f^{-1}(0)$ is a smooth mfld.

Pf: Implicit function theorem (exercise).

Once we have a smooth structure, we can locally do calculus like it's $\mathbb{R}^n$: Define: $C^\infty(M)$ = set of smooth fns on M

($f : M \to \mathbb{R}$ is smooth iff $f \circ \varphi_\alpha$ is smooth $\forall \alpha$)

— $C^\infty(M, N)$: likewise.

— define tangent spaces: For $x \in M$

$$T_xM = \{\gamma : (-\varepsilon, \varepsilon) \to M \mid \gamma(0) = x\} / \sim$$

where $\gamma_1 \sim \gamma_2 \iff (\varphi_\alpha \circ \gamma_1)'(0) = (\varphi_\alpha \circ \gamma_2)'(0)$ for some α

$\iff$ " " for all α

$\iff (f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for all $f \in C^\infty(M)$

then for $\gamma : \mathbb{R} \to M$, $\gamma'(t)$ is well-defined elt of $T_{\gamma(t)}M$

This gives rise to another way to define: given $V \in T_xM$, the directional deriv of f in the direction of V is well-defined, denoted Vf. This satisfies product rule: $V(f \cdot g) = Vf \cdot g(x) + f(x) \cdot Vg$.

In fact, $T_xM = \{D : C^\infty(M) \to \mathbb{R}$ s.t. D is linear, $D(f \cdot g) = D(f)g(x) + f(x)D(g)\}$

so T_xM has structure of a vector space.

— Let TM = tangent bundle of M = collection of all the tangent spaces.

— Let $\mathcal{V}(M) = \{$smooth vector fields on $M\}$ - smooth maps $X : M \to TM$ s.t. $X(p) \in T_pM$ $\forall p$.

~~This lets us d~~ ~~Then we can define~~

For $f \in C^\infty(M,N)$, we define the derivative $f_*: TM \to TN$ or Df. $\quad f_*: T_xM \to T_{f(x)}N$

Several ways to define: 1) Velocity of curves – for $v \in T_xM$, let $\gamma: (-\varepsilon,\varepsilon) \to M$, $\gamma(0)=x$, $\gamma'(0)=v$, define $f_*(v) = (f\circ\gamma)'(0)$.

2) Differential operators: for ~~f~~ $h \in TN$, $v \in T_xM$ let $f_*(v)h = V(f\circ h)$ when all else fails, we write this in coords.

3) Coordinates charts: Let $x \in M$, $\varphi: U \to B \subset \mathbb{R}^d$ a chart for a nbhd of x. ~~Then Let $x^1, \dots, x^d$~~ ~~Suppose $\varphi = (x_1, \dots, x_d)$~~. Let $\varphi = (x^1, \dots, x^d)$ be the components of φ. Then $\frac{\partial}{\partial x^i}$ is a vector field ~~on~~ ~~M~~ U - namely, the standard basis of $\mathbb{R}^d$, pushed forward under φ.
(Specifically, the vector field corresp to curves of the form $\gamma(t) = (c^1, \dots, c^{i-1}, t, c^{i+1}, \dots, c^d)$)

~~The fields $\partial_i = \frac{\partial}{\partial x^i}$ form a basis of ~~ frame field.~~

~~$\forall u \in U$, $(\partial_1)_u, \dots, (\partial_d)_u$ form a basis of T_uM.~~

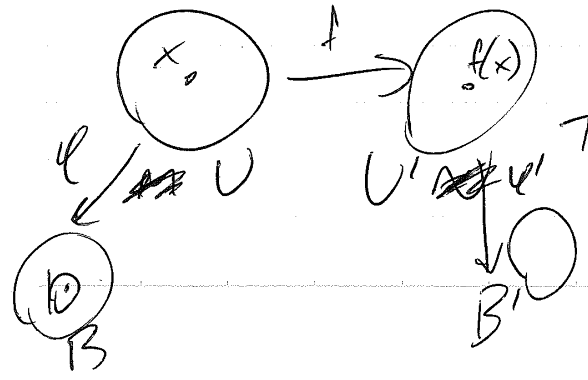
Let $\partial_i = \partial/\partial x^i$ ~~and~~ then $\forall u \in U$, $(\partial_1)_u, \dots, (\partial_d)_u$ form a basis.

Let ~~$\lambda = (y_1, \dots, y_{d'})$~~: $U' \subset N$ ~~be~~ and $\varphi': U' \to B' \subset \mathbb{R}^{d'}$ a chart for $f(x)$. $= (y_1, \dots, y_{d'})$ Then likewise $\partial_i' = \partial/\partial y_i \in \mathcal{V}(U')$ ~~form a basis~~ ~~are~~

~~Let Then~~ Then we can write f in coordinates
$\varphi' \circ f \circ \varphi^{-1}: x \in W \subset U \to U'$ and

$$D(\varphi' \circ f \circ \varphi^{-1}) \in \mathbb{R}^{d\times d'}$$

3) Coordinate charts: when all else fails, we should always be able to fall back to a coordinate system.



Let $\varphi: U \to B$ be a chart near x, $\varphi': U' \to B'$ a chart near $f(x)$.
Then $\varphi' \circ f \circ \varphi^{-1}: W \to B'$ is defined on a small nbhd of x, smooth, so it has a derivative $D(\varphi' \circ f \circ \varphi^{-1}) \in \mathbb{R}^{d\times d'} = M$
(Matrix: takes $\mathbb{R}^d \to \mathbb{R}^{d'}$)

How is this related to f_*? ~~Let $\partial_1, \dots, \partial_d \in V(U)$~~

Basis of T_xM: Let $\partial_1, \dots, \partial_d \in V(U)$ be image of standard basis. (Why this notation? Vectors are differential operators. ~~Let $\varphi = (x^1, \dots, x^d)$ — then $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^d}$~~

Let $\varphi = (x^1, \dots, x^d) : U \to \mathbb{R}^d$. Then $\frac{\partial}{\partial x^i}$ is deriv along curves of the form $\gamma(t) = (x_1^1, \dots, t, \dots, x^d)$, so coordinate vector fields)

Let $\partial_1', \dots, \partial_{d'}' \in V(U')$ be image of standard basis.

Then $f_*(v^1 \partial_1 + \dots + v^d \partial_d) = w^1 \partial_1' + \dots + w^{d'} \partial_{d'}'$, where

$$\begin{pmatrix} w^1 \\ \vdots \\ w^{d'} \end{pmatrix} = M \begin{pmatrix} v^1 \\ \vdots \\ v^d \end{pmatrix}$$

Cumbersome: Let m_i^j s.t. $f_*(\partial_i) = \sum_j m_i^j \partial_j'$ (coeffs of M): $M = \begin{pmatrix} m_1^1 & m_2^1 & \cdots \\ m_1^2 & & \\ \vdots & & \end{pmatrix}$ Then

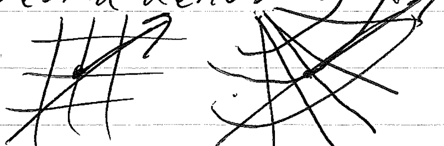
$f_*(\sum_i v^i \partial_i) = \sum_{ij} v^i m_i^j \partial_j'$. ~~(This is why~~ we use the

which we abbreviate $f_*(v^i \partial_i) = v^i m_i^j \partial_j'$

(Einstein summation convention: if a term has i in a subscript and superscript, we sum over i) — ∂_i - vector, v^i - covector.)

~~This is the basics of smooth manifolds. We're interested in Riemannian mflds.~~

Last time: started to do geom based on second derivatives of curves. Problem: Second derivs on smooth mflds aren't well defined. That is, if



we consider the same curve in two coord systems, the first deriv in coordinates just changes by a change of basis:

But the second doesn't, because fundamentally, second deriv compares vectors in diff't spaces

To deal with that, we define connections.

A connection on M is a collection of fns $\nabla : T_pM \times V(M) \to T_pM$ for $p \in M$, written $\nabla_{X_p} Y$ for $X_p \in T_pM$, $Y \in V(M)$, s.t.:

- ∇ is bilinear in X_p and Y
- (smoothness) if $X, Y \in V(M)$, then $\nabla_X Y = (p \longmapsto \nabla_{X_p} Y) \in V(M)$.
- (Leibniz): $\nabla_{X_p} fY = (X_p f) Y + f \nabla_{X_p} Y$.

Leibniz rule means that it acts like a derivative —
in this case, like a directional derivative of Y in direct of X.

Ex: If $M \subset \mathbb{R}^n$ is a smooth submanifold, we define the <u>tangential connection</u> ~~$\nabla^T_{X_p} Y = \frac{d}{dt} Y(\gamma(t))|_{t=0}$, where~~

~~$= \pi_{TM} \frac{d}{dt} = \pi_{TM}((Y \circ \gamma)'(0))$ where $\gamma'(0) = X_p$.~~

for $Y \in \mathcal{V}(M)$ extend Y to ~~a smooth~~ $\overline{Y} \in \mathcal{V}(\cancel{\mathbb{R}}U)$, where U is a nbhd of M. Let $\nabla^T_{X_p} Y = \pi_{TM}(X_p \overline{Y})$.
We can view $\overline{Y} : U \to \mathbb{R}^n$

~~Then~~ Check: — this is well-defined
— bilinear — Leibniz.

Can use this to define geodesic curvature.

But: There are <u>lots</u> of connections. Next time: ~~define~~ define more connections, explain how to pick the "best" one.

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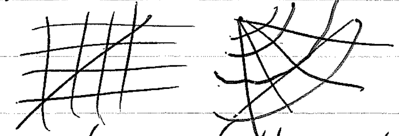
Next year, maybe swap this w/ introduction of Riem mflds

Last time: Smooth manifolds, ~~smooth maps~~, tangent spaces and derivatives of smooth maps.

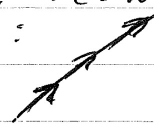
Today: ~~derivatives of vector fields~~ ~~curvature~~ second derivatives of maps.

Problem: ~~first~~ derivatives ~~of~~ are well-defined ~~up to~~ tangent vectors:



same curve in two coord systems, ~~there~~ the derivatives of coordinate fns ~~one coord system~~ differ by a change of basis.

But the second derivatives don't, because 2nd deriv compares vectors in different tangent spaces:



How can we compare diff't tangent spaces?

Connections: A connection on M is a collection of functions $\nabla: T_pM \times \mathcal{V}(M) \to T_pM$ for $p \in M$, written $\nabla_{X_p} Y$ for $X \in T_pM$, $Y \in \mathcal{V}(M)$ s.t.

- ∇ is bilinear in X_p and Y.
- (smoothness): If $X, Y \in \mathcal{V}(M)$, then $(p \mapsto \nabla_{X_p} Y)$ is smooth. We write this as $\nabla_X Y$
- (Leibniz): for $f \in C^\infty(M)$, $Y \in \mathcal{V}(M)$, $\nabla_{X_p}(fY) = f(p)\nabla_{X_p} Y + (X_p f) Y_p$ i.e., $\nabla_X fY = f\nabla_X Y + (Xf)Y$.

∇_{X_p} acts like ~~a derivative~~ a directional derivative of Y in direction of X.

Ex: If $M = \mathbb{R}^n$, define the trivial connection: for $X_p \in T_p\mathbb{R}^n \cong \mathbb{R}^n$, $Y \in \mathcal{V}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n, \mathbb{R}^n)$

$$\nabla_{X_p} Y = \frac{d}{dt}\Big|_{t=0} Y_{p+tX_p}$$

- check bilinear, Leibniz.

Ex: If $M \subset \mathbb{R}^n$ is a smooth submanifold, can define tangential connection: for $Y \in \mathcal{V}(M)$ let $\tilde{Y} \in \mathcal{V}(U)$ be an extension of Y to a tubular nbhd $M \subset U \subset \mathbb{R}^n$

Let $\nabla^T_{X_p} Y = \pi_{T_pM}(\nabla_{X_p} \tilde{Y})$ Then we can diff $\tilde{Y}$ in direction of X_p, but gen. not in tangent space. So we project.

Check: bilinear, Lipschitz

But there are lots of these! Let $\varphi = (x^1, \dots, x^n): U \to \mathbb{R}^n$ be a coordinate chart, let $\partial_1, \dots, \partial_n \in \mathcal{V}(U)$ corresp v. fields.
Let Γ^k_{ij} ~~be~~: $U \to \mathbb{R}$ be arbitrary smooth functions

(oops, this isn't quite accurate — you need a PoU to construct connections)

~~Then $\exists$ a connection ∇ s.t.~~ Then $\exists$ connection ∇ s.t. $\nabla_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k \partial_k$.

~~For $X = f^i\partial_i$, $Y = g^j\partial_j$~~

Namely, if $X = \sum f^i \partial_i$, $Y = \sum g^j \partial_j$, then Leib + bilin imply

$$\nabla_X Y = \nabla_{\cancel{\sum f^i\partial_i}X} \sum(g^j\partial_j) = \cancel{\sum f^i \nabla_{\partial_i}\sum g^j\partial_j}$$

$$\cancel{= \sum f^i(\partial_i g^j)\partial_j}$$

$$= \sum (Xg^j)\partial_j + \sum g^j \nabla_X \partial_j$$

$$= \sum (Xg^j)\partial_j + \sum g^j f^i \nabla_{\partial_i}\partial_j$$

$$= \sum (Xg^j)\partial_j + \sum g^j f^i \Gamma_{ij}^k \partial_k$$

(Check that this satisfies definition) ~~What can we do w/ connections?~~

~~And we use this to differentiate along curves.~~

~~One of the main~~ Two big uses of connections: ① to define derivative of v. field along a curve. For γ: ~~$[0,1] \to M$~~, $I \to M$, let $\mathfrak{X}(\gamma) =$ {smooth v. fields along γ}, ~~i.e.~~ ~~That is, $V \in \mathfrak{X}(M)$~~

That is, $\mathfrak{X}(\gamma) = \{V : I \to TM \mid V(t) \in T_{\gamma(t)}M$, V is smooth

$V \in \mathfrak{X}(\gamma)$ ⟺ (i.e. $t \longmapsto V(t)f$ is smooth $\forall f \in C^\infty(M)\}$

or $\forall t_0$ $\exists$ chart $U \ni \gamma(t_0)$, smooth fns $v^1, \dots, v^n$ s.t.

$V(t) = \sum v^i(\partial_i)$ for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$.

Then: ~~Given~~

Lemma: Given a connection ∇ and a curve $\gamma : [0,1] \to M$, $\exists!$ <u>covariant derivative</u> $D_t : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$ s.t.

① D_t is linear.

② $\forall f \in C^\infty([0,1])$ $D_t(fV) = \frac{df}{dt}V + fD_tV$. ~~$\forall f \in$~~

③ If V is the restriction of $\overline{V} \in \mathfrak{X}(M)$, then $D_tV = \nabla_{\gamma'(t)}\overline{V}$.

Pf: In any patch, ③ implies $D_t\partial_i = \nabla_{\gamma'(t)}\partial_i$, so by ①, ②,

(✱) $D_tV = D_t(\sum v^i\partial_i) = \sum \frac{dv^i}{dt}\partial_i + \sum v^i \nabla_{\gamma'(t)}\partial_i$.

So if D_t exists, it's unique. Ex: Check that (✱) satisfies the def. So we can define D_t on any coord patch.

Cover γ by patches. Define D_t on each. By uniqueness, these defs agree on overlap.

② Parallel fields: $V \in \mathfrak{X}(\gamma)$ is parallel along γ if $D_t V = 0$

~~Note:~~ $\cancel{[0,1]} \xrightarrow{\gamma} M$ $\qquad t_0 \in I$

Lemma: Let γ be a smooth curve. Then $\forall\, c \in T_{\gamma(t_0)}M$, $\exists!\, V \in \mathfrak{X}(\gamma)$ s.t. $D_t V = 0$ and $V(t_0) = c$.

Pf: If γ lies in a patch, we can write $V = \sum v^i \partial_i$ and $\gamma = (\gamma^1, \dots, \gamma^n)$. Then

$$D_t V = 0 \iff 0 = D_t\left(\sum v^i \partial_i\right) = \frac{dv^i}{dt}\partial_i + v^i D_t \partial_i$$

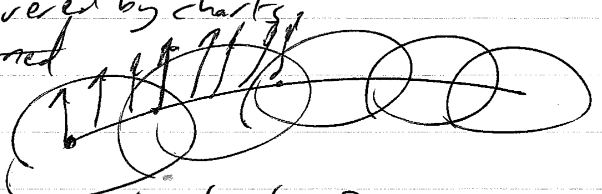
$$\sum -\frac{dv^i}{dt}\partial_i = \sum v^i \nabla_{\gamma'(t)} \partial_i = \sum v^i \nabla_{\frac{d\gamma^j}{dt}\partial_j} \partial_i$$

$$-\frac{dv^i}{dt}\partial_i = v^i \frac{d\gamma^j}{dt} \Gamma^k_{ji}\, \partial_k$$

$$-\frac{dv^k}{dt}\partial_k = v^i \frac{d\gamma^j}{dt} \Gamma^k_{ji}\, \partial_k$$

This is a system of 1st order ODEs so it has a unique solution for any initial cond.

As before, γ ~~is contained~~ can be covered by charts, so

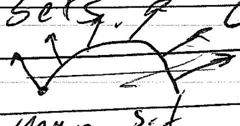


we can define a parallel field in each chart. By uniqueness, these agree on overlaps. //

③ Parallel transport: Let $\gamma : I \to M$, $t_0, t_1 \in M$.

~~Lem Cor Cor: $\forall t_0$, Let $t_0, t_1 \in I$. Then $\exists$ a parallel transport~~

~~Note: Generally, we can only define parallel fields on curves, not on open sets, a Connections have curvature:~~



~~Let ... $P_{t_0,t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$ be the map s.t. $\forall\, c \in T_{\gamma(t_0)}M$, if V_c is the parallel field s.t. $V_c(t_0) = c$ then $V_c(t_1) = P_{t_0,t_1}(c) = V_c(t_1)$~~ Then

For $c \in T_{\gamma(t_0)}M$ let V_c be the parallel field s.t. $V_c(t_0) = c$.

Let $P_{t_0,t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$ be the map $P_{t_0,t_1}(c) = V_c(t_1)$.

Then:

~~① P_{t_0,t_1} is an linear isomorphism~~ ① $\forall\, t_0, t_1, t_2 \in I$,
$$P_{t_0,t_2} = P_{t_1,t_2} \circ P_{t_0,t_1}.$$

② P_{t_0,t_1} is a linear isomorphism.

③ ~~$D_t(V)$~~ $D_t V(t_0) = \frac{d}{dt} P_{t,t_0}(V_t)\big|_{t=0}$.

Pf: As much as possible.

No.
Date 2024 02 01

Recap: ~~Not sure I've done the best job of explaining how things relate,~~ so I wanted a quick recap:

— ~~Ex If $M \subset \mathbb{R}^n$, we can define geodesics in M as curves s.t. $\gamma''(t)$ is orthogonal to $T_{\gamma(t)}M$. We can define curvature~~

~~Last time: Connections: differentiate a vector field on M over γ.~~
~~In particular, $V \in \mathcal{V}(\gamma)$ is parallel if $D_t V = 0$.~~

Recap: We started by considering ~~curves~~ submanifolds in $\mathbb{R}^n$ — we said that if $h_u : (-\varepsilon, \varepsilon) \times [0,1] \to M$ is a variation of $\gamma = h_0$, $U = \frac{\partial h}{\partial u}$, $U(0) = U(1) = 0$,
then $\frac{d}{du} E(h_u) = -\int_0^1 \langle U, \gamma'' \rangle \, dt = -\int_0^1 \langle U, \pi_{TM}(\gamma'') \rangle \, dt$

~~γ is a geodesic if $\pi_{TM}(\gamma'') = 0$~~, Geodesic curvature of $\gamma = \pi_{TM}(\gamma'')$

γ is a geodesic if $\pi_{TM}(\gamma'') = 0$, ~~and~~

— geodesic curvature is preserved by length-preserving maps. Why?

Last time: Connections. We can formalize this w/ connections:

— let ∇^T be the tangential connection so that $D_t^T V = \pi_{TM}\left(\frac{dV}{dt}\right)$

Then $D_t^T \gamma' = \pi_{TM}(\gamma'')$ is geodesic curvature.

γ is a geodesic $\iff D_t^T \gamma' = 0 \iff \gamma' \in \mathcal{V}(\gamma)$ is a parallel field.

— Claim: ∇^T is preserved by length-preserving maps.

Q: What makes ∇^T special? How can we generalize to arbitrary mflds?

Fundamental Lemma of Riemannian Geometry: Let (M, g) be a Riemannian manifold. Then $\exists !$ connection ∇ on M which is torsion-free and compatible with the metric. When $M \subset \mathbb{R}^n$, ~~this is the tangent~~ $\nabla = \nabla^T$.

Today: Explain the pieces of this lemma.

Riemannian manifold: How many have heard the definition?

Let M be a smooth manifold. A Riemannian metric on M assigns a positive-definite inner product to each tangent space T_pM in a smooth way: — for each $x \in M$, there is a ~~symmetric bilinear form g_x s.t. $\forall u, v \in T_xM$~~

— ~~symmetry~~ $g_x : T_xM \times T_xM \to \mathbb{R}$ s.t. $\forall u, v, w \in T_xM$, $\forall a, b \in \mathbb{R}$,

$g_x(u,v) = g_x(v,u)$ $\quad g_x(au + bv, w) = a\, g_x(u,w) + b\, g_x(v,w)$.

$g_x(u,u) \geq 0$ and $g_x(u,u) = 0 \Rightarrow u = 0$.

(Symmetric, bilinear, positive-definite)

- (smooth): $\forall U, V \in \mathcal{V}(M)$, $x \mapsto g_x(U_x, V_x)$ is smooth.

We'll write $g_x(u,v)$ as $\langle u|v\rangle$. (This lets us measure length: $\|u\| = \sqrt{\langle u,u\rangle}$ and angle: $\cos\angle(u,v) = \frac{\langle u,v\rangle}{\|u\|\|v\|}$.)

(Ex: If $M \subset \mathbb{R}$ is a smooth submanifold, then $g_x(u,v) = \langle u|v\rangle_{Euc.}$ is a Riem. metric (induced metric))

If g is a Riemannian metric, $\varphi: U \to \mathbb{R}^n$ is a chart w/ vecs $\partial_1, \dots, \partial_n$, we let $g_{ij} := \langle \partial_i | \partial_j \rangle$ - these n^2 smooth fns describe g. $= g_{ji}$

~~We say that ∇ is compatible with g~~ if

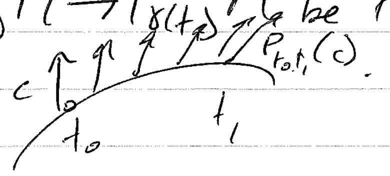
Compatible with the metric is based on parallel fields, parallel transport.

Parallel transport: Recall: $V \in \mathcal{V}(\gamma)$ is parallel if $D_t V = 0$

Lemma: $\forall t_0 \in I$, $\forall c \in T_{\gamma(t_0)}M$, $\exists!$ $V_c \in \mathcal{V}(\gamma)$ s.t. V_c is parallel, $V_c(t_0) = c$.

For $t_0, t_1 \in I$, let $P_{t_0,t_1}: T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$ be the map

$$P_{t_0,t_1}(c) = V_c(t_1)$$



~~Lemma~~: Then: P_{t_0,t_1} is linear

Pf: $V_a + V_b$ is parallel, so $V_a + V_b = V_{a+b}$.

- If $t_0, t_1, t_2 \in I$ then $P_{t_0,t_2} = P_{t_1,t_2} \circ P_{t_0,t_1}$.
- So P_{t_0,t_1} is an isomorphism.
- (Exercise: $D_t V(t_0) = \frac{d}{dt}\Big|_{t=t_0} P_{t,t_0}(V)$ - ∇ determined by parallel transport

We say that ∇ is compatible with g if P_{t_0,t_1} is an isometry for all γ, all t_0, t_1

Lemma: TFAE: ① ∇ is compatible with g.

② - $\forall \gamma$, If $V_1, V_2 \in \mathcal{V}(\gamma)$ are parallel, then $\langle V_1 | V_2 \rangle$ is const.

③ - $\forall V$, if V is parallel, then $\|V\|$ is constant.

④ - $\forall V, W \in \mathcal{V}(\gamma)$, $\frac{d}{dt}\langle V|W\rangle = \langle D_t V | W\rangle + \langle V | D_t W\rangle$

⑤ - $\forall X, Y, Z \in \mathcal{V}(M)$, $X\langle Y|Z\rangle = \langle \nabla_X Y | Z\rangle + \langle Y | \nabla_X Z\rangle$

Pf: ①$\Leftrightarrow$②$\Leftrightarrow$③ Exercise, ④$\Leftrightarrow$⑤ Exercise.

④$\Rightarrow$②: if V_1, V_2 parallel, then $\frac{d}{dt}\langle V_1 | V_2\rangle = \langle D_t V_1 | V_2\rangle + \langle V_1 | D_t V_2\rangle$
$= 0 \Rightarrow$ ②.

②⇒④: ~~Use~~ As always, use coordinates. If ② holds, there are orthonormal parallel fields $E_1, \dots, E_n \in \mathcal{V}(\gamma)$, and we can write $V = \sum v^i E_i$, $W = w^j E_j$. Then

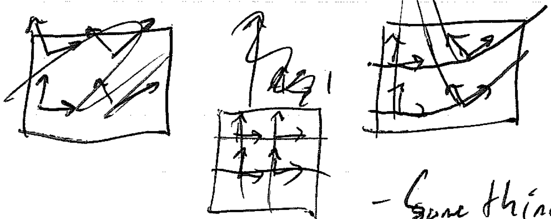
$$D_t V = \sum \frac{dv^i}{dt} E_i + \cancel{v^i D_t E_i} \to 0, \quad D_t V = \frac{dw^j}{dt} E_j.$$

$$\frac{d}{dt}\langle V|W\rangle = \frac{d}{dt}\underbrace{\sum_i v^i w^i}_{\sum v^i \delta_{ij} w^j} = \sum_i \frac{dv^i}{dt} w^i + \sum_i v^i \frac{dw^i}{dt} = \langle V|D_t W\rangle + \langle D_t V|W\rangle$$

Ex: The tangential connection on $M \subset \mathbb{R}^n$ is compatible with the induced metric.

~~One more term: torsion~~ → ~~free~~. Is there a unique ∇ which is compat w/ metric? No: Take Euc metric on $\mathbb{R}^n$. Let $E_1, \dots, E_n \in \mathcal{V}(\mathbb{R}^n)$ be orthonormal fields. Then $\exists$ a connection s.t. the E_i's are parallel — this is compatible w/ metric, but it's twisted. The torsion of ∇ measures that twisting.

Torsi:



~~Torsion~~ What makes this twisted?
- can't extend to a grid (i.e., are coord vectors)
- don't commute.
- Something, measured by the Lie bracket.

Lie bracket: ~~Commutativity expressed~~ whether two v. fields "commute".

~~What is~~ Def: If $X, Y \in \mathcal{V}(M)$ let $[X, Y] = XY - YX$.

That is, X, Y are 1st order diff operators, XYf is a second deriv of f, so is YXf, but the difference is 1st order — a derivation. (exercise), so it's a vector field.

Further, $[\ ,\ ]$ is — alternating — bilinear over $\mathbb{R}$, — satisfies a product rule. Namely if $\mathcal{L}_X(Y) = [X, Y]$, then

$$\mathcal{L}_X(fY) = (Xf)Y + f\mathcal{L}_X(Y) \quad \text{(exercise)}$$
$$[X, fY] = (Xf)Y + f[X, Y]$$

— measures whether X, Y can be coordinate vectors. $\varphi = (x^1, \dots, x^n): U \to \mathbb{R}^n$

~~Lemma (Frobenius)~~: On one hand, if $\partial_1, \dots, \partial_n$ coord fields, then

$$[\partial_i, \partial_j] f = \partial_i \partial_j f + \partial_j \partial_i f = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i} f = 0. \Rightarrow$$
$$[\partial_i, \partial_j] = 0. \quad \text{OTOH} \ \cancel{\text{Frob}}$$

Thm (Frobenius): Suppose $E_1, \dots, E_k \in \mathcal{V}(M)$ commute. Then $\forall p \in M$, $\exists f: (-\varepsilon, \varepsilon)^k \to M$ s.t. $\frac{\partial f}{\partial x^i} = E_i$ $\forall i$.

Over thm: ~~Ex~~ Lie bracket examples, $\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$

No.
Date: 2024 02 06

Last time: FLRG: Let (M,g) be a Riemannian manifold. Then $\exists!$ connection ∇ on M which is torsion-free, compat with g.

- Compat w/ g: parallel fields on any curve γ have constant length.
- Torsion-free: "~~locally parallel fields at locally commute~~ at p

"fields that are locally parallel at p commute at p"

That is, if $p \in M$, and if $X \in \mathcal{V}(M)$, we say X is locally parallel at p if $\nabla_{V_p} X = 0$ $\forall V_p \in T_pM$. ∇ is torsion-free $\Leftrightarrow$

$[X,Y](p) = 0$ $\forall X, Y \in \mathcal{V}(M)$ that are locally parallel at p.

Today: explain this.

Last time: for $X, Y \in \mathcal{V}(M)$, $[X,Y] = XY - YX \in \mathcal{V}(M)$
= Lie bracket of X and Y.

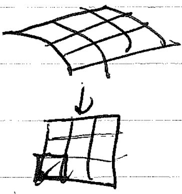
Props: ① alternating: $[X,Y] = [Y,X]$

② bilinear over $\mathbb{R}$: $[aX, aY + bZ] = a[X,Y] + b[X,Z]$

③ Leibniz: Let $\mathcal{L}_X(Y) = [X,Y]$. Then $\mathcal{L}_X(fY) = (Xf)\cdot Y + f\mathcal{L}_X(Y)$
$= (Xf)\cdot Y + f[X,Y]$.

④ If $\varphi: U \to \mathbb{R}^n$, $\partial_1, \dots, \partial_n \in \mathcal{V}(U)$, then

$$[\partial_i, \partial_j] = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i} = 0$$



~~In particular, if $X = v^i\partial_i$, $W = w^j\partial_j$, then ④ imply~~

~~$[X,Y] = X^i[\partial_i \dots]$~~

~~$[X,Y] = \mathcal{L}_X(y^j\partial_j)$~~ Std. Example: $X_{(x,y,z)} = (1,0,0)$, $Y_{(x,y,z)} = (0,1,x)$.

~~$[X,Y] = (X^j\partial_j Y^i - Y^j\partial_j X^i)\partial_i$~~ $[X,Y] = [\partial_x, \partial_y + x\partial_z]$

~~$= (X y^i - Y x^i)\partial_i$~~ $= [\partial_x, \partial_y] + \partial_x x\,\partial_z$

~~(Ex.) $[V,W] = (v^j\partial_j w^i - w^j\partial_j v^i)\partial_i = (Vw^i - Wv^i)\partial_i$~~ $+ x[\partial_x, \partial_z]$

Couple ways to view this:

- As in PS $\Phi_Y^{-t} \circ \Phi_X^{-t} \circ \Phi_Y^{t} \circ \Phi_X^{t} \approx \Phi_{[X,Y]}^{t^2}$ (Any v. field X has a flow: if $K \subset M$ cpt, a family of maps $\Phi_X^t : K \to M$ for $t \in (-\varepsilon, \varepsilon)$ s.t. $\frac{d}{dt}[\Phi_X^t(p)] = X(\Phi_X^t(p))$ $\forall p \in K$, $\forall t \in (-\varepsilon, \varepsilon)$.)

~~- If $[X,Y] = 0$ on U, then $\forall p \in U$, $\forall$ suff small s,t,~~

~~$\mathcal{L}_X(Y)$~~ - If $[X,Y] = 0$ on U, $p \in U$, then $\forall$ suff small s,t.

$$\Phi_X^s \circ \Phi_Y^t = \Phi_Y^t \circ \Phi_X^s$$

Thm (Frobenius Lemma): If $[X_i, X_j] = 0$ $\forall i,j$ on U, ~~then~~ and $p \in U$, then $\exists$ a map $f(x^1, \dots, x^k) = \Phi_{X_1}^{x^1} \circ \dots \circ \Phi_{X_k}^{x^k}(p)$ s.t. $\frac{\partial f}{\partial x^i} = X_i$ $\forall i$.

So if ~~$[X_i, X_j$~~ $X_1, \dots, X_n$ commute, $X_1(p), \dots, X_n(p)$ a basis of T_pM, then $\exists$ chart s.t. $\partial_i = X_i$ near p.

— $[X,Y]$ measures whether we can extend X, Y to coordinate fields

Def: The torsion of a connection ∇ is ~~$\tau(X,Y)$ & def~~ $\forall X, Y \in \mathcal{V}(M)$, $\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$

This is a priori a fn $\tau: \mathcal{V}(M) \times \mathcal{V}(M) \longrightarrow \mathcal{V}(M)$. But ~~it is tensorial~~: in fact, its really a fn $T_pM \times T_pM \to T_pM$.

Lemma: $\tau(X,Y)_p$ is determined by X_p and Y_p.

Pf: Check that τ is alternating, bilinear over $\mathbb{R}$.

Let $f \in C^\infty(M)$ Claim: $\tau(X, fY) = f\tau(X,Y)$.

$$\tau(X, fY) = \nabla_X fY - \nabla_{fY} X - [X, fY]$$
$$= \cancel{Xf \cdot Y} + f\nabla_X Y - f\nabla_Y X - f[X,Y]$$
$$= f\nabla_X Y - f\nabla_Y X - f[X,Y] = f\tau(X,Y).$$

Therefore, if $X = \sum f^i \partial_i$, $Y = g^j \partial_j$, then

$$\tau(X,Y) = \sum \tau(X, \sum_j g^j \partial_j) = \sum g^j \tau(X, \partial_j)$$
$$= \sum g^j \tau(\sum f^i \partial_i, \partial_j)$$
$$= \sum_{ij} f^i g^j \tau(\partial_i, \partial_j).$$

~~But $f(p), g(p)$ are det~~

$$\tau(X,Y)_p = f^i(p) g^j(p) \tau(\partial_i, \partial_j),$$

and $f^i(p)$ and $g^j(p)$ are determined by X_p and Y_p. //

So τ assigns a bilinear $\tau_p: T_pM \times T_pM \to T_pM$ to each point — τ is a tensor. How to think about it? Well, it doesn't matter what fields, take the most convenient: make $\nabla_X Y$ vanish?

~~Further,~~ If $X, Y \in \mathcal{V}(M)$ are locally parallel at p, then
$$\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = -[X,Y]$$
torsion measures whether locally parallel fields commute.

Proof of FLRG: Let ∇ be a connection, let $\varphi: U \to \mathbb{R}^n$ be a coord chart.

Claim: ~~If ∇ is torsion-free, compat, it is unique.~~

~~Pf:~~ $\exists!$ torsion-free, compatible connection on U.

First, suppose ∇ exists.

Then $\tau(\partial_i, \partial_j) = \nabla_{\partial_i}\partial_j - \nabla_{\partial_j}\partial_i - [\partial_i, \partial_j] = 0$

Torsion-free $\Rightarrow \tau(\partial_i, \partial_j) = 0 \Rightarrow \nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i$

Compatible: $\partial_i \langle \partial_j | \partial_k \rangle = \langle \nabla_{\partial_i}\partial_j | \partial_k \rangle + \langle \partial_j | \nabla_{\partial_i}\partial_k \rangle$

Permute: $\partial_j \langle \partial_k | \partial_i \rangle = \langle \nabla_{\partial_j}\partial_k | \partial_i \rangle + \langle \partial_k | \nabla_{\partial_j}\partial_i \rangle$

$\partial_k \langle \partial_i | \partial_j \rangle = \langle \nabla_{\partial_k}\partial_i | \partial_j \rangle + \langle \partial_i | \nabla_{\partial_k}\partial_j \rangle$

These have common terms & b.c. torsion-free. So:

$$\partial_i\langle \partial_j | \partial_k \rangle + \partial_j \langle \partial_k | \partial_i \rangle - \partial_k \langle \partial_i | \partial_j \rangle = 2\langle \nabla_{\partial_i}\partial_j | \partial_k \rangle$$

That is, $\langle \nabla_{\partial_i}\partial_j | \partial_k \rangle$ is determined by the metric.
But ∂_k is a basis, so $\nabla_{\partial_i}\partial_j$ is determined by the metric.
$\Rightarrow \nabla$ is unique.

Existence on U: Any smooth fns $\Gamma^k_{ij} \in C_\infty(U)$ determine a connection.
Let $g_{ij} = \langle \partial_i | \partial_j \rangle$ and let $g^{ij} = (g_{ij})^{-1}$ (matrix inverse)
$g_{ij} g^{jk} = \{1 \text{ if } i=k,\ 0 \text{ if } i \neq k\} = \delta_i^k$ ~~Then $g_{ij} g^{jk} = \sum_j g_{ij} g^{jk} = \{1\ i=k,\ 0\ i \neq k$~~ $= \delta_i^k$

Then we can rewrite:

$$\tfrac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}\right) = \langle \nabla_{\partial_i}\partial_j | \partial_k \rangle = \langle \Gamma^l_{ij}\partial_l | \partial_k \rangle = g_{kl}\Gamma^l_{ij}$$

- metric mtx

$$g^{mk} g_{kl} \Gamma^l_{ij} = \delta^m_l \Gamma^l_{ij} = \Gamma^m_{ij}.$$

So $\tfrac{1}{2} g^{mk}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}) = g^{mk} g_{kl}\Gamma^l_{ij} = \Gamma^m_{ij}$.

— for any metric g, $\exists!$ a connection on U.

Construct ∇ on every chart. By uniqueness, these agree on overlapping charts $\Rightarrow \nabla$ exists on all of M. //
This is called the <u>Levi-Civita connection</u>.

Overflow: Γ^m_{ij} depends only on g_{ij}, $\partial_i g_{jk}$. ~~Cut a strip of orange peel. Lay it out flat.~~ Take an orange. Draw ~~a curve on the orange, draw a field.~~ Peel the orange in a long strip. Draw a ~~field~~ curve on center of strip, v. field along that curve. What's Levi-Civ?