2024 02 08

The Levi-Civita Connection.

Last time: FLRG: Let M be a Riem. mfld. $\exists!$ connection ∇ on M which is compatible w/ g, torsion-free.

Last time: Uniqueness: If ∇ exists, then for any chart, then

$$\partial_i \langle \partial_j | \partial_k \rangle + \partial_j \langle \partial_k | \partial_i \rangle - \partial_k \langle \partial_i | \partial_j \rangle = 2 \langle \nabla_{\partial_i} \partial_j | \partial_k \rangle$$

(So if there's another, then $\langle \nabla_{\partial_i} \partial_j - \nabla'_{\partial_i} \partial_j | \partial_k \rangle = 0 \ \forall k \Rightarrow \nabla = \nabla'$)

Existence: find a formula for $\nabla_{\partial_i} \partial_j$.

Need a side note about matrix multiplication:

Let $g_{ij} = \langle \partial_i | \partial_j \rangle$. Then $\begin{pmatrix} g_{11} & \cdots & g_{n1} \\ \vdots & & \vdots \\ g_{1n} & \cdots & g_{nn} \end{pmatrix} = M$ represents the metric: $g_{ij} = g_{ji}$ and

if $v = v^i \partial_i$, $w = w^j \partial_j$, then $\langle v | w \rangle = \cancel{\sum} v^i w^j \underline{g_{ij}} = (v^1 \cdots v^n) M \begin{pmatrix} w^1 \\ \vdots \\ w^n \end{pmatrix}$

Define g^{ij} so that $M^{-1} = \begin{pmatrix} g^{11} & \cdots & g^{n1} \\ \vdots & & \vdots \\ g^{1n} & \cdots & g^{nn} \end{pmatrix}$ — we can write matrix mult in Einstein notation

Then $M^{-1} M = I_n = (\delta^i_j)$ where $\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

And $g_{ij} g^{jk} = (I_n)^k_i = \delta^k_i$ So:

$$\frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}) = \langle \nabla_{\partial_i} \partial_j | \partial_k \rangle = \langle \Gamma^l_{ij} \partial_l | \partial_k \rangle = \Gamma^l_{ij} g_{lk}$$

$$\frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}) g^{km} = \Gamma^l_{ij} g_{lk} g^{km} = \Gamma^l_{ij} \delta^m_l = \Gamma^m_{ij}$$

(Check: This ~~formula~~ connection is torsion-free, compatible.)

And we can finish like a lot of these: on any chart, $\exists!$. So we construct on every chart, these agree on overlap. //

This is called the <u>Levi-Civita connection</u>. There's a v. nice geometric interpretation ~~for~~:

- To characterize a connection, enough to consider parallel fields along curves. (by PSet)
- By the formula, connection depends on g_{ij} and $\partial_i g_{jk}$, so if two metrics agree up to first order at a point, ~~they have the same connection at that pt.~~ the L-C connections agree.

No.
Date.

In particular, if two surfs are tangent: $v \in T_{\gamma(0)}M$

Say we have a curve γ on a mfld and a vector – how do we extend to a parallel field? Nontrivial, but not hard – ~~we can map~~ there's a "best" map of a nbhd of γ to $\mathbb{R}^n$.

(You've seen this if you've ever peeled a apple/orange/potato – that strip of peel lays flat on cutting board.)

Under that chart, $g_{ij} = \delta_{ij}$ and $\cancel{\gamma}\ \partial_i g_{jk} = 0$ on γ. – so parallel fields on the strip are the same as parallel fields in $\mathbb{R}^n$:

– draw curve, vector. – peel strip. – complete field – replace strip. So those are the parallel fields!

What's more, you can see the holonomy: if you take a vector, parallel transport around a closed curve, you get a diff't vector.

Ex: square

Ie, let γ ~~be~~ $: [0,1] \to M$ closed curve, $\gamma(0) = \gamma(1)$.

then $P_{0,1} : T_{\gamma(0)}M \to T_{\gamma(1)}M = T_{\gamma(0)}M$ is an endomorphism.

By compatibility, it is an isometry from $T_{\gamma(0)}M \to T_{\gamma(0)}M$.

In this case, because M is 2-d, just a rotation, which makes the theory particularly nice.

~~Special case: if $\dim M = 2$ and M is orientable then $P_{0,1}$ is a rotation~~

So let's talk about this in the special case of surfaces:

Geometry of surfaces: Previously, discussed geodesic curv, so let's define ~~it~~

~~Def: Let $\gamma: [0,L] \to$~~ Let M be an orientable 2-manifold.

Def. Let $\gamma: \cancel{[0,L]}\ I \to M$ be a smooth curve parameterized s.t. $\|\gamma'(t)\| = 1\ \forall t$.

Let $V = \gamma'(t) \in V(\gamma)$, let N = unit positive normal = rotate V ccw $90° = R_{\frac{\pi}{2}} V$

Then ~~$\langle D_t V, $~~ $\langle V|V\rangle = 1$, so $\frac{d}{dt}\langle V|V\rangle = 0$, by product rule,

$\frac{d}{dt}\langle V|V\rangle = 2\langle D_t V|V\rangle = 0 \Rightarrow D_t V$ is orthogonal to V

$\Rightarrow \exists\ \kappa: I \to \mathbb{R}$ s.t. $D_t V = \kappa N$. We call κ the geodesic curvature of γ. (Turning speed of γ)

If $\gamma: S^1 \to \mathbb{R}^2$ is a simple closed curve with positive orientation, then $\int_{S^1} \kappa\, dt = 2\pi$.

~~In general, not nec. true, because of holonomy.~~

~~Now that we have parallel transport,~~ can interpret in terms of ~~transport~~.

How to generalize? Parallel fields.

Let $\gamma:[0,L]\to M$,

Let W ~~P~~ $\in V(\gamma)$ be field s.t. $D_t W = 0$, ~~$P(0)=V(0)\gamma'(0)$~~,

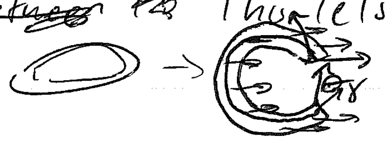
~~Then~~ By compat, P_{t_0,t_1} is an orientation-preserving isometry, $\forall\theta$, $P_{t_0,t_1}(W(t_0)) = W(t_1)$, so $P_{t_0,t_1}(R_\theta W(t_0)) = R_\theta W(t_1) \Rightarrow R_\theta W$ is parallel.

So κ is turning wrt parallel fields: if $V=\gamma'(t)$, $\exists\, \phi: I\to\mathbb{R}$ s.t. $V = R_{\phi(t)}P$. ~~Then~~ $= \cos\phi(t)\, P + \sin\phi(t)\, R_{\frac{\pi}{2}} P$

$$D_t V = \phi'(t)\left(-\sin\phi(t)\, P + \cos\phi(t)\, R_{\frac{\pi}{2}}P\right)$$
$$= \phi'(t)\, R_{\frac{\pi}{2}} V = \phi'(t) N \Rightarrow \kappa = \phi'(t).$$

$\Rightarrow \kappa$ is turning wrt a parallel field.

~~Gaussian curvature: This lets us characterize~~ ~~This gives us a connection~~ ~~between~~ geodesic curvature, ~~Gaussian curv~~ : ~~Gauss noticed remarkable connect.~~ ~~between~~ This lets us calculate $\int \kappa\, dt$ using paper strips:

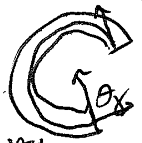


$\int \kappa\, dt$ = turning number of curve.

~~In particular~~ If $\gamma:[a,b]\to M$ is a smooth closed curve, ~~then~~

~~$P_{a,b} = P_\gamma = R_\theta$ where $\theta = \int \kappa\, dt - 2\pi$~~ let θ_γ = turning defect $= 2\pi - \int \kappa\, dt$

- how much extra turning



Turning defect is additive:

$\theta_{\gamma_3} = \theta_{\gamma_2} + \theta_{\gamma_1}$

~~M~~ So really, there's a function K on M s.t. if $\gamma = \partial D$, then $\theta_\gamma = \int K\, dA$. — this is the Gaussian curvature of M.

On the sphere of radius r the equator has $\theta = 2\pi \Rightarrow K = \frac{2\pi}{2\pi r^2} = \frac{1}{r^2}$.

~~Spaces with~~ Saddle has negative curvature.

Finally Thm (Gauss-Bonnet): Let $M \cong S^2$. Then $\int K\, dA = 4\pi$.

Pf: Let γ (unit speed) cut M into discs D_1, D_2. Then:

$$\theta_\gamma = 2\pi - \int_\gamma \kappa\, dt = \int_{D_1} K\, dA$$

$$\theta_{-\gamma} = 2\pi - \int_{-\gamma} \kappa\, dt = 2\pi + \int_\gamma \kappa\, dt = \int_{D_2} K\, dA$$

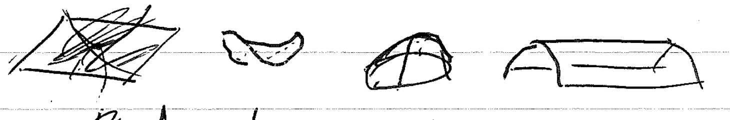
$$4\pi = \int_{S^2} K\, dA$$

Date. 2024 02 13

Last time: Gaussian and geodesic curvature for surfaces in $\mathbb{R}^3$: three related quantities:

 $\theta_\gamma = 2\pi - \int \kappa \, dL$ = ~~amount of ro~~ = turning defect

$$P_\gamma : T_{\gamma(0)}M \to T_{\gamma(0)}M = R_{\theta_\gamma} \qquad \theta_\gamma = \int K \, dA$$

(You may also have seen: 
at every pt $p \in M$, ~~M has~~
~~two principal curvatures k_1, k_2 measuring~~
can rotate, translate so that $p=0$ and near p, $M = \{z = f(x,y)\}$, where $Df = 0$. Let $H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$. Then the eigenvalues of H_f are the principal curvatures k_1, k_2, and $K = k_1 k_2$. }, so in partic $-, +, 0$)

Today: generalize to higher dims. Def: The curvature tensor of a connection ∇ is the map $R : \mathcal{V}(M) \times \mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{V}(M)$,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Then: ① R is trilinear over $\mathbb{R}$ ② R is alternating: $R(X,Y)Z = -R(Y,X)Z$.

Lemma: $(R(X,Y)Z)_p$ depends only on X_p, Y_p, Z_p. (R is a tensor)

Pf: Let $f \in C^\infty(M)$. Then we check product rules:

$$R(X, fY)Z = \nabla_X \nabla_{fY} Z - \nabla_{fY} \nabla_X Z - \nabla_{[X, fY]} Z$$

$$= \nabla_X [f \nabla_Y Z] - f \nabla_Y \nabla_X Z - \nabla_{f[X,Y] + (Xf)Y} Z$$

$$= (Xf) \nabla_Y Z + f \nabla_X \nabla_Y Z - f \nabla_Y \nabla_X Z - f \nabla_{[X,Y]} Z - (Xf) \nabla_Y Z$$

$$= f R(X,Y)Z.$$

Likewise, ~~claim that~~ exercise that $R(fX, Y)Z = R(X,Y)[fZ] = f R(X,Y)Z$

Therefore, if $X = f^i \partial_i$, $Y = g^j \partial_j$, $Z = h^k \partial_k$,

Then $R(X,Y)Z = R(\sum f^i \partial_i, \sum g^j \partial_j)[\sum h^k \partial_k]$
$= f^i g^j h^k R(\partial_i, \partial_j)\partial_k$.

$(R(X,Y)Z)_p = f^i(p)\, g^j(p)\, h^k(p)\, R(\partial_i, \partial_j)\partial_k$. — and these depend only on X_p, Y_p, Z_p. // Can define for any connection. When ∇ is L-C, ~~Furthermore,~~ R satisfies several symmetries:

Prop: ~~① : −R~~ $\forall X, Y, Z, W \in \mathfrak{V}(M)$,

① $-R(X,Y)Z = R(Y,X)Z$

② $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

③ $\langle R(X,Y)Z \,|\, W\rangle = -\langle R(X,Y)W \,|\, Z\rangle$

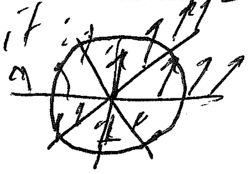
④ $\langle R(X,Y)Z \,|\, W\rangle = \langle R(Z,W)X \,|\, Y\rangle$

Pf: Exercise. ~~Further, because ∇ is~~

What does R measure?

② Whether parallel fields exist:

In general, ~~parall~~ we can construct fields parallel along lines, but that's it:



R gives a necessary cond for parallel field to exist:

swap

Suppose $U \subset M$, Z is parallel on U (i.e., $\nabla_X Z = 0 \ \forall X$)

Then $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = 0$.

Further, by symmetries, $R(Z,X)Y = R(Y,Z)X = 0$ — if parallel fields exist then R vanishes. Conversely, prove lemma.

① Holonomy of parallel transport:

Ex: If $X_u, Y_u \in T_u M$ and $f: \mathbb{R}^2 \to M$ is a map s.t. $f(0) = u$, $\partial_x(u) = X_u$, $\partial_y(u) = Y_u$, then $R(X,Y)Z = \lim_{s \to 0} \frac{Z - P_{\gamma_s}(Z)}{s^2}$, where γ_s is the image of the $s \times s$ square and P_{γ_s} is parallel-transport around γ_s — as we saw on the sphere $\approx \frac{\theta_{\gamma_s}}{s^2}$

~~PM~~ Pf: problem set.

This explains some symmetries: ~~$P_{\gamma_s}(Z) \in O(n)$,~~ ~~$\langle R(X,Y)Z \,|\, W\rangle = -\langle R(X,Y)W \,|\, Z\rangle$~~

~~$\Rightarrow R(X,Y)$ is~~ $P_{\gamma_s}(Z) \in O(n)$, so

$R(X,Y) = \lim_{s \to 0} \frac{I - P_{\gamma_s}}{s^2} \in T_I O(n)$. But ~~M B∈O(n)~~ B

~~$I+tB$~~ $B \in T_I O(n) \iff \frac{d}{dt}\Big|_{t=0} \langle (I+tB)W \mid (I+tB)Z\rangle = 0 \ \forall V,W$

$\iff \langle BW \mid Z\rangle + \langle W \mid BZ\rangle = 0 \ \forall V,W$

(i.e. B is antisymmetric)

$\Rightarrow \langle R(X,Y)Z \mid W\rangle = \langle R(X,Y)Z \mid W\rangle$

③ ~~Commutativity of parallel transport.~~

~~What does~~ ③ ~~Distance from flatness.~~ Local obstruction to flatness.

Thm: $R \equiv 0 \iff M$ is locally isometric to $\mathbb{R}^n$.

Pf: ($\Leftarrow$) $\forall p \in M$, $X_p \in T_pM$, there is a nbhd $U \ni p$ s.t. X_p extends to a parallel field $\Rightarrow R(X_p,Y)Z = 0$.

($\Rightarrow$) Step 1: Construct parallel fields.

Lemma: Suppose $R \equiv 0$. Then $\forall p \in M$, $\forall X_p \in T_pM$, $\exists U$ s.t. X_p extends to a parallel field on U, $X \in \mathfrak{X}(U)$.

Pf: Let $\varphi: U \to B_1(0) \subset \mathbb{R}^n$ be a chart near p.

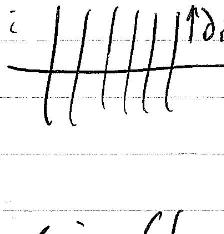
For $q = (q^1, \dots, q^n)$ $(u^1,\dots,u^n)$, let γ_q be the path $(0,0,\dots,0) \to (q^1,0,\dots,0) \to (q^1,q^2,0,\dots,0) \to \cdots \to q$. Let $X_q = P_{\gamma_q}(X_p)$. This is smooth.

~~Claim: X is parallel on U_i.~~ Claim by induction X is parallel on U_i:

Let $U_i = u^1 \cdots u^i$-plane. ~~By construction,~~

By construction, parallel on U_1. Suppose parallel on U_{i-1}.

Then:



$\uparrow \partial_i$ U_{i-1} So $\nabla_{\partial_i} X = 0$ on U_i

$\nabla_{\partial_j} X = 0$ on U_{i-1}, $\forall j < i$.

Let $j < i$. Claim: $\nabla_{\partial_j} X = 0$ on U_i. Calculate:

$\cancel{R(\partial_i,\partial_j)X} = \nabla_{\partial_i}\nabla_{\partial_j}X - \nabla_{\partial_j}\nabla_{\partial_i}X - \underbrace{\nabla_{[\partial_i,\partial_j]}X}_{0} \Rightarrow \nabla_{\partial_i}\nabla_{\partial_j}X = \nabla_{\partial_j}\nabla_{\partial_i}X = 0$

So $\nabla_{\partial_j} X = 0$ is parallel along vertical curves and $\nabla_{\partial_j} X = 0$ on U_{i-1}

$\Rightarrow \nabla_{\partial_j} X = 0$ on U_i. By induction, X is parallel on U_n.

③ Distance from flatness, state thm.

Overflow: orthonormal parallel fields commute $\Rightarrow$ flatness

No.
Date.

Sidebar: Differential topology vs. Diff Geo: Something that came up at the end of last class: If $\varphi: M \to N$ is a diffeo, then M and N are "the same" smooth mfld. What does that mean?

~~① All the structures we defined using just smooth func~~

① – Same smooth functions: $C^\infty(M) \cong C^\infty(N)$, so all the objects def'd using smooth fns should be the same: $TM \cong TN$, $\mathfrak{X}(M) \cong \mathfrak{X}(N)$. Explicitly: $f \in C^\infty(M) \overset{\cong}{\longmapsto} f \circ \varphi^{-1} \in C^\infty(N)$

~~– Operations should be the same~~

$V_p \overset{\in T_pM}{\longmapsto} D\varphi(V_p) \in T_{\varphi(p)}N$,

$$\begin{tikzcd} TM \arrow[r, "D\varphi"] & TN \\ M \arrow[u, "V"] \arrow[r, "\varphi"] & N \end{tikzcd}$$

$V \in \mathfrak{X}(M) \longmapsto D\varphi \circ V \circ \varphi^{-1} \in \mathfrak{X}(N)$. "$\varphi_* V$"

– Same operations: $V \in \mathfrak{X}(M)$, $f \in C^\infty(M)$, $Vf \in C^\infty(M)$

$$\underset{\mathfrak{X}(N)}{\varphi_* V}[\underset{C^\infty(N)}{f \circ \varphi^{-1}}] = (Vf) \circ \varphi^{-1}.$$

$\Rightarrow [\varphi_* X, \varphi_* Y] = \varphi_*[X, Y]$.

This is topology: donut is diffeo to coffee mug, so they're topolog. the same

Diff Geo asks: How can we distinguish these? How is donut diff't from mug?

– ~~Metrics~~ ~~If~~ Add additional structure – lengths, angles, etc.

– metrics: $g_p: T_pN \times T_pN \to \mathbb{R}$. If $\varphi: M \to N$, then $\exists$ $\varphi^* g$ on M s.t. $\langle V | W \rangle_{\varphi^* g} = \langle \varphi_* V | \varphi_* W \rangle_g$.

We say (M, h), (N, g) are <u>isometric</u> if $\exists$ $\varphi: M \to N$ a diffeo s.t. $\varphi^* g = h$.

Fundamental problem: How do we tell whether M, N are isometric? If they are iso, construct an isometry. If not? – define invariants that help us distinguish – volume, length of shortest closed curve, <u>curvature</u> – that's where we are now.

Sidenote while we're here: ~~Trick~~ What if $\varphi: M \to N$ is not a diffeo, (say $\dim M = 2$, $\dim N = 3$?). Some of this doesn't work anymore: e.g. $D\varphi: TM \to TN$, but no good maps between $\mathfrak{X}(M)$ and $\mathfrak{X}(N)$:

$$\begin{tikzcd} TM \arrow[r] & TN \\ M \arrow[u, "V"] \arrow[r] & N \arrow[u, "W"] \end{tikzcd}$$

$TM \times TM$

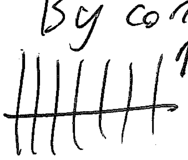
But: ~~can still define~~ if g is a metric on N, $g: TN \times TN \to \mathbb{R}$.
then still define $\varphi^* g$ on M: $\langle V | W \rangle_{\varphi^* g} = \langle D\varphi V | D\varphi W \rangle_g$.
If $\varphi: (M,h) \to (N,g)$, $h = \varphi^* g$, we call φ an <u>isometric immersion</u>.

Alternatively, $V \in \mathcal{V}(M)$ and $W \in \mathcal{V}(N)$ are φ-related if $\forall p \in M$. $\varphi_*(V_p) = W_{\varphi(p)}$. Ex: if V is φ-related to $\bar{V}$, W to $\bar{W}$, then $[V,W]_M$ is φ-rel' to $[\bar{V},\bar{W}]_N$

Okay. So we've defined the Riemann curvature tensor R – ~~what does it tell~~
It's a metric invariant: invt under isometries. What does it tell us?
Thm: $R \equiv 0 \iff M$ is locally isometric to $\mathbb{R}^n$.
Lemma: Suppose $R \equiv 0$. Then $\forall p \in M$, $\forall V_p \in T_pM$, $\exists$ a nbhd $U \ni p$ and $W \in \mathcal{V}(U)$ s.t. $W_p = V_p$ and $\nabla_X W = 0$ $\forall X \in \mathcal{V}(U)$.

Pf: R measures commutativity of parallel transport.
Let $\varphi = (u^1, \dots, u^n): U \to B \subset \mathbb{R}^n$ be a chart s.t. B = unit ball, $\varphi(p) = 0$.
We'll write $(a^1, \dots, a^n)$ for the pt $\varphi^{-1}(a^1, \dots, a^n) \in U$.
For $q = (q^1, \dots, q^n)$, let γ_q be the path $(0, \dots, 0) - (q^1, 0, \dots, 0) - \dots - (q^1, \dots, q^n)$
Let $W_q = P_{\gamma_q}(V_p)$. Then $W \in \mathcal{V}(U)$.

Let $U_i = u_1 \cdots u_i$-plane. Claim W is parallel along U_i.
By construction, W is parallel along U_1. Suppose parallel on U_{i-1}.



Then pic is like figure, ~~Further~~ So $\nabla_{\partial_i} W = 0$ on U_i
– construct by parallel transp. along ∂_i dir from U_{i-1}. $\nabla_{\partial_j} W = 0$ on U_{i-1} for $j < i$
Let $j < i$. Claim $\nabla_{\partial_j} W = 0$ on U_i: We have
$R(\partial_i, \partial_j) W = \nabla_{\partial_i} \nabla_{\partial_j} W - \nabla_{\partial_j} \nabla_{\partial_i} W - \nabla_{[\partial_i, \partial_j]} W \to 0 \implies \nabla_{\partial_i} \nabla_{\partial_j} W = 0$ on
~~$\nabla_{\partial_i} \nabla_{\partial_j} W = \nabla_{\partial_j} \nabla_i W + \nabla \to 0$~~

So $\nabla_{\partial_j} W$ is parallel along any vert line, but $\nabla_{\partial_j} W = 0$ on U_{i-1}, so it's 0 everywhere //

Now prove theorem. We need

Thm (Frobenius Lemma): if ~~$[E_i, E_j]$~~ $E_1, \dots, E_n \in \mathfrak{V}(U)$ are linearly independent, $[E_i, E_j] = 0$, then $\forall p \in U$, $\exists$ chart $\varphi : U' \to B \subset \mathbb{R}^n$ s.t. $\partial_i = E_i$, $\varphi(p) = 0$.

Pf of Thm: Let $p \in M$. By lemma, ~~$\exists U \ni p, W_1, \dots, W_n \in \mathfrak{V}(U)$ s.t. W_i is parallel~~, we can choose an orth. frame ~~$(V_i)_p$~~ $v_1, \dots, v_n \in T_pM$, and extend to parallel fields $W_1, \dots, W_n \in \mathfrak{V}(U)$. Since orthonormal at p, orthonormal ~~ever~~ on U.

Since ∇ is torsion-free, $\tau(W_i, W_j) = \nabla_{W_i} W_j - \nabla_{W_j} W_i - [W_i, W_j] = 0$
$\Rightarrow [W_i, W_j] = 0$.
So $\exists$ a $U' \ni p$ and a diffeo $\varphi : U' \to B \subset \mathbb{R}^n$ s.t. $\varphi(p) = 0$, $\partial_i = W_i$. Thus, $\varphi_* W_1, \dots, \varphi_* W_n$ are coord fields on $\mathbb{R}^n$ which are orthonormal. Since φ sends orthonormal fields to orthonormal fields, it's an isometry //

~~Next~~: Sketch of where we go next: Geodesics and distances:
If g is a Riem metric on M, we can define a distance fn: if
$\gamma : [0,1] \to M$, $\ell(\gamma) = \int_0^1 \|\gamma'(t)\| \, dt$.
$d(v, w) = \inf_{\substack{\gamma(0)=v \\ \gamma(1)=w}} \ell(\gamma)$. How does this distance behave?

In order to understand: geodesics: γ is a geodesic if $\gamma' \in \mathfrak{V}(\gamma)$ is a parallel field. Will show: - use geods to define <u>normal coordinates</u>
~~At~~ Will show: - any shortest path from v to w is a repar. of a geodesic (first variation formula)
- If M is complete, then $\exists$ a geod between any pair of pts. (Hopf-Rinow)
~~curvature affects how lengths of geodesics~~
- second variation formula: ~~curvature affects~~
- Can use geodesics to define <u>normal coordinates</u> on M
- second variation formula + Jacobi fields to understand how geodesics change as you move the endpoints, you need curvature.

2024 02 20

Geodesics:

in particular, $\|\gamma'\|$ is constant.

We say $\gamma: I \to M$ is a geodesic if $\gamma' \in \mathcal{V}(\gamma)$ is parallel along γ. i.e. $D_t \gamma' = 0$. In Euc. space, these are straight lines — ~~length minimizing~~ length-min connecting ~~any two~~ pairs of pts, ~~connect any two pts~~. Does this generalize to M? (Not quite: $M = \mathbb{R}^2 \setminus 0$ is smooth, ~~...~~ closed curves not length min. But, locally, yes.)

~~First, some basics and technical lemmas~~ First, need some basics: ~~First, need some basics:~~ ~~Local~~ Lemma: (Uniform local existence) $\forall x_0 \in M$, $\exists W \ni x_0$, $\varepsilon > 0$ s.t. $\forall x \in W$, $\forall v \in T_x M$ s.t. $\|v\| < \varepsilon$ $\exists!$ geod. $\gamma_v : (-2,2) \to M$ s.t. $\gamma_v(0) = x$, $\gamma_v'(0) = v$, and γ_v depends smoothly on v, x.

Need more basic stuff: reparametrization, $\exp(tv) = \gamma_v(t)$.

Pf: Let $(u^1, \dots, u^n) : U \to \mathbb{R}^n$ a coord chart, let $\gamma(t) = (x^1(t), \dots, x^n(t))$, and let $v^i = \frac{dx^i}{dt}$. Then we write geod eqns:

$$0 = D_t \frac{d\gamma}{dt} = D_t [v^i \partial_i] = \frac{dv^i}{dt} \partial_i + v^i D_t \partial_i$$
$$= \frac{dv^i}{dt} \partial_i + v^i \nabla_{v^j \partial_j} \partial_i$$

this juggles quantifiers too much - next you split this into two equations, then define the exponential map!

$$0 = \frac{dv^i}{dt} \partial_i + v^i v^j \Gamma_{ij}^k \partial_k \quad (\text{recall } \Gamma_{ij}^k = \Gamma_{ji}^k)$$

$$\Rightarrow \frac{dv^k}{dt} \partial_k = -v^i v^j \Gamma_{ij}^k \partial_k \Rightarrow \left.\begin{aligned} \frac{dv^k}{dt} &= -v^i v^j \Gamma_{ij}^k(x^1, \dots, x^n) \\ \frac{dx^k}{dt} &= v^k \end{aligned}\right\} \text{system of ODEs.}$$

So local solution exists for every initial cond ~~...~~ on a compact set ~~$v_0 = 0 \in T_pM$~~.

That is, $\exists \delta > 0$, $W \subset U$ s.t. $\forall x \in W$, $v \in T_x M$ $\|v\| < 1$, if sol exists, then unique $\exists!$ ~~sol~~ $\gamma_v : (-\delta, \delta) \to M$ s.t. $\gamma_v(0) = x$, $\gamma_v'(0) = v$. Not quite as stated - need to exist on $(-2,2)$ for $\|v\| < \varepsilon$.

Let $\varepsilon = \delta/4$. For $\|v\| < \varepsilon$, let $\gamma_v(t) = \gamma_{\frac{2v}{\delta}}(\frac{\delta}{2} t)$ Def'd for $t \in (-2,2)$, check that this is still a geodesic //

Let's us define: Exponential map:

~~Exp~~ ~~Prop~~ Def: Let $p \in M$. The exponential map $\exp_p : T_p M \to M$ is the map $\exp_p(v) = \gamma_v(1)$. ~~...~~ (technically this is only def'd on some nbhd of 0 — ~~we just~~ but it's standard to think of it as a map $T_pM \to M$).

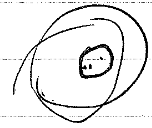
Thm: $D\exp_p : T_0(T_pM) = T_pM \to T_pM$
$D\exp_p(v) \overset{?}{=} \gamma_v'(0) = v \Rightarrow D\exp_p = id_{T_pM}$. So by IFT, $\exp_p$ is a local diffeo near $0 \Rightarrow \exists U \ni p$ s.t. $\forall q \in U$, $\exists v \in T_pM$ s.t. $\exp_p(v) = q$ — i.e. γ_v is a geod from p to q. It's convenient to have sthing a little stronger.

Thm (Uniformly normal nbhd theorem): $\forall p \in M\ \exists W, \varepsilon > 0$ s.t.

① – $\forall x, y \in W$, x, y are connected by a unique geod γ of length $< \varepsilon$.

② – γ depends smoothly on x, y.

③ – $\forall x \in W$, $\exp_x$ sends $B_\varepsilon(0)$ diffeomorphically to a nbhd U_x of x s.t. $W \subset U_x$.



Pf: Consider $F: TM \to M \times M$

$(x, v) \mapsto (x, \exp_x(v))$. Then F is smooth on nbhd of $(p, 0)$

and if we parametrize TM as $(x^1, \dots, x^n, v^1, \dots, v^n) \mapsto ((x^1, \dots, x^n), v^i \partial_i)$

then $\cancel{DF_{(p,0)}(\partial_{x^i}) = (}$ $F(x, 0) = (x, x) \Rightarrow DF_{(p,0)}(\partial_{x^i}) = (\partial_i, \partial_i)$

$F(x, v) = (\cancel{\exp} x, \exp_x(v)) \Rightarrow DF_{(p,0)}(\partial_{v^i}) = (0, \partial_i)$.

so $DF_{(p,0)} = \begin{pmatrix} I & I \\ & I \end{pmatrix}$ is invertible

Choose ε, W small enough that ~~$F|_W$ is invertible on W_ε where~~ F is invert on W_ε, where

$W_\varepsilon = \{(q, v) \in TM \mid q \in W, \|v\| < \varepsilon\}$. Choose ~~$W \subset V$~~ s.t. $V \ni p$ s.t. $W \times W \subset F(V_\varepsilon)$.

~~$\exp_x$ is a diffeo on $B_\varepsilon(0)$ $\forall x \in V \Rightarrow$ ② ③ are immed.~~

Then ①: $\forall (x, y) \in W$, $F^{-1}(x, y) = (x, v)$, so $\exp_x(v) = y$, $\gamma_{x,y} = \gamma_v$.

~~Any geod of length $< \varepsilon$ from x ends at some point exp~~

~~Further, $\exp_x: B_\varepsilon(0) \to$ is a diffeo~~ ~~Further v is un~~

Further, $\exp_x$ is a diffeo on $B_\varepsilon(0)$, so γ_v is unique geod of length $< \varepsilon$ from x to y. //

So, local existence, strict length minimization!

Further, use geodesics to work with length metric:

~~$\forall x, y \in M$, let d $\forall \gamma \in M$, let $\ell(\gamma) =$~~ piecewise smooth

$\forall\ \gamma: I \to M$ let $\ell(\gamma) = \int_I \|\gamma'(t)\| \, dt$.

$\forall x, y \in M$, let ~~$d(x,y) = \inf_\gamma \ell(\gamma)$ where $\gamma(0) = x, \gamma(1) = y$.~~

$d(x, y) = \inf\{\ell(\gamma) : \gamma: [0,1] \to M \text{ piecewise smooth}, \gamma(0) = x, \gamma(1) = y\}$

Clearly satisfies triangle ineq, symmetry. Positivity? Not immediate

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~~Pf: Gauss's Lemma:~~ Goal:

Thm: Let $x \in M$, let $r_0 > 0$ s.t. $\exp_x$ is a diffeo $B_{r_0}(0) \xrightarrow{\exp} B_{r_0}(0) \cong W$.
Let $y \in W$, let γ be unique geod of length $< r_0$ from x to y. Then $d(x,y) = \ell(\gamma)$ and γ is unique shortest path from x to y.
Conversely, if $d(x,z) < r_0$, then $z \in W$.
— so $\exp_x$ sends lines through origin to geods, sends spheres to metric spheres, by thm, ought to be metric spheres)

~~Pf~~ Need: Gauss's Lemma: Let W, r_0 as above, let $r < r_0$.) Let $S_r = \{ v \in T_xM : \|v\| = r \}$, so $\exp(S_r)$ is a sphere embedded in W. Then $\forall v \in T_xM$, γ_v is orthog. to $\exp S_r$.
~~(like polar coords:~~

Pf: Let $v \in S_r$, let $\theta : (-1,1) \to S_1$ be a curve with $\theta(0) = v$.
~~Claim that $t \mapsto \exp_x(r\theta(t))$ intersects γ_v orthogonally.~~
Let $\theta_r(t) = \exp_x(r\theta(t))$. Claim θ_r is orthog to γ_v
(like polar coords:)

Let $f(\rho, t) = \exp_x(\rho\theta(t))$, so $\gamma_v(t) = f(t, 0)$
$\theta_r(t) = f(r, t)$

Consider $\langle \frac{\partial f}{\partial \rho} | \frac{\partial f}{\partial t} \rangle = \langle \partial_\rho | \partial_t \rangle$:
Let D_ρ, D_t be covariant derivs in ρ, t-directions.
Then $\partial_\rho \langle \partial_\rho | \partial_t \rangle = \langle \underbrace{D_\rho \partial_\rho}_{\text{geod.} \Rightarrow 0} | \partial_t \rangle + \langle \partial_\rho | D_\rho \partial_t \rangle$

Further, $\tau(\partial_\rho, \partial_t) = D_\rho \partial_t - D_t \partial_\rho - \underbrace{[\partial_\rho, \partial_t]}_{0} = 0 \Rightarrow$

$$\partial_\rho \langle \partial_\rho | \partial_t \rangle = \langle \partial_\rho | D_t \partial_\rho \rangle = \frac{1}{2} \partial_t \langle \partial_\rho | \partial_\rho \rangle = 0$$

(because $\partial_\rho = \gamma'$ ~~is constant~~ has constant length)
So $\langle \partial_\rho | \partial_t \rangle$ is constant in ρ. If $\rho = 0$, then $\partial_t = 0$,
so $\langle \partial_\rho | \partial_t \rangle = 0$. But $\theta_r'(t) = \partial_t$, $\gamma_v'(t) = \partial_\rho$,
so θ_r, γ_v are orthogonal //

~~Overflow: (Next: Next step.~~
~~Overflow: next: proof of thm.~~

Lem: Let $0 < a < b < r_0$. If $w(t) = \exp_x(\rho(t)\theta(t))$ ~~$\rho : [0,1] \to [a,b]$~~, is a path from $\exp S_a$ to $\exp S_b$, then $\ell(w) \geq b - a$, with equality iff θ is constant, ρ is monotone — i.e. w reparameterizes a geodesic

Pf: Let $f(\rho, t) = \exp_x \rho\theta(t)$. Then $\partial_\rho \langle \frac{\partial f}{\partial \rho} | \frac{\partial f}{\partial t} \rangle = 0$ and $\|\frac{\partial f}{\partial \rho}\| = 1$.

~~So $w'(t) = \rho'(t)\frac{\partial f}{\partial \rho} + \theta'(t)\frac{\partial f}{\partial \theta}$~~

Since $w(t) = f(\rho(t), t)$, $w'(t) = \rho'(t)\frac{\partial f}{\partial \rho} + \frac{\partial f}{\partial t}$.

$$\|w'(t)\| = \sqrt{|\rho'(t)|^2 + \|\tfrac{\partial f}{\partial t}\|^2} \geq |\rho'(t)|.$$

So $l(w) \geq \int_0^1 |\rho'(t)|\, dt \geq \rho(1) - \rho(0) = b - a$.

If $l(w) = b - a$, then these ineqs are sharp: $\rho'(t) \geq 0$ and $\|\frac{\partial f}{\partial t}\| = 0$

$\Rightarrow \theta$ constant, ρ monotone //.

Pf of Thm: Suppose $\exp_x$ is a diffeo on $B_r(0)$. Let $y \in \exp_x B_r(0)$, and $r = |\exp_x^{-1}(y)|$, let w connect x to y. Then w contains a segment from $\exp S_\varepsilon$ to $\exp S_r$, so $l(w) \geq r - \varepsilon \Rightarrow l(w) \geq r \Rightarrow d(x,y) \geq r$.

Suppose $l(w) = r$. Then the segment from $\exp S_\varepsilon$ to $\exp S_r$ is a segment of a geodesic $\Rightarrow$ ~~w is a geodesic~~ w is a reparametrized (reparam) geodesic. //

~~We define injectivity r~~

~~We define the injectivity radius of M at x to be the~~ by keep.

~~$\operatorname{injrad}(x) = \sup\{r > 0 : \exp_x \text{ is a diffeo on } B_r(0)\}$. Then~~

~~If $d(x,y) < \operatorname{injrad}(x)$, then x, y are connected by a unique length-minimizing~~ geodesic.

~~So every $x \in M$ has a nbhd where there's a unique geod from x to y.~~ W s.t. $\forall y \in W$, $\exists!$ length-minimizing

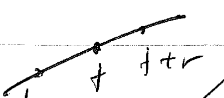
Lemma: If $d(x,y) < \operatorname{injrad}(x)$, then x, y connected by unique len-minimizing geodesic.

Further: ~~Cor~~: If $\gamma: [0, L] \to M$ is piecewise-smooth, length-minimizing, constant-speed, then γ is a geodesic.

~~WLOG, say $\|\gamma'\| = 1$.~~ After rescaling, say $\gamma: [0, L] \to M$, $\|\gamma'\| = 1$.

Pf: Geodesic $\Leftrightarrow D_t \gamma' = 0$, so it's enough to prove that γ is locally a geodesic.

Let $0 < t < L$. Let ~~W be a~~ W, ε as in uniformly normal nbhd thm, so $\forall x \in W$ $\exp_x$ is diffeo on $B_\varepsilon(0) \subset T_x M$. ~~Let $r < \frac{\varepsilon}{3}$, st. $\gamma(t \pm r) \in W$. Then $\gamma(t-r)$ and $\gamma(t+r)$ are~~



$\Rightarrow \operatorname{injrad}(w) \geq \varepsilon$ for all $w \in W$.

Let $r < \varepsilon/3$ s.t. $\gamma(t \pm r) \in W$. ~~Then~~ then $d(\gamma(t-r), \gamma(t+r)) \leq 2r < \varepsilon$, so $\gamma(t-r)$ and $\gamma(t+r)$ are connected by a unique length-minimizing geodesic. So $\gamma|_{[t-r, t+r]}$ must be this geodesic, parameterized w/ constant speed.

$\Rightarrow D_t \gamma'(t) = 0$, $\forall t$ so γ is a geodesic //

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~~When is there a length-minimizing path?~~ Helps to define:

Def: Let $x \in M$. The <u>injectivity radius</u> at x is
$\mathrm{inj\,rad}(x) = \sup\{r : \exp_x \text{ is a diffeo on } B_r(0)\}$
— by thm, if $d(x,y) < \mathrm{injrad}(x)$, then $\exists!$ length-minimizing geod from x to y.

Furthermore, by Uniformly Normal Nbhd Thm, $\forall x \in M$, $\exists W \ni x$, $\varepsilon > 0$ s.t. $\mathrm{injrad}(w) > \varepsilon \ \forall w \in W$. So we can prove:

Lemma: Let $\gamma : [0,1] \to M$ be a piecewise-smooth, length-minimizing path w/ constant speed. Then γ is a geodesic.
After rescaling, suppose $\gamma : [0,L] \to M$ and $\|\gamma'\| = 1$.
Pf: Geod $\iff D_t \gamma' = 0$, so suffices to show that γ is locally a geod.
By ~~compactness~~ above, $\forall t \in [0,L]$, $\exists$ nbhd W, $\varepsilon > 0$ s.t. $\mathrm{injrad}(w) \geq \varepsilon \ \forall w \in W$. By compactness $\exists \varepsilon > 0$ s.t. $\mathrm{injrad}(\gamma(t)) \geq \varepsilon \ \forall t \in [0,L]$.
Let $t \in (0,L)$, let $r = \min\{t, L-t, \frac{\varepsilon}{3}\}$. ~~Then~~ Consider $\gamma(t-r)$, $\gamma(t+r)$. Then $d(\gamma(t-r), \gamma(t+r)) \leq \frac{2\varepsilon}{3} < \mathrm{injrad}(\gamma(t-r))$, so $\exists$ unique length-minimizing geod from $\gamma(t-r)$ to $\gamma(t+r)$, and $\gamma|_{[t-r,t+r]}$ parametrizes this geodesic with const speed
$\Rightarrow D_t \gamma'(t) = 0 \ \forall t \Rightarrow \gamma$ is a geodesic //
(lets us construct geodesics by finding shortest paths)

Alternatively, can characterize by calculus of variations:
Def: For $p, q \in M$, let $\Omega_{p,q} = \{$piecewise ~~smooth~~ paths from p to $q\}$, with the C^1-topology — γ is close to λ if $d(\gamma(t), \lambda(t))$ is small for all t and $\gamma'(t)$ is close to $\lambda'(t)$ $\forall t$.
~~This is kind of like a manifold. Like a mfld, we~~
~~This~~ has a "tangent space":

When does a length-minimizing path from p to q exist?
Def: M is <u>geodesically complete</u> if $\forall x \in M$, $\forall v \in T_x M$ $\exists \gamma_v : \mathbb{R} \to M$ a geod s.t. $\gamma_v(0) = x$, $\gamma_v'(0) = v$ — i.e. γ_v defined for all time.

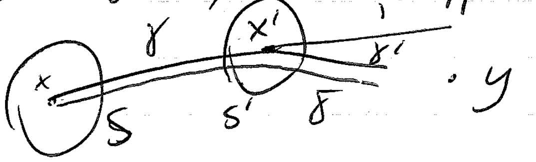
Thm (Hopf-Rinow): If M is geodesically complete, then $\forall x,y\in M$, $\exists$ a length-minimizing geodesic from x to y.

Pf: Let $L=d(x,y)$. Let $r_0=\text{injrad}(x)$. If ~~$y\in \exp_x B_{r_0}(0)$, we're done~~ $L<r_0$, we're done.
~~Otherwise~~ Suppose $L>r_0$. Let $0<\delta<r_0$, let $S=\exp_x S_\delta$.
S is compact, so $\exists\ z\in S$ s.t. $d(z,y)=\min_{s\in S} d(s,y)$.
Let γ be unit-speed geod s.t. $\gamma(0)=x$, $\gamma(\delta)=z$. Claim: $\gamma(L)=y$.

ETS that $d(\gamma(t),y)=L-t\ \ \forall\ t\in[\delta,L]$. "Induction":

First: ~~Let~~ $t=\delta$: By Δ ineq,
$$d(x,y)\le \min_{s\in S}(d(x,s)+d(s,y)).$$
But any path from x to y crosses S, so in fact,
$$d(x,y)=\min_{s\in S} d(x,s)+d(s,y)\quad \forall s\in S,$$
$d(x,s)=\delta$, so
$$L=\delta+\min_{s\in S} d(s,y)=\delta+d(z,y). \checkmark$$

Now, "inductive step". Let $t_0=\sup\{t\in[\delta,L]\mid d(\gamma(t),y)=L-t\}$.
If $t_0=L$, done. Suppose DWOC that $t_0<L$. Let $x'=\gamma(t_0)$.



Let S' be a small sphere around x' of radius δ'. Let $z'\in S'$ s.t. $d(z',y)=\min_{s'\in S'} d(s',y)$, let γ' be a geod from x' through z'.
Then, as before, $d(x',y)=\min_{s'\in S'} d(x,s')+d(s',y)$ ~~$=\delta+d(z,y)$~~.
$$L-t_0=\delta'+d(z',y)\Rightarrow d(z',y)=L-t_0-\delta'$$
~~Let $\bar\gamma=\gamma([0,t_0])\cup\gamma'([0,\delta'])$~~ $\Rightarrow d(x,z')\ge t_0+\delta'$.
But $\bar\gamma=\gamma([0,t_0])\cup\gamma'([0,\delta'])$ is a path of length $t_0+\delta'$.
So $\bar\gamma$ is a geodesic $\Rightarrow$ $\bar\gamma$ w/ same start as γ ~~is smooth at x'~~ $\Rightarrow \bar\gamma=\gamma$
$$\Rightarrow \gamma(t_0+\delta')=\gamma'(\delta')\Rightarrow d(y,\gamma(t_0+\delta'))=L-t_0-\delta' \quad \divideontimes //$$