Geodesics

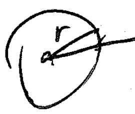
Last time: Geodesics and length-minimizers.

- ~~$\forall x,y\in M$, if $d(x,y)<\text{inj rad}(x)$, then $\exists$ length-minimizing curve from x to y~~
- if γ is a length-minimizing const. speed curve, it is a geodesic.
- $\forall x,y\in M$, if $d(x,y)<\text{inj rad}(x)$ then $\exists!$ length-minimizing curve from x to y
- $\forall x\in M$, if $r<\text{inj rad}(x)$, then $B_r(x)=\{y \mid d(x,y)<r\}$
  $= \exp_x(B_r(0))$.

~~Cor: $d: M\times M\to\mathbb{R}$ is a continuous function~~

Today: Thm (Hopf-Rinow): If M is geodesically complete, then $\forall x,y$, $\exists$ a length-minimizing curve from x to y (a geod from x to y of length $d(x,y)$). (Recall: geod. complete means that $\forall x\in M$, $\forall v\in T_xM$ $\exists$ a geod $\gamma:\mathbb{R}\to M$ with $\gamma'(0)=v$.)

~~Shooting method~~: Uses following lemma: Let $x,y\in M$, let $L=d(x,y)$, ~~let~~ suppose, $0<r<\cancel{d(x,y)}L$ and $r<\text{inj rad}(x)$. Then $\exists z\in \cancel{\exp_x(S_r)}M$ s.t. $d(x,z)+d(z,y)=\cancel{d(x,y)}L$ (in ~~fact,~~ $d(x,z)=r$, $d(z,y)=\cancel{d(x,y)}L-r$)



Pf: Let $\Sigma=\exp_x(S_r)$ = sphere of radius r. This is compact, so $\exists z\in\Sigma$ s.t. $d(z,y)=\min_{s\in\Sigma} d(s,y)$. Then $d(z,x)=r$.

~~By triangle ineq $d(y,z)\geq d(x,y)-r$.~~ By triangle ineq,
$d(x,y)\leq \min_{s\in\Sigma}(d(x,s)+d(s,y))$. But every path from x to y crosses Σ, so $d(x,y)\geq \min_{s\in\Sigma}(d(x,s)+d(s,y))$. $\Rightarrow L=\min_s d(x,s)+d(s,y)$
$=\min_s(r+d(s,y))=r+d(z,y)$, so $d(z,y)=L-r$ //

Suggests a strategy: $x \quad z_1 \quad z_2 \quad z_3 \quad \cdots \quad y$ ~~doesn't quite almost works~~ more detail have – why geod? ~~needs to be tweaked.~~

– but you'll note we haven't used geodesic completeness yet. So:

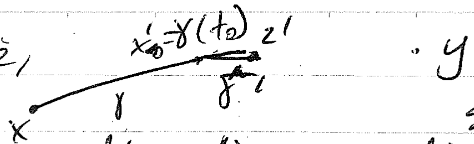
Pf of Hopf-Rinow: Let $x,y\in M$. ~~Let $r>0$, ... $L=d(x,y)$, let $r>0$ and $z\in M$ s.t. $d(x,z)=r$, $d(z,y$~~ Let $L=d(x,y)$. ~~If $L<\text{injrad}(x)$ done. Otherwise, let $z\in M$ s.t. $d(x,z)=r$, $d(z,y)=L-r$.~~ If $L<\text{injrad}(x)$, done. ~~Let $0<r<\text{inj rad}$~~ Otherwise, let $0<r<\text{injrad}(x)$, let z s.t. $d(x,z)=r$, $d(z,y)=L-r$. ~~Let γ $\exists!$ unit geod of length r from x to z. By completeness, can extend~~

Then $\exists!$ unit speed geod γ s.t. $\gamma(0)=x$, $\gamma(r)=z$. ~~Claim: $\gamma(L)=y$~~.

Claim: $\forall t\in[0,L]$, $d(\gamma(t),y)=L-t$. – then $d(\gamma(L),y)=0\Rightarrow\gamma(L)=y$.

Let $t_0=\sup_{\cancel{t\in[0,L]}}\{t\in[0,L] \mid d(\gamma(t),y)=L-t\}$. Note: $t_0\geq r$ by Δ-ineq. If $t_0=L$, we're done.

Let $x_0' = \gamma(t_0)$

Otherwise, 

Let $0 < r_0' < \text{inj rad}(x_0')$

~~st~~ $z_0' \in M$ s.t. ~~$d(x_0, y) - d(x_0, z_0)$~~

$d(x_0', z_0') = r_0'$, $d(z_0', y) = d(x_0', y) - r_0' = L - t_0 - r'$

~~Then~~ Consider the curve $\bar{\gamma} = \gamma([0,t_0]) \cup \gamma'([0,r'])$. ~~This has~~ from x to z'. This has length $t_0 + r'$, but ~~$d(x,z') \neq t_0 + r'$~~. By Δ, $d(x,z') + d(z',y) \geq L$

$$d(x,z') \geq L - (L - t_0 - r') = t_0 + r'.$$

So $\bar{\gamma}$ is length-minimizing! So $\bar{\gamma}$ is a geodesic $\Rightarrow z' = \gamma'(r') = \gamma(t_0 + r')$ $\Rightarrow d(\gamma(t_0 + r'), y) = L - t_0 - r'$ which contradicts the maximality of t_0. //

Cor: If M is geodesically complete, then $\forall x \in M$, $r > 0$, ~~the metric ball~~ and we let $\bar{B}_r(x) = \{y \mid d(x,y) \leq r\}$, ~~is~~ then ~~equal to e~~ $\bar{B}_r(x) = \exp_x(\bar{B}_r(0))$. In particular, $\bar{B}_r(x)$ is ~~complete~~ compact.

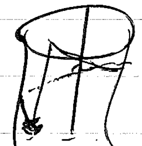
Cor: M complete $\Leftrightarrow$ M geod complete

Cor: If M is compact, then any ~~two~~ $x, y \in M$ are connected by a length-min geodesic.

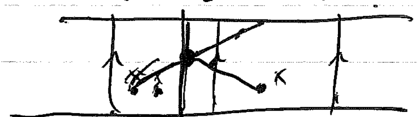
(break)

~~When is a geodesic length-minimizing?~~

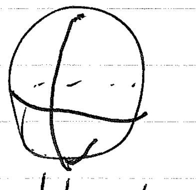
When is a geodesic minimal/not minimal? why is this a geodesic

Two things can happen: 1) ~~loss of global minimality~~:  M = cylinder

Geodesics are helixes (Easiest to see by unrolling:



A geodesic that goes more than halfway isn't globally minimal, but it is locally minimal (because ~~straight lines are~~ any perturb increases length.)

2) $M = S^2$  Geodesics of length $< \pi$ are minimal, but geodesics of length $> \pi$ are no longer locally minimal (a rubber band around the equator wants to slip off — even a rubber band wrapped 3/4 around wants to slip.).

What's the difference? Calculus of variations.

For $p, q \in M$, let $\Omega_{p,q} = \{$piecewise-smooth paths from p to $q\}$, with the C^1_* topology (γ is close to λ if ~~$d(\gamma(t), \lambda(t))$ is small~~ $\gamma(t)$ close to $\lambda(t)$ and $\gamma'(t)$ close to $\lambda'(t)$ $\forall t$)

This makes $l : \Omega_{p,q} \to \mathbb{R}$ cts. We can treat this ~~as a~~ like an ∞-dim mfld.

In particular, ~~we define T_γ~~ for $\gamma \in \Omega_{p,q}$, let

$T_\gamma \Omega = \{$piecewise-smooth v.fields on γ with $V(0) = V(1) = 0\}$

A *variation* of $\gamma : [0,1] \to M$ is a path $\bar{\alpha} : (-\varepsilon, \varepsilon) \to \Omega_{p,q}$ s.t.
① $\bar{\alpha}(0) = \gamma$ ② $\exists$ partition $0 = t_0 < t_1 < \cdots < t_n = 1$ s.t.
~~$\alpha(u,t)$~~ $\alpha : (u,t) \mapsto \bar{\alpha}_u(t)$ is smooth on each $(-\varepsilon, \varepsilon) \times [t_i, t_{i+1}]$.

This ensures that $\frac{\partial \alpha}{\partial u}$ is defined and $\frac{\partial \alpha}{\partial u}\Big|_{u=0} \in T_\gamma \Omega$.

Let $E : \Omega_{p,q} \to \mathbb{R}$, $E(\gamma) = \frac{1}{2} \int_0^1 \|\gamma'(t)\|^2 dt$.

~~Thm: γ is a geodesic $\Leftrightarrow$ γ is a critical pt of E. That is $\forall$ variations $\bar{\alpha}$, $\frac{d}{du} E(\bar{\alpha}_u)\Big|_{t=0} = 0$~~ Then ~~$\gamma$ be piecewise smooth~~,

First Variation Formula: ~~Let $\bar{\alpha}$ be a variation of γ, let $W = \frac{\partial \alpha}{\partial u}\Big|_{u=0}$.~~
~~Then $\frac{d}{du} E(\bar{\alpha}_u)\Big|_{u=0} =$~~ Let γ be piecewise-smooth, say smooth except at $t_1, \ldots, t_n$. Let $V = \frac{d\gamma}{dt}$, write $V(t_i^{\pm}) = \lim_{t \to t_i^{\pm}} V(t)$.
Let $A = D_t V$, for all i, let $\Delta_{t_i} V = V(t_i^+) - V(t_i^-)$, $\Delta_t V = 0$ otherwise
Let $\bar{\alpha}$ be a variation of γ, let $W = \frac{d\alpha}{du}\Big|_{u=0}$. Then

$$\frac{d}{du}\left[E(\bar{\alpha}_u)\right]_{u=0} = -\sum_i \langle W(t_i) | \Delta_{t_i} V \rangle - \int_0^1 \langle W | A \rangle dt$$

$$= \underbrace{-\sum_t \langle W | \Delta_t V \rangle}_{\text{corners}} - \int_0^1 \langle W | A \rangle dt.$$

~~$\Delta_t V$ & A represent curvature of γ: $\Delta_t V$ at corners, A elsewhere~~

Essentially, an integral of ~~the~~ inner product of W with "curvature" of V — ~~curv~~ $D_t V$ on smooth parts & point mass at corners.

Overflow: in particular, geod $\Leftrightarrow$ crit pt.

Calculus of Variations

~~For~~ Last time: $\Omega_{p,q} = \{$ piecewise-smooth curves from p to q $\}$

$T_\gamma\Omega = \{$ piecewise smooth vector fields $\mid V(0)=0, V(1)=0 \}$

A variation of γ is a map $\bar\alpha : (-\varepsilon,\varepsilon) \to \Omega_{p,q}$ s.t. $\bar\alpha_0 = \gamma$,
$\alpha(t,u) = \bar\alpha_u(t)$ is piecewise smooth —
smooth on $(-\varepsilon,\varepsilon)\times[t_i, t_{i+1}]$ for all i.
in particular, all $\bar\alpha_u$'s are smooth on the same partitions.

~~How does length change when~~ Let $E : \Omega_{p,q} \to \mathbb{R}$,
$E(\gamma) = \frac{1}{2}\int_0^1 \|\gamma'(t)\|^2 dt$. Q: How does $E(\gamma)$ change as γ changes?

Thm: γ is a geodesic $\iff$ γ is a critical point of E
(i.e., $\forall$ variations $\bar\alpha$, $\frac{d}{du}E(\bar\alpha_u)\big|_{u=0} = 0$.)

Pf: ~~Suppose Calculate~~ First Variation Formula:
Let $\bar\alpha$ be a variation of γ. Let $\alpha(t,u) = \bar\alpha_u(t)$. Then

$\bar\alpha_u' = \frac{\partial\alpha}{\partial t}$, $E(\bar\alpha_u) = \frac{1}{2}\frac{d}{du}\int_0^1 \langle \frac{\partial\alpha}{\partial t} \mid \frac{\partial\alpha}{\partial t}\rangle dt$

$= \int_0^1 \langle D_u \frac{\partial\alpha}{\partial t} \mid \frac{\partial\alpha}{\partial t}\rangle dt$

$= \int_0^1 \langle D_t \frac{\partial\alpha}{\partial u} \mid \frac{\partial\alpha}{\partial t}\rangle dt$.

$$\boxed{\begin{array}{l}\text{Further,}\\ 0 = T(\frac{\partial\alpha}{\partial u}, \frac{\partial\alpha}{\partial t}) = \nabla_u \frac{\partial\alpha}{\partial t} - \nabla_t \frac{\partial\alpha}{\partial u} - [\frac{\partial\alpha}{\partial u}, \frac{\partial\alpha}{\partial t}] \\ \Rightarrow \nabla_u \frac{\partial\alpha}{\partial t} = D_t \frac{\partial\alpha}{\partial u}.\end{array}}$$

Integrate by parts:

$$\langle D_t \frac{\partial\alpha}{\partial u} \mid \frac{\partial\alpha}{\partial t}\rangle = \frac{d}{dt}\langle \frac{\partial\alpha}{\partial u} \mid \frac{\partial\alpha}{\partial t}\rangle - \langle \frac{\partial\alpha}{\partial u} \mid D_t \frac{\partial\alpha}{\partial t}\rangle$$

~~except at the t_i's~~ So. — except there's a discont at t_i's

So $\frac{d}{du}\cancel{E(\bar\alpha_u)} = \int_0^1 \langle D_t \frac{\partial\alpha}{\partial u} \mid \frac{\partial\alpha}{\partial t}\rangle = \sum_{i=1}^n \langle \frac{\partial\alpha}{\partial u} \mid \frac{\partial\alpha}{\partial t}\rangle \Big|_{t_{i-1}^+}^{t_i} - \int_0^1 \langle \frac{\partial\alpha}{\partial u} \mid D_t \frac{\partial\alpha}{\partial t}\rangle$

Let $W = \frac{\partial\alpha}{\partial u}\big|_{u=0}$ (variation field), $V = \frac{\partial\alpha}{\partial t}\big|_{u=0}$ (velocity), let $V(t_i^\pm) = \lim_{t\to t_i^\pm} V(t)$.
Let $A = D_t V$, $\Delta_t V = V(t^+) - V(t^-)$ (note that $\Delta_t V = 0$ except when $t = t_i$)

Then: $\frac{d}{du}E(\bar\alpha_u)\big|_{u=0} = -\int_0^1 \langle W \mid D_t V\rangle dt + \langle W(t_n) \mid V(t_n^-)\rangle$
$- \langle W(t_{n-1}) \mid V(t_{n-1}^+)\rangle + \langle W(t_{n-1}) \mid V(t_{n-1}^-)\rangle$
$\cdots - \langle W(t_1) \mid V(t_1^+)\rangle + \langle W(t_1) \mid V(t_1^-)\rangle - \langle W(t_0) \mid V(t_0^+)\rangle$

$= -\int_0^1 \langle W \mid D_t V\rangle dt - \sum_{i=1}^{n-1} \langle W(t_i) \mid \Delta_t V(t_i)\rangle$

$= -\int_0^1 \langle W \mid dK(t)\rangle$, where ~~Rst~~

where dK is a "curvature measure" - cts part $D_t V$, point measure $\Delta_t V_t$.
This is the First Variation Formula: $\frac{d}{du} E(\bar\alpha_u)$ is C^1, and formula for the deriv.

If γ is a geodesic then $D_t V = 0$, $\Delta_t V = 0 \Rightarrow \frac{d}{du} E(\bar\alpha_u)|_{u=0} = 0$.
Conversely, if ~~$\frac{d}{du}E(\bar\alpha_u)$~~ $\frac{dE}{du}|_{u=0} = 0$ $\forall \bar\alpha$, then

$$-\int_0^1 \langle W \mid D_t V\rangle dt - \sum_t \langle W \mid \Delta_t V\rangle = 0 \quad \forall W.$$

$\Rightarrow D_t V = \Delta_t V = 0$, i.e., γ is C^1 and $D_t\gamma' = 0 \Rightarrow \gamma$ a geod //.

~~Now that we've done this, we can go further:~~
Okay: geodesics are critical pts of energy. If we want to understand length-minimisers we need more - what sort of critical points are they?
Second Variation Formula: If γ is a geodesic, $W_1, W_2 \in T_\gamma\Omega$, and $\bar\alpha : (-\varepsilon,\varepsilon)^2 \to \Omega$ is s.t. $\alpha(0,0) = \gamma$, $\frac{\partial\alpha}{\partial u_i} = W_i$, then $E(\bar\alpha_{u_1,u_2})$ is C^2 and

$$H(E)(W_1,W_2) = \frac{\partial^2 E}{\partial u_1 \partial u_2}(0,0) = -\sum_t \langle W_2 \mid \Delta_t D_t W_1\rangle - \int_0^1 \langle W_2 \mid D_t^2 W_1 - R(V,W_1)V\rangle,$$

where $\Delta_t D_t W_1(t) = D_t W_1(t^+) - D_t W_1(t^-)$.
- i.e., inner product with "second derivative" plus curv.

Pf: Differentiate $\frac{\partial E}{\partial u_2} = -\sum_t \langle \frac{\partial\alpha}{\partial u_2} \mid \Delta_t \frac{\partial\alpha}{\partial t}\rangle - \int_0^1 \langle \frac{\partial\alpha}{\partial u_2} \mid D_t \frac{\partial\alpha}{\partial t}\rangle$

$$\frac{\partial^2 E}{\partial u_1 \partial u_2} = -\sum_t \langle \frac{\partial^2\alpha}{\partial u_1 \partial u_2} \mid \underbrace{\Delta_t \frac{\partial\alpha}{\partial t}}_{0}\rangle - \sum_t \langle \frac{\partial\alpha}{\partial u_2} \mid D_{u_1} \Delta_t \frac{\partial\alpha}{\partial t}\rangle$$
$$-\int_0^1 \langle \frac{\partial^2\alpha}{\partial u_1 \partial u_2} \mid \underbrace{D_t \frac{\partial\alpha}{\partial t}}_{0}\rangle - \langle \frac{\partial\alpha}{\partial u_2} \mid D_{u_1} D_t \frac{\partial\alpha}{\partial t}\rangle$$

At $(0,0)$,
$$\frac{\partial^2 E}{\partial u_1 \partial u_2}(0,0) = -\sum_t \langle W_2 \mid D_{u_1} \Delta_t \frac{\partial\alpha}{\partial t}\rangle - \int_0^1 \langle W_2 \mid D_{u_1} D_t \frac{\partial\alpha}{\partial t}\rangle$$
$$= -\sum_t \langle W_2 \mid \Delta_t D_{u_1} \frac{\partial\alpha}{\partial t}\rangle - \int_0^1 \langle W_2 \mid D_t D_{u_1} \frac{\partial\alpha}{\partial t} - R(V,W_1)V\rangle$$
(by linearity)
$$= -\sum_t \langle W_2 \mid \Delta_t D_t \frac{\partial\alpha}{\partial u_1}\rangle - \int_0^1 \langle W_2 \mid D_t^2 \frac{\partial\alpha}{\partial u_1} - R(V,W_1)V\rangle \quad //$$
(torsion) (torsion)

(Note that since $E(\bar\alpha)$ is C^2, this has to be symmetric in W_1 and W_2 -

It is, but you need some int by parts to see it.

~~How do we study this? This is a symmetric bilinear form — $\mathbb{R}^n$. If $F: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a symmetric biline form, then we can write: $F(v,w) = v^T M w$ for some symmetric matrix M. The behavior of F depends on eigenspaces of M.~~

How do we study this? Remember the theory in $\mathbb{R}^n$:
if $f: \mathbb{R}^n \to \mathbb{R}$, is smooth, $Df = 0$, then ~~$f(x) = H(f)(x,x) + O(\|x\|^3)$~~

~~$H(f)(v,w)$ is a matrix~~ then $H(f) = \left(\frac{\partial f}{\partial x_1^2} \cdots\right)$

~~And we characterize $H(f)$ by its nullspace~~ is a symmetric matrix,
and ~~if~~ if $v_1, v_2 \in \mathbb{R}^n$, then the second partial

$$v_1 v_2 f(0) = \frac{\partial^2}{\partial u_1 \partial u_2}\left[f(u_1 v_1 + u_2 v_2)\right]_{(0,0)} = v_1^T H(f) v_2$$

In particular, $Q(v) = \frac{1}{2} v^T H_f v = \frac{1}{2} v^2 f(0,0)$ is a quadratic describing f near 0: $f(x) = Q(x) + O(\|x\|^3)$,
and we describe Q by eigenspaces of H_f:
if $v_1, \dots, v_n$ are an eigenbasis of H_f, then $v_i \cdot v_j = 0,\ i \neq j$,
so $Q(a_1 v_1 + \dots + a_n v_n) = \lambda_1 a_1^2 + \dots + \lambda_n a_n^2$ — if $\lambda_i > 0$, then f has a local minimum at 0, if $\lambda_i < 0$ a local max.

So H_f is characterized by # of pos/neg/zero eigvals. Similar here:

~~Further, define the nullspace of H as~~
~~$\mathrm{null}(H_f) = \{v \mid H_f v = 0\}$ (different from zeroes!)~~

The nullspace of $H(E)$ is the space

$$\mathrm{null}(H(E)) = \{W \in T_\gamma \Omega \mid H(E)(V, W) = 0\ \forall V\}$$
$$= \{W \mid W \text{ is } C^2 \text{ and } D_t^2 W - R(V,W)V = 0\}.$$

These are important to studying minimality. In fact, if we drop a condition: We say $J \in \mathcal{V}(\gamma)$ is a Jacobi field, on a geodesic γ if $J \in \mathcal{V}(\gamma)$ and $D_t^2 W = R(V,W)V$.

Next time: Explain why these are important

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Jacobi fields

Last time: Second Variation Formula: If ~~φ~~ γ is a geodesic, $W_1, W_2 \in T_\gamma \Omega$, $\bar{\alpha} : (-\varepsilon, \varepsilon)^2 \to \Omega$ is s.t. ~~$\alpha(0,0)=\delta$~~, $\bar{\alpha}_{(0,0)} = \gamma$, $\frac{\partial \alpha}{\partial u_i}\big|_{(0,0)} = W_i$, then

$$H(E)(W_1, W_2) = \frac{d^2 E}{du_1 du_2}(0,0) = -\sum_t \langle W_2 | \Delta_t D_t W_1 \rangle - \int_0^1 \langle W_2 | D_t^2 W_1 - R(V, W_1)V \rangle$$

In particular, if $\bar{B} : (-\varepsilon, \varepsilon) \to \Omega$ is a variation w/ $\frac{\partial B}{\partial u} = W$, then $\frac{d}{du} E(\bar{B}_u) = 0$, $\frac{d^2}{du^2} E(\bar{B}_u) = H(E)(W, W)$

~~$= -\sum \langle W | \Delta_t D_t W \rangle - \int_0^1 \langle W | D_t^2 W - R(V,W)V \rangle$~~

~~If W is smooth~~ $\Rightarrow E(\bar{B}_u) = \frac{1}{2} H(E)(W,W) u^2 + O(u^3)$ – how does $H(E)(W,W)$ behave?

Sidenote: This is a quadratic form. In $\mathbb{R}^n$, a quadratic form can be written $Q(v) = v^T M v$ for some symmetric matrix $M = M^T$ – ~~Then M is diagonalizable – $\exists$ orthogonal matrix~~

Lemma: Let $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ be eigvals of M – then $\exists$ orthogonal ~~transf.~~ F s.t. $Q \circ F(x^1, x^2, \dots, x^n) = \lambda_1 (x^1)^2 + \dots + \lambda_n (x^n)^2$ – we understand Q via eigspaces of M. Likewise, understand $H(E)$ via operator $W \longmapsto D_t^2 W - R(V,W)V$.

In particular, if W is smooth, $D_t^2 W = R(V, W)V$, then we call W a **Jacobi field** – these are the nullspace of the operator.

~~(Ex: $M = \mathbb{R}^n$, $D_t^2 W = 0$, i.e., W is of the form $tX + Y$)~~

Thm: W is a Jacobi field if and only if W is the variation field of a variation through geodesics (a variation $\bar{\alpha} : (-\varepsilon, \varepsilon) \times [0,1] \to M$ s.t. $\bar{\alpha}_u$ is a geodesic for all u) (Note: endpoints need not be fixed)

Pf: ($\Leftarrow$) Suppose $\bar{\alpha}$ is a variation through geodesics.

Claim that $\frac{\partial \alpha}{\partial u}$ is Jacobi:

$$D_t^2 \frac{\partial \alpha}{\partial u} = D_t D_t \frac{\partial \alpha}{\partial u} = D_t D_u \frac{\partial \alpha}{\partial t} = \cancel{D_u D_t \frac{\partial \alpha}{\partial t}} + R\left(\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial u}\right)\frac{\partial \alpha}{\partial t}$$

$= R(V, W)V \Rightarrow W$ is Jacobi.

($\Rightarrow$) Count dimensions: A solution to $D_t^2 W = R(V,W)V$ is determined by $W(0)$, $D_t W(0)$, so $\dim\{\text{Jacobi fields on } \gamma\} = 2n$.

Claim: $\exists$ $2n$ dims worth of variations through geodesics

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Let U ~~be~~ be a unit normal nbhd of $\gamma(0)$, let $\varepsilon>0$ s.t. $\gamma(\varepsilon)\in U$. $\forall v_1\in T_{\gamma(0)}M$, $v_2\in T_{\gamma(\varepsilon)}M$, let λ_1, λ_2 be curves s.th $\lambda_1(0)=\gamma(0)$, $\lambda_1'(0)=v_1$, $\lambda_2(0)=\gamma(\varepsilon)$, $\lambda_2'(0)=v_2$, and let $\alpha_u=\gamma_{\lambda_1(u),\lambda_2(u)}$ be the geod ~~from~~ s.t. $\alpha_u(0)=\lambda_1(u)$, $\alpha_u(\varepsilon)=\lambda_2(u)$.

For u suff. small, this extends to $[0,1]$. ~~Then $\frac{\partial\alpha}{\partial u}$ is a Jacobi field.~~ Further, α_u depends smoothly on u. So $\frac{\partial\alpha}{\partial u}$ is Jacobi. Further, $\frac{\partial\alpha}{\partial u}(0)=\lambda_1'(0)=v_1$, $\frac{\partial\alpha}{\partial u}(\varepsilon)=v_2$. This gives an injective map $T_{\gamma(0)}M\times T_{\gamma(\varepsilon)}M\to\{\text{Jacobi fields}\}$, whose image consists of variations of geodesics. Since ~~$T_{\gamma(0)}M\times$~~ $\dim(T_{\gamma(0)}M\times T_{\gamma(\varepsilon)}M)=2n$, this is surjective //

Ex: $M=\mathbb{R}^n$: ~~$R(V,W)V=0\Rightarrow W$ is Ja~~
$D_t^2W=\underbrace{R(V,W)V}_{\to 0}=0\Rightarrow W=X+tY$ for $X,Y\in\mathbb{R}^n$

Ex: $M=S^n$:
① Calculate geodesics on S^n. ②: Sectional curvature of S^n is constant
~~i.e. $\langle R(X,Y)Y|X\rangle=1$ $\forall$ orthonormal X,Y~~. When X,Y orthonormal,
~~and $\langle R(X,Y)X|X\rangle=0\Rightarrow R(X,Y)X=X$~~

That is, $K(X,Y)=\langle R(X,Y)Y|X\rangle=1$ $\Rightarrow R(X,Y)Y=X$
$\langle R(X,Y)X|Y\rangle=0$ $\quad R(X,Y)X=-Y$.

Let γ be equator, N= northward field, E= eastward field $=\gamma'$.
~~Then~~ Let $W=fN+gE$. Then

$D_t^2W=R(V,W)V$ Since N,E parallel,

$f''N+g''E=$ ~~$R(f\cdot R(E,fN)E+gE)E+gR($~~
$=R(E,fN+gE)E=fR(E,N)E=-fN.$

$\Rightarrow g''=0$, $f''=-f$. $\quad g=a+bt$
$f=c\cos(t)+d\sin(t)$.

~~So we decompose W~~ If $f=0$, $W=(a+bt)E$.
a tangential Jacobi field. Generally ~~boring~~ dull:
these come from variations of form
$\alpha_u(t)=\gamma(t+$ ~~$au+but$~~ $(a+bt)u)$

If $g=0$, $W=(c\cos(t)+d\sin(t))$ – normal Jacobi field

~~Describes geods close to γ.~~ What can we do with these?:

~~Properti~~ Prop: Let γ be a geodesic:

① The Jacobi fields on γ form a $2n$-dim subspace of $V(\gamma)$.

② Any Jacobi field J decomposes as $J = J^{\perp} + J^{\parallel}$, where $J^{\parallel}, J^{\perp}$ ~~is~~ are Jacobi fields, ~~$J^{\parallel}$ is par~~ $J^{\perp}$ is orthogonal to γ', $J^{\parallel}$ is parallel to γ'. In fact, $\exists\, a,b \in \mathbb{R}$ s.t. $J^{\parallel} = (a+bt)\cdot\gamma'$.

③: Jacobi fields describe how nearby geods behave. In particular, if $p \in M$, $V \in T_pM$, $\gamma(t) = \exp_p(tV)$, ~~then~~ and $J \in V(\gamma)$ is a Jacobi field with $J(0)=0$, $D_tJ(0)=W$, then $(D\exp_p)_V(W) = \frac{d}{du}\left[\exp_p(V+uW)\right]\Big|_{u=0} = J(1)$.

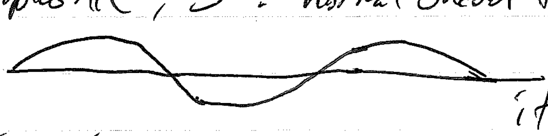
Pf: ①,②: exercise. ③: Let α be the variation ~~$\alpha_u(t) = \exp$~~ $\alpha_u(t) = \exp_p(V+uW)t$. This is a variation through geodesics, so $\frac{\partial\alpha}{\partial u}\big|_{u=0}$ is Jacobi. Let $K = \frac{\partial\alpha}{\partial u}\big|_{u=0}$. ~~ETS $\frac{\partial\alpha}{\partial u}(0)\big|_{u=0} = 0$~~ and Claim $K = J$. ETS $K(0)=0$, $D_tK(0)=W$.

$$K(0) = \frac{\partial}{\partial u}\Big|_{(0,0)} \exp(V+uW)t = 0.$$

$$D_tK(0) = D_t\frac{\partial\alpha}{\partial u}\Big|_{(0,0)} = D_u\frac{\partial\alpha}{\partial t}\Big|_{(0,0)} = \frac{\partial}{\partial u}[V+uW]_0 = W.$$

So $J = K$. In particular $K(t) = \frac{\partial\alpha}{\partial u}\Big|_{(0,t)} = \frac{d}{du}\left[\exp_p(V+uW)\right]\Big|_{u=0} = (D\exp_p)_V(W)$. //

Examples: $\mathbb{R}^n$, S^2: normal Jacobi fields describe nearby geodesics:



if we rotate great circle slightly, it oscillates around the equator — intersecting every length π. ~~so $D\exp_p t$~~

Jacobi fields are 0 at $\pi, 2\pi, 3\pi$, etc $\Leftrightarrow$ $\exp_p$ is singular at πE, $2\pi E$, etc...

— these are conjugate points, we'll discuss them Thurs.

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Spaces with constant curvature

Last time: Jacobi fields: let γ be a geodesic. $J \in \mathcal{V}(\gamma)$ is a Jacobi field if $D_t^2 J = R(V, J)V$.

Thm: J is a Jacobi field $\Leftrightarrow$ J is the variation field of a variation through geodesics.

Prop: If $V, W \in T_pM$, $\gamma(t) = \exp_p(tV)$, $J \in \mathcal{V}(\gamma)$ Jacobi w/ $J(0) = 0$, $D_t J(0) = W$, then $(d\exp_p)_V(W) = J(1)$.

Today: use this to describe metric near p.

Normal coordinates: Recall: $\operatorname{inj\,rad}(p) = \sup\{r \mid \exp_p \text{ is a diffeo on } B_r(0)\}$.
If $0 < r < \operatorname{inj\,rad}(p)$, then $\exp^{-1} : B_r(p) \to T_pM$ is ~~called~~ a diffeo ~~a normal coordinate chart on M~~. Let $(E_1)_p, \dots, (E_n)_p$ be an orthonormal basis for T_pM, let $E : T_pM \to \mathbb{R}^n$ send $a^i (E_i)_p$ to $(a^1, \dots, a^n)$. Then $\varphi = E \circ \exp^{-1} : B_r(p) \to \mathbb{R}^n$ is called a normal coordinate chart on M.

Then $\forall\ V_p = v^i (E_i)_p$, $\exp(V_p)$ has coords $\varphi \circ \exp(V_p) = E(\exp \circ \exp^{-1})(V_p)$
$= E(V_p) = (v^1, \dots, v^n)$.

$\gamma_V(t) = \exp(tV) = (tv^1, \dots, tv^n)$ — geods through p map to lines through origin. So if $\alpha_u = \gamma_{V + uW}$ is a variation through geodesics, we have ~~$\frac{\partial \alpha}{\partial u} = \frac{\partial}{\partial u}(t(v^1 \dots$~~

~~$\frac{\partial \alpha}{\partial u} = \frac{\partial}{\partial u}(t(v^1 + uw^1), \dots, t(v^n + uw^n)) = (tw^1, \dots, tw^n)$~~

$$\frac{\partial \alpha}{\partial u} = \frac{\partial}{\partial u}(t(v^1 + uw^1), \dots, t(v^n + uw^n))$$
$= ~~(tw^1, \dots, tw^n)~~\ t w^i \partial_i$ — this is the field s.t. $D_t \frac{\partial \alpha}{\partial u} = W$.

$\Rightarrow$ $J_i(t) = \partial_i t$ is Jacobi for all t — these generate an n-dim subspace.

Today: use to study metric.

~~Riemannian metric in normal coords:~~

~~Spaces with constant curvature:~~ Recall:

$K(X, \ldots$ — Spaces with constant curvature:

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Let $X, Y \in T_pM$ - then $P_{X,Y} = \exp_p($linearly indep $\operatorname{span}(X,Y))$ is a surface.
One can show that
$$K(X,Y) = \frac{\langle R(X,Y)Y|X\rangle}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}$$
is the Gaussian curvature of $P_{X,Y}$ at p. - how does this affect the metric?

Easiest to see for spaces w/ constant curvature:

~~Suppose M is su~~

Thm: $\forall m \geq 2$, $K \in \mathbb{R}$, $\exists!$ complete, simply-connected Riem. m-mfld. $M = M^m_K$ s.t. $\forall p \in M$, $X, Y \in T_pM$, $K(X,Y) = K$.

$$K > 0 \Rightarrow M^m_K = S^m(\tfrac{1}{\sqrt{K}})$$
$$K = 0 \Rightarrow M^m_K = \mathbb{R}^m$$
$$K < 0 \Rightarrow M^m_K = \text{hyperbolic } m\text{-space}.$$

In polar coords,
$$K > 0 \Rightarrow dg^2 = dr^2 + (\tfrac{1}{\sqrt{K}}\sin(r\sqrt{K})\,d\theta)^2$$
$$K = 0 \Rightarrow dg^2 = dr^2 + (r\,d\theta)^2$$
$$K > 0 \Rightarrow dg^2 = dr^2 + (\tfrac{1}{\sqrt{-K}}\sinh(r\sqrt{-K})\,d\theta)^2$$

i.e., circles of radius r have circumference
$$\frac{2\pi}{\sqrt{K}}\sin(r\sqrt{K}) < 2\pi r < \frac{2\pi}{\sqrt{K}}\sinh(r\sqrt{-K})$$

Pf: Existence: ~~prove that these are~~ calculate the curvatures.
Uniqueness is more interesting: let $p \in M$, ~~$\forall \in T_pM$~~

~~Suppose M is such a space, let $\varphi = E \circ \exp_p^{-1}: B_r(M) \to \mathbb{R}^m$ be normal coords s.t. $(\partial_i)_p = V_i$. What are the Express dg^2 in polar coords. Let $\gamma(t) = \exp_p(t\partial_1)$ - geodesic in ∂_1-direction. Then $J_1(t) = t\partial_1$ are Jacobi fields along γ and $J_2(t) = t\partial_2$ $R(\partial_1, \partial_2)\partial_1 =$~~

Lemma: Suppose M is such a space. ~~Then~~ If X, Y orthonormal, then ~~$\langle R(X,Y)X|X\rangle = K^2$~~. $R(X,Y)X = -KY$.

Pf: ~~Like PS, When $m=2$ problem~~

~~We have~~ Recall $\langle R(X,Y)Z|W\rangle = \langle R(Z,W)X|Y\rangle = -\langle R(X,Y)W|Z\rangle$

Then $\langle R(X,Y)X|X\rangle = 0$ $\langle R(X,Y)X|Y\rangle = -K$

Suppose Z is orthonormal to X, Y. Then
$$\langle R(X,Y+Z)X|Y+Z\rangle = \langle R(X,Y)X|Y\rangle + \langle R(X,Y)X|Z\rangle + \langle R(X,Z)X|Y\rangle + \langle R(X,Z)X|Z\rangle$$
$$\|X\|^2\|Y+Z\|^2K = \|X\|^2\|Y\|^2K + 2\langle R(X,Y)X|Z\rangle + \|X\|^2\|Z\|^2K$$
$$0 = 2\langle R(X,Y)X|Z\rangle.$$

So $R(X,Y)X = -\kappa Y$. //

~~So let's write the metric in polar coords:~~

~~Lemma:~~ That lets us compute Jacobi fields:

~~Lemma: Let γ be a unit geodesic, J a normal Jacobi field on γ, $J(0)=0$.~~
~~Then $\exists$ a parallel field W on γ s.t. $J = fW$ where $f'' = -\kappa f$.~~
~~Pf: Let $E_0, \dots, E_n \in \mathcal{V}(\gamma)$ a set of orthonormal parallel fields s.t. $\gamma' = E_0$. Suppose $J = f^i E_i$ is normal Jacobi. Then~~
~~$D_t^2 J = (f^i)'' E_i = R(E_0, f^i E_i) E_0 = -\kappa f^i E_i$.~~
~~$\Rightarrow (f^i)'' = -\kappa f^i \quad \forall i$~~

~~So if~~ Lemma: Let γ be a unit geod. ~~Then the normal Jacobi fields on γ are spanned by~~ J a normal Jacobi on γ s.t. $J(0)=0$, $\|D_t J(0)\| = 1$.
Then $\exists$ a parallel field W on γ s.t. $J = fW$ where $f'' = -\kappa f$.
Pf: Let $E_1, \dots, E_n \in \mathcal{V}(\gamma)$ ~~s.t. $\gamma' = E_1$,~~ orthonormal parallel fields s.t. $\gamma' = E_1$, $D_t J(0) = E_2$. Suppose $J = f^i E_i$ ~~Then~~ w/ $f^1 = 0$.
Then $D_t^2 J = (f^i)'' E_i = R(E_1, f^i E_i) E_1 = -\kappa f^i E_i$ ~~$= -\kappa J$~~
$\Rightarrow (f^i)'' = -\kappa f^i \quad \forall i$. Since $f^i(0) = (f^i)'(0) = 0$ for $i \geq 3$,

$$J = f E_2 \text{ where } f'' = -\kappa f \quad //$$

In fact, ~~$J = D_t J(0) f =$~~
$$f = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(t\sqrt{\kappa}) & \kappa > 0 \\ t & \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh(t\sqrt{-\kappa}) & \kappa < 0 \end{cases} = f_\kappa$$

So, let's write the metric on M: Let $\gamma(t) = \exp(V \cdot t)$ ~~Let $\varphi = E \circ \exp^{-1} : B_r(M) \to \mathbb{R}^m$~~ be ~~a system of normal coords,~~ ~~let $\gamma(t) = \exp t$~~ with $(\partial_1)_p = V$.
~~Then be one to Let J be a Jacobi field with $D_t J(0)$~~ $J(0) = 0$, $D_t J(0) = \partial_2$. Then on one hand,
~~$J(t) = t\partial_2$~~ ~~$\partial_t \partial_1$~~ and $\|D_t J(0)\| = \|\partial_2\| = 1$.
So ~~$\partial_t \partial_1$~~, $\exists$ parallel field E_2 s.t. $J(t) =$

Let $p \in M$, let $X, Y \in T_pM$ orthonormal vectors, let $\gamma(t) = \exp(rX)$,
$\alpha(\theta, r) = \exp_p r(X\cos\theta + Y\sin\theta)$. This is a variation through geodesics, so $\partial_\theta = \frac{\partial \alpha}{\partial \theta}\big|_{\theta=0}$ is a Jacobi field. By Gauss's Lemma, normal Jacobi field, so orthogonal to $\frac{\partial \alpha}{\partial r}\big|_{\theta=0} = X$. By above,
~~$\|\frac{\partial \alpha}{\partial \theta}\| =$~~ $\|\partial_\theta\| = f_\kappa$ thus $dg^2 = dr^2 + d\theta^2 f_\kappa^2$.

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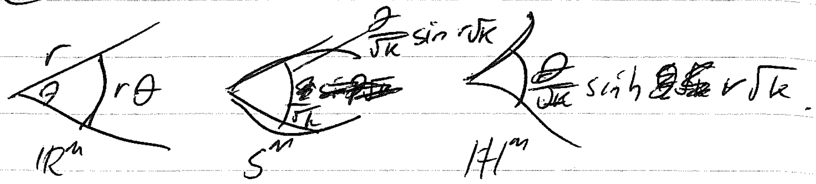
Normal coordinates

Last time: Model spaces – spaces with constant curvature:

$K = 0$ $\quad M^n_K = \mathbb{R}^n$ $\quad$ Characterized in terms of geodesics:

$K > 0$ $\quad M^n_K = S^n$

$K < 0$ $\quad M^n_K = \mathbb{H}^n$



Today: nonconstant curvature:

What can we say about ~~g~~ g near p given ~~R_{ij}~~, ~~Rat~~ $(R_p(X,Y)Z)_p$?

Need some notation: Def: Let $p \in M$, let $E_1, \dots, E_n \in T_pM$ an orthonormal basis. Let $0 < r < \operatorname{inj\,rad}(p)$, let

$\alpha : B_r(0) \subset \mathbb{R}^n \to M \quad \alpha(x^1, \dots, x^n) = \exp_p(\sum x^i E_i)$.

Then α is a diffeo and $\varphi = \alpha^{-1} : B_r(p) \to \mathbb{R}^n$ is called a normal coord chart centered at p. ~~Let~~

Very convenient chart: ① Geodesics through p map to lines through 0.

② By Gauss's lemma, geometric sphere of radius $r' < r$ maps to round sphere in $\mathbb{R}^n$.

③ ~~Jacobi fields~~ If $\forall\, V \in T_pM$, $\gamma_V(t) = \exp_p(tV)$, then $\forall i$,

$J_i(t) = t\,\partial_i$ is a Jacobi field on γ_V.

(Specifically, $J_i = \frac{\partial \alpha_i}{\partial u}\big|_{u=0}$, where ~~$(\alpha_i)_u(t) = \gamma_{V+uE_i}(t)$~~ $(\overline{\alpha}_i)_u = \gamma_{V+uE_i}$)

~~So the coeffs of the metric are given by g_{ij}~~ Estimate g: Let $(x^1, \dots, x^n) \in \mathbb{R}^n$, let $V = x^i \partial_i$.

Recall that $g_{ij} = \langle \partial_i | \partial_j \rangle$ ~~$g_{ij}(\gamma_V(t)) =$~~

so $\langle J_i(t) | J_j(t) \rangle = t^2 \langle \partial_i | \partial_j \rangle = t^2 g_{ij}(\gamma_V(t))$

$= t^2 g_{ij}(tq)$

~~And we can approximate the left side – unfortunately~~

Expand in t? (will have to expand to fourth order):

$$\frac{d^k}{dt^k} \langle J_i(t) | J_j(t) \rangle = \sum_l \binom{k}{l} \langle D_t^l J_i(t) | D^{k-l} J_j(t) \rangle$$

So we need $D_t^l J_i(0)$: $\quad J_i = t\,\partial_i \quad D_t J_i = \partial_i \quad D_t^2 J_i = R(\gamma', J_i)\gamma'$

$= t\, R(\gamma', \partial_i)\gamma'$

$D_t^3 J_i = R(\gamma', \partial_i)\gamma' + t \frac{d}{dt} R(\gamma', \partial_i)\gamma'$

$= t\, R(V, \partial_i) V$

Let $d_{ij} = t^2 g_{ij}$:

$J_i(0) = 0$

$D_t J_i(0) = \partial_i$

$D_t^2 J_i(0) = 0$

$D_t^3 J_i(0) = R(V, \partial_i)V$

$\cancel{t^2 g_{ij}(0) = 0}$ $\quad f(0) = 0$

$\frac{d}{dt} t^2 g_{ij}(0) = \cancel{\frac{df}{dt}(0) = \langle J_i(0) | D_t J_j(0) \rangle + D_t J_j(0)} |_p$

$\frac{df}{dt}(0) = \langle J_i | D_t J_j \rangle + \langle D_t J_i | J_j \rangle |_0 = 0.$

$\frac{d f^2}{dt^2} = \cancel{\langle J_i | D_t^2 J_j \rangle} + 2 \langle D_t J_i | D_t J_j \rangle + \cancel{\langle D_t^2 J_i | J_j \rangle}|_0$

$= 2 \langle \partial_i | \partial_j \rangle = 2\delta_{ij}$

$\frac{d^3 f}{dt^3}\Big|_0 = 0 + 0 + 0 + 0 = 0$

$\frac{d^4 f}{dt^4} = 4 \langle D_t J_i | D_t^3 J_j \rangle + 4 \langle D_t^3 J_j | D_t J_i \rangle |_0$

$= 4 \langle R(\gamma', \partial_j)\gamma' | \partial_i \rangle + 4 \langle R(\gamma', \partial_i)\gamma' | \partial_j \rangle |_0$

$= 8 \langle R(\gamma', \partial_j)\gamma' | \partial_i \rangle$

i.e. $t^2 g_{ij}(\gamma_v(t)) = \cancel{\frac{t^2}{2} 2\delta_{ij}} \; \frac{t^2}{2} \cdot 2\delta_{ij} + \frac{t^4}{24} \cdot 8 \langle R(\gamma', \partial_j)\gamma' | \partial_i \rangle + O(t^5)$

$g_{ij}(\gamma_v(t)) = \delta_{ij} + \frac{t^2}{3} \langle R(V, \partial_j) V | \partial_i \rangle + O(t^3)$

~~But let~~ $\cancel{g_{ij}(tx^1, \dots, tx^n) = \delta_{ij} + \frac{1}{3} \langle R(x^i \partial_i}$

$= \delta_{ij} + \frac{t^2}{3} \langle R(x^k \partial_k, \partial_j) x^l \partial_l | \partial_i \rangle + O(t^3)$

$\cancel{g_{ij}(y^1, \dots, y^n) = \delta_{ij} + \frac{1}{3} y^k y^l \langle R(\partial_k, \partial_j)\partial_l | \partial_i \rangle + O(\|y\|^3)}$

R represents second derivative of metric

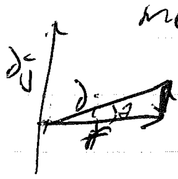
set $t = 1$: $g_{ij}(\exp(V)) = \delta_{ij} + \frac{1}{3} \langle R(V, \partial_j) V | \partial_i \rangle + O(\|V\|^3)$

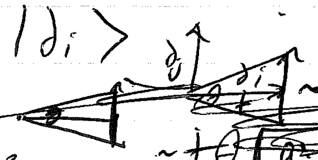
~~Can use this to show, e.g. $vol(B_r(p)) = vol(B_r(0))(1+$~~

We use this to define some other ~~curvature tensors~~ curvatures: sectional, Ricci, scalar.

Sectional curvature: $K(\partial_i, \partial_j)_p = \langle R(\partial_i, \partial_j) \partial_j | \partial_i \rangle$

measures spread of geodesics:





$\sim r\theta \sqrt{g_{jj}}$ ~~take $V = \partial_i$ then~~

$\sim t\theta \sqrt{g_{jj}(\exp t\partial_i)}$

$g_{jj}(0, \dots, 0, \overset{i}{t}, 0, \dots, 0) = 1 + \frac{1}{3} t^2 \langle R(\partial_i, \partial_j) \partial_i | \partial_j \rangle + O(t^3)$

$= 1 - \frac{1}{3} t^2 K(\partial_i, \partial_j) + O(t^3)$

$\sqrt{g_{jj}} = 1 - \frac{1}{6} t^2 K(\partial_i, \partial_j) + O(t^3)$

Ricci curvature: $Ric_p(Y, Z) = \sum_k \langle R(E_k, Y) Z | E_k \rangle$ ($\forall$ orth. basis E_i) is a symmetric bilinear form — maybe easier to discuss as a quadratic form: take $E_1 = X$, $E_2, \dots, E_n$ orthog.

$Ric_p(X, X) = \sum_k \langle R(E_k, X) X | E_k \rangle = \sum_{k=2}^n K(X, E_k)$

① Al ~~so measures spread of geodesics: The volume~~ ~~of~~

$\lim_{\varepsilon \to 0}$ $\frac{vol(B_\varepsilon(p))}{\varepsilon^n \omega_n}$ ~~$\approx vol(B_\varepsilon(0))$~~ $\sqrt{\det(g_{ij}(\exp(V))}$

vol of Euclidean ball

$\approx 1 + \frac{1}{2} tr((g_{ij}) - I)$

$\approx 1 + \frac{1}{6} \sum_i \langle R(\partial_i, X) X | \partial_i \rangle$

Also measures spread: Let $F: \mathbb{R}^n \to M$

$F(x^1, \dots, x^n) = \exp(x^i E_i)$ as before. Then

for $U \subset \mathbb{R}^n$ $\mathrm{vol}(F(U)) = \int_U \sqrt{\det((g_{ij}))}\, dx^1 \wedge \dots \wedge dx^n$, i.e.,

$\sqrt{\det((g_{ij}))}\, dx^1 \wedge \dots \wedge dx^n$ is the volume form of M.

When X is small, $(g_{ij}(X)) \approx I$ and $\det(g_{ij}) \approx 1 + \mathrm{tr}(g_{ij} - I)$

$$\sqrt{\det(g_{ij}(X))} \approx 1 + \tfrac{1}{2}\mathrm{tr}((g_{ij}(X)) - I)$$
$$\approx 1 + \tfrac{1}{2} \sum_k \tfrac{1}{3} \langle R(X, \partial_k)X \mid \partial_k \rangle$$
$$\approx 1 - \tfrac{1}{6} Ric(X, X)$$

Interp:

(When $n=3$, Ric determines R — in higher dims, mflds can be Ricci-flat but not flat.)

Scalar curvature: $Scal(p) = \mathrm{tr}_g Ric_p = \sum_i Ric(E_i, E_i)$

where $E_1, \dots, E_n \in T_pM$ is any orthonormal basis.

— measures volume growth:

$$\mathrm{vol}(B_\varepsilon(p)) \approx \underbrace{\varepsilon^n \omega_n}_{\text{vol of Eucl. ball}} \left(1 - \frac{Scal(p)}{6(n+2)}\varepsilon^2 + O(\varepsilon^3)\right)$$

(Overflow: Questions before midterm)

(Next: Understand singularities of the exponential map:

Sketch of Morse Index Thm: Conjugate pts:

~~Singularities of exp correspond to Jacobi fields with~~

~~If $(D\exp_p)_V$ singular, then there's a nonzero Jacobi field along γ_V with~~
~~$J(p) = 0$, $J(q) = 0$ — if~~

Def: p and q are conjugate along γ if $\exists$ a Jacobi field J s.t. $J(p) = 0$, $J(q) = 0$, but J is not identically zero.

— this is a singularity of the exponential map: if $q = \exp_p(V)$, then rank$(D\exp_p)_V < n$. ~~The order of conj~~ — ~~Morse Index theorem~~

These are v. closely connected to the question of when the geodesic γ is minimizing — will cover this after break.