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2nd Variation and minimal geodesics

Recall: 2nd Variation Formula: $E(\gamma) = \frac{1}{2}\int_0^1 \|\gamma'\|^2 dt \quad \ell(\gamma) \leq \sqrt{2E(\gamma)}$ with equality iff γ const-speed

$H(E)(W_1, W_2) = \frac{\partial^2}{\partial u_1 \partial u_2} E(\bar{\alpha}_{u_1,u_2}) = -\sum_t \langle W_2 | \Delta_t D_t W_1 \rangle$
$- \int_0^1 \langle W_2 | D_t^2 W_1 - R(V, W_1)V \rangle dt$.

γ is a geodesic $\Leftrightarrow$ γ is a critical point of E.

~~Th~~ γ is a length-min geod $\Leftrightarrow$ γ is a minimum of E among all curves from p to q.

$\Rightarrow H(E)(W,W) \geq 0 \quad \forall W \in T_\gamma \Omega$ — Today: use 2nd Var to study minimizers

When does this happen? ~~First, Thm (Myers): Suppose M a cplt Riem mfld, K(X,Y) ≥ 1/r² ∀X,Y. Then diam M ≤ πr. Pf: Let γ a geodesic of length L.~~ $\boxed{\text{First, Myers}}$

Def: If $\gamma: \mathbb{R} \to M$ is a geod, $\gamma(0) = p$, $\gamma(t) = q$, $t > 0$, we say q is conjugate to p along γ if $\exists$ a Jacobi field W on γ s.t. $W(0) = 0$, $W(t) = 0$, but $W \not\equiv 0$. The order of conjugacy of q is

$$\mathrm{ord}_p(q) = \dim \{W \in V(\gamma) \mid W \text{ is Jacobi}, W(0) = 0, W(t) = 0\}.$$

~~Recall:~~

Thm: - If p, q are not conjugate, then any Jacobi field W is det by $W(p), W(q)$ (because $W \mapsto (W(p), W(q))$ is a bijection)

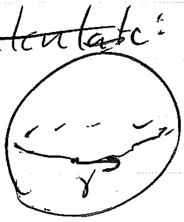
- If W is Jacobi, $W(0) = 0$, $W(t) = 0$, ~~then~~ $X = D_t W(0)$, then $(D\exp_p)_{tV}(X) = 0$ — so $V = \gamma'(0)$

$$\mathrm{ord}_p(q) = \dim \ker (D\exp)_{\exp^{-1}(q)}.$$

- so conjugate pts are measure zero by Sard.

On the sphere, conj. pts are closely connected to minimizing geodesics:

~~Calculate~~: $M = S^n$, $\gamma(0) = p$, γ unit speed. (Ex)



Can calculate: ~~if W is Jacobi~~, if X is parallel along γ, $\langle X | \gamma' \rangle = 0$ then $W(t) = (\sin(t))X$ is a Jacobi field, and every normal Jacobi field along γ ~~is of the~~ with $W(0) = 0$ is of this form. ~~So~~ ~~W(π)=0~~ Further, $W(t) = 0 \Leftrightarrow t = 0, 2\pi ?$...

so $\mathrm{ord}_p(-p) = \dim$ ~~$\langle \gamma'(0) \rangle^\perp$~~ $\gamma'(0)^\perp = n - 1$

~~Cool~~ Before the first conjugate point, γ is length-minimizing. After, it's not. In fact, that's a general phenomenon

~~Recall:~~

Thm: If γ is a ~~loca~~ length-minimizing geodesic from p to q, then the interior of γ contains no points conjugate to p.

Pf: ~~Suppose~~ Take $\gamma(0)=p$, $\gamma(1)=q$. Suppose $t_0\in(0,1)$, p conjugate to $\gamma(t_0)$. Claim: $\exists\, Z\in T_\gamma\Omega$ s.t. $H(E)(Z,Z)<0$.

Let $q_0=\gamma(t_0)$, let X a nonzero Jacobi field st. $X(0)=0, X(t_0)=0$.
Let $Y(t)=\begin{cases} X(t) & t\le t_0 \\ 0 & t>t_0\end{cases}$

Then Y satisfies:

$$H(W,Y) = -\sum_t \langle W \mid \Delta_t D_t Y\rangle - \int_0^1 \langle W \mid \cancel{D_t^2 Y - R(\gamma',Y)\gamma'}\rangle$$

$$= -\langle W(t_0) \mid \Delta_{t_0} D_t Y(t_0)\rangle$$

$$= \langle W(t_0) \mid D_t X(t_0)\rangle$$ — since $X\equiv 0$, $D_t X(t_0)\neq 0$

And we use this to craft Z: let U be a smooth field w/ $U(t_0)=-D_t X(t_0)$, $U(0)=U(1)=0$. Let $Z=Y+\varepsilon U$. Then:

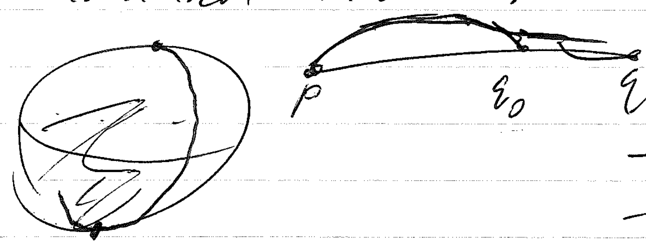
$$H(Z,Z)=H(Y,Y)+2\varepsilon H(Y,U)+\varepsilon^2 H(U,U)$$

$$=0+2\varepsilon\langle D_t X(t_0) \mid -D_t X(t_0)\rangle+\varepsilon^2 H(U,U)$$

$$=-2\varepsilon\,\|D_t X(t_0)\|^2+\varepsilon^2 H(U,U)<0 \text{ if } \varepsilon \text{ small.}$$

So γ is not a local minimum //

Pic:



- Adjust geod on $[0,t_0]$
- create corner at t_0
- perturb corner away
- decrease energy.

~~We can do the~~ In fact, we can quantify the extent to which γ fails to be minimal. $Q:\mathbb{R}^n\to\mathbb{R}$, $F(v,w)=v^TMw$ is a bilinear form.

Recall: if $Q(v)=v^TMv$ is a general quadratic form, $M^T=M$,
then $\exists$ a basis s.t. $Q(v^i e_i)=\sum a_i (v^i)^2$, where $a_i\in$ ~~$\{-1,0,1\}$~~ $\mathbb{R}$.
In fact, generally many such bases, ~~all with the same a_i~~
but ~~# positive a_i's, # zero, # ne~~
of a_i's which are positive, zero, negative remains same.

index(Q) = # of $a_i<0$

nullity(Q) = # of $a_i=0$.

$F: V\times V \to \mathbb{R}$

~~Ex~~ ~~Morse~~ ~~spoken~~ $\cancel{Q}: V \to \cancel{\mathbb{R}}$,

nullity(F) = ~~dim null(F)~~ $= \dim\{v\in V \mid F(v,w)=0\ \forall w\in V\}$.

index(F) = max dim$\{P \mid F(x,x)<0\ \forall x\in P\setminus 0\}$.

Ex: $Q(x,y)=x^2-y^2$

nullity = 0
index = 1

note: if $x\in \text{null}(F)$ then $Q(x+y) = F(x+y,x+y)=F(y,y) = Q(y)$
— Q splits as a product

~~Morse Index~~ Theorem:

~~More generally~~: Prop: Let F be a symmetric bilinear form on V. Then $\exists$ a decomp $V=V_-\oplus V_0\oplus V_+$ s.t.

- F is pos. def on V_+: $F(v,v)>0\ \forall v\in V_+\setminus 0$.
- F is neg. def on V_-: $F(v,v)<0\ \forall v\in V_-\setminus 0$
- $V_0=\text{null}(F)$: $\forall v_0\in V_0,\ v\in V,\ F(v,v_0)=0$.

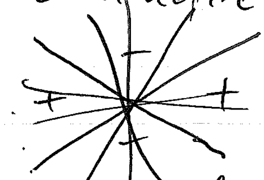
Thus, $F(v+v_0,v+v_0)=F(v,v)$.

- If $W_-, W_+\subset V$, $F|_{W_-}<0$, $F|_{W_+}>0$, then can extend to $V_-\supset W_-$, $V_+\supset W_+$ s.t. $V=V_-\oplus V_+\oplus V_0$.

~~$= \text{index}(F)=\dim(V_-)$~~ Further, all V_-, V_+, V_0 have same dimension.

So we can define index$(F)=\dim(V_-)$ = max dim of a neg-def. subspace of V.

Ex:



$F(v,v)=x^2-y^2$

Thm (Morse Index Theorem): Let $\gamma:[0,1]\to X$ be a geodesic.
Let index(γ) = index$(H(E)_\gamma)$. Then ~~index~~

$$\text{index}(\gamma)=\sum_{t\in[0,1]}\text{order}_p\,\gamma(t).$$

Further index$(\gamma)<\infty$.

Overflow: Applications ~~Thm (Bonnet-Myers)~~ Recall that $Ric(U_1,U_2)=\sum\langle R(U_1,E_i)E_i, U_2\rangle$ where E_i is an orthonormal basis ~~and if $\|U\|$~~

if $\|U\|=1$, we can take $U, E_1, \dots, E_{n-1}$ orthonormal and get

$Ric(U,U)=\sum\langle R(U,E_i)E_i, U\rangle=\sum K(U,E_i)$ = total sectional curvature.

Then Thm (Bonnet-Myers): Suppose M is an n-manifold, $r>0$. Suppose $\forall U\in TM$, $\|U\|=1$, $Ric(U,U)\geq\frac{n-1}{r^2}$. Then ~~$\text{diam}(M)\leq$~~ any geodesic of length $> \pi r$ is not minimizing

Pf: ~~$\forall p$~~ Let $\gamma:[0,1]\to M$ a geod. of length $L>\pi r$, let $E_0, \dots, E_{n-1}$ parallel orthonormal, $E_0=\frac{\gamma'}{\|\gamma'\|}$.

$W_i = \sin(\pi t)E_i$. Claim: $H(W_i,W_i)\leq 0$ for some i. Write $H(W_i,W_i)=\int_0^1 \pi^2\sin^2\pi t - L^2\sin^2(\pi t)\langle R(E_0,E_i)E_i, E_0\rangle$ and sum to get Ric. Cor: If M complete, $Ric\geq\frac{n-1}{r^2}$, then $\text{diam}\,M\leq\pi r$, $\pi_1(M)$ finite

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Morse Index Theorem

Last time: index of a form: if F symmetric bilinear form on V,
~~Def:~~ $\text{index}(F) = \max \dim\{P \subset V \mid F(p) < 0 \; \forall p \in P \setminus 0\}$
~~Can be tricky to calculate.~~ To calculate, use ~~following~~ either:
- diagonalize F: $F(v,v) = a_i \sum_i a_i (v^i)^2$, then index = # of negative a_i
Or: Prop: Let F be a symmetric bilinear form on V finite dim. Then $\exists$ a decomp $V = V_+ \oplus V_0 \oplus V_-$ s.t.
  - F is positive def on V_+: $F(v,v) > 0 \; \forall v \in V_+ \setminus 0$.
  - Negative def on V_-: $F(v,v) < 0 \; \forall v \in V_- \setminus 0$.
  - $V_0 = \text{null}(F) = \{v \in V \mid F(v,w) = 0 \; \forall w \in V\}$

(In particular, $F(v+w, v+w) = F(w,w) \; \forall w \in V$.)

Further, if $F|_{W_-} < 0$, $F|_{W_+} > 0$, then $\exists$ such a decomp w/ $W_- \subset V_-$, $W_+ \subset V_+$.

So: last time $F(v,v) = x^2 - y^2 - z^2$ has index 2:
$F|_{\langle \partial_y, \partial_z \rangle} < 0$ so ind ≥ 2, $F|_{\langle \partial_x \rangle} > 0$ so index ≤ 3 ~~$\mathbb{R}^3 = \langle x \rangle \oplus \langle y, z \rangle$~~

Morse Index thm:
Let $\gamma : [0,1] \to M$ a geo from p to q. Then
$$\text{index } H(E)_\gamma = \sum_{t \in [0,1)} \text{order}_p(\gamma(t))$$
How to prove, how to calculate index?

Pf of Morse Index: 1. reduce to finite dimensions.
Claim: $\exists$ a v. space $W \subset T_\gamma \Omega$ s.t. $H|_W > 0$ and W has finite codim.
Pf: ~~Let $\varepsilon > 0$ s.t. $\forall$~~
Let $\varepsilon > 0$ small enough that $\forall t \in [0,1]$ $B_\varepsilon(\gamma(t))$ is contained in a unit normal nbhd. Let $0 = t_0 < t_1 < \cdots < t_k = 1$ s.t. $|t_i - t_{i-1}| < \varepsilon \; \forall$. Then each segment $\gamma([t_i, t_{i+1}])$ is minimal, and $\gamma(t_i)$ not conj to $\gamma(t_{i+1})$.

state as a lemma.

Let $J = J(t_0, \ldots, t_k) = \{$broken Jacobi fields$\}$
$= \{W \in T_\gamma \Omega \mid W|_{[t_i, t_{i+1}]} \text{ is Jacobi}\}$.
Then an element $W \in J$ is determined by $W(t_0), \ldots, W(t_k)$ — $\dim J < \infty$. Let $J^\perp = \{W \in T_\gamma \Omega \mid W(t_i) = 0 \; \forall i\}$. Then
$J \cap J^\perp = 0$, $T_\gamma \Omega = J \oplus J^\perp$, and if $W_1 \in J$, $W_2 \in J^\perp$, then
$$H(W_1, W_2) = -\sum_t \cancel{\langle W_2 \mid \Delta_t D_t W_1 \rangle_0} - \int_0^1 \langle W_2 \mid \cancel{D_t^2 W_1 - R(\gamma', W_1)\gamma'} \rangle_0$$
$W_2(t_i) = 0$. Jacobi

i.e. $H(J, J^\perp) = 0$

~~Claim~~ Claim: $H|_{J^\perp} \geq 0$. Let $X \in J^\perp$, let $X_i := X|_{[t_i, t_{i+1}]}$.
Then $\cancel{H(W_i, W_i)}$ $\gamma|_{[t_i, t_{i+1}]}$ is minimal, so $H(X_i, X_i) \geq 0$.

Suppose $X \in J^\perp$, $H(X, X) = 0$. On one hand $H(X, J) = 0$. OTOH, suppose $Y \in J^\perp$.
Then $X + \varepsilon Y \in J^\perp$, so

$$f(\varepsilon) = H(X+\varepsilon Y, X+\varepsilon Y) = H(X,X) + 2\varepsilon H(X,Y) + \varepsilon^2 H(Y,Y)$$
$$= 2\varepsilon H(X,Y) + \varepsilon^2 H(Y,Y) \geq 0 \quad \forall \varepsilon$$

$f(0) = 0$, ~~so $f'(0) = 0$~~ and $f(\varepsilon) \geq 0$ $\forall \varepsilon$ $\Rightarrow H(X,Y) = 0$.

That is $H(X, Z) = 0$ $\forall Z \in T_\gamma \Omega \Rightarrow X \in \text{null}(H) = \{\text{Jacobi}\}$

Since $X(t_i) = 0$ $\forall i$, $X = 0$, so $H|_{J^\perp} > 0$. Thus,
$\text{index}(H) \leq \dim(J) < \infty$

In fact, $\text{index}(H) = \text{index}(H|_J)$.

pf: $T_\gamma\Omega = J + J^\perp$, let p, $p^\perp$ be projections. Then
$\forall$ $W \in T_\gamma \Omega$, $H(W, W) = H(p(W), p(W)) + 2\,\cancel{H(p(W), p^\perp(W))}$
$+ H(p^\perp(W), p^\perp(W))$
$\geq H(p(W), p(W))$.

So if $H|_N < 0$, then $H|_{p(N)} < 0$ and $N \cap J^\perp = 0$, so p is inj.
$\Rightarrow \text{index}(H) = \text{index}(H|_J)$.

Likewise $\text{null}(H) \subset J \Rightarrow \text{nullity}(H|_J) = \text{nullity}(H)$.

So we can consider $H|_J = H|_{J(t_0, t_1, \ldots, t_k)}$

Step 2: ~~How does $H|_{J(t_0, t_1, \ldots, t_k)}$ this~~ depend on t_k?

Let $\gamma_\tau = \gamma|_{[0,\tau]}$, $H_\tau = H(E)_{\gamma_\tau}$. $\lambda(\tau) = \text{index}\, H_\tau$.

How does λ depend on τ?

1 – $\lambda(\tau)$ is nondecreasing (a v. field on γ_τ extends to a field on $\gamma_{\tau+\varepsilon}$.

2 – $\lambda(\tau) = 0$ for $\tau < \varepsilon$

Index $H_\tau = \text{index}\, H|_{J(0,\tau)}$ but $J(0,\tau) = 0$.

~~For larger~~

Claim: λ only changes at conjugate points.

Lemma: $\forall \tau$, $\lambda(\tau - \delta) = \lambda(\tau)$ for suff small δ.

pf: Partition so $0 = \tau_0 \leq \cdots < t_i < \tau < t_{i+1} \cdots$

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Let $J_\tau = J(t_1, \dots, t_{i-1}, \tau)$ — then for small δ,
$$J_\tau \cong J_{\tau-\delta} \cong T_{\gamma(t_1)}M \oplus \dots \oplus T_{\gamma(t_{i-1})}M.$$
Call this Σ.
Then $H_{\tau-\delta}$ is a form on Σ that varies ctsly.
In particular, if $N \subset \Sigma$, $H_\tau|_N < 0$, then $H_{\tau-\delta}|_N < 0$ $\forall$ suff small δ $\Rightarrow \lambda(\tau-\delta) \geq \lambda(\tau) \Rightarrow \lambda(\tau-\varepsilon) = \lambda(\tau)$ //
(~~Remember: $\Sigma = \Sigma_+ \oplus \Sigma_0$~~)
So when can λ change? Recall: $\Sigma = \Sigma_+ \oplus \Sigma_0 \oplus \Sigma_-$
If we perturb H, to $\overline{H}$, then $\overline{H}|_{\Sigma_+} > 0$, $\overline{H}_{\Sigma_-}| < 0$.
But $\overline{H}|_{\Sigma_0}$ can change signs.

plan to stop here - ish
expand previous slightly

Lemma: Let $n = \text{nullity}(H_\tau)$. Then $\forall$ suff small $\delta > 0$,
$$\lambda(\tau+\delta) = \lambda(\tau) + n.$$
Pf: ~~By the above, if ε suff small, then~~ ~~plan to stop here~~
(Upper) Let Σ as above, let $\Sigma = \Sigma_- \oplus \Sigma_0 \oplus \Sigma_+$ (wrt H_τ).
Then if ε suff small, $H_{\tau+\varepsilon}|_{\Sigma_+} > 0 \Rightarrow$

$$\text{index } H_{\tau+\varepsilon} \leq \dim \Sigma_0 + \dim \Sigma_- = \text{index}(H_\tau)$$
(Lower): let $J_1, \dots, J_n$ Jacobi fields, $J_i(0) = J_i(\tau) = 0$, ~~lin ind~~ lin ind, n = nullity. n = nullity(H_τ)
Let $W_1, \dots, W_k \in T_\gamma\Omega$ span a neg. def subspace, k = index.
Claim: ~~$\langle J_1, \dots, J_n, W_1, \dots$~~ $S = \langle W_1, \dots, W_k, J_1, \dots, J_n \rangle$ can be perturbed to be a neg. def subspace.
We have $H(J_i, W_k) = 0$
$$H|_S = \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix} \text{ where } M < 0$$
$H(J_i, J_j) = 0$, so. Let $z_i = \Delta_t D_t J_i(\tau)$.
For $Y \in T_\gamma\Omega$, $H(J_i, Y) = \langle z_i | Y(\tau) \rangle$.
Let $y_j \in T_{\gamma(\tau)}M$ be vec's st. $\langle z_i | y_j \rangle = \delta_{ij}$. and let Y_j fields s.t. $Y_j(\tau) = y_j \Rightarrow H(J_i, Y_j) = -\delta_{ij}$.

Let $S' = \langle W_1, \dots, W_k, c^{-1}J_1 + cY_1, \dots, c^{-1}J_n + cY_n \rangle$. — then
$$H|_{S'} = \left(\begin{array}{c|c} M & cH(W_i, Y_j) \\ \hline \text{''} & -2I + c^2 H(Y_i, Y_j) \end{array} \right).$$ As $c \to 0$, converges to $\left(\begin{array}{c|c} M & 0 \\ \hline 0 & -2I \end{array} \right)$
which is neg. def. //