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This is Diff. Geo II.

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Bring paper for email list

- Office hrs? (after ~~class?~~ Monday, 2-3?

Jeff said you covered - metrics, connections, Levi-Civita connection, curvature tensor.

(Do you know how to: - calculate lengths of curves? Suppose I give you a surface in $\mathbb{R}^3$ - say, a torus.

- express a metric in a coordinate patch?
- describe a connection in a coordinate patch?
- ~~Prove the uniqueness~~ Calculate the L-C connection? ~~Can you~~ cal~~culate metric~~
- Calculate the curvature tensor?

What can you calculate?)

~~What I want to do this sem.~~

These are essentially all local: Jeff said that he showed you how to use curvature to calculate Taylor expansion of metric, right? — ~~too~~ this is local structure. This sem: continue studying local structure, ~~how local st~~ but also focus on how local structure determines global structure — if you know something about curvature (pos. curv, neg. curv, flat, etc) what does that say about the global structure of the mfld?

Let's start by reviewing some of the basics. I like to work, ~~with~~ at least as a motivating example, with submanifolds of $\mathbb{R}^n$.

(Nash showed that every closed manifold embeds smoothly, isometrically in $\mathbb{R}^n$ for large enough n, so this is not much of a restriction)

Q: How can we find shortest paths in $\mathbb{R}^n$?

Calculus of variations: Let $\gamma : [a,b] \to \mathbb{R}^n$ be a smooth path, define $E_a^b(\gamma) = \int_a^b \|\frac{d\gamma}{dt}\|^2 dt$.

By C-S, $\underbrace{\int_a^b \|\frac{d\gamma}{dt}\| dt}_{\text{length}} \leq \sqrt{\int_a^b 1\, dt} \sqrt{\int_a^b \|\frac{d\gamma}{dt}\|^2} = \sqrt{b-a}\sqrt{E_a^b(\gamma)}$

So $L(\gamma) \leq \sqrt{b-a}\sqrt{E_a^b(\gamma)}$, and E is minimized when γ is minimal length, constant speed.

~~Var~~ How do variations affect E? Let $E = E_0^1$

Let $h_u : [0,1] \to \mathbb{R}^n$ be a smooth htpy of γ

What is $\frac{d}{du} E(h_u)$? $\frac{1}{2}\frac{d}{du} E(h_u) = \frac{1}{2}\int \frac{\partial}{\partial u}\langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial t}\rangle dt$

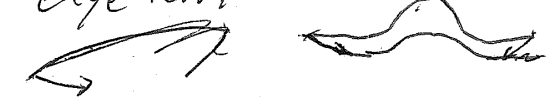
$= \int \langle \frac{\partial^2 h}{\partial u \partial t}, \frac{\partial h}{\partial t}\rangle dt$

$$\boxed{\frac{\partial}{\partial t}\left\langle \frac{\partial h}{\partial u}, \frac{\partial h}{\partial t}\right\rangle = \left\langle \frac{\partial^2 h}{\partial u \partial t}, \frac{\partial h}{\partial t}\right\rangle + \left\langle \frac{\partial h}{\partial u}, \frac{\partial^2 h}{\partial t^2}\right\rangle}$$

$$= \left\langle \frac{\partial h}{\partial u}, \frac{\partial h}{\partial t}\right\rangle\Big|_0^1 - \int_0^1 \left\langle \frac{\partial h}{\partial u}, \frac{\partial^2 h}{\partial t^2}\right\rangle dt$$

Let $U = \frac{\partial h}{\partial u}$, $V = \frac{\partial h}{\partial t}$, $A = \frac{\partial^2 h}{\partial t^2}$

$$= \underbrace{\langle U(1), V(1)\rangle - \langle U(0), V(0)\rangle}_{\text{edge term}} - \underbrace{\int_0^1 \langle U, A\rangle dt}_{\text{curvature term}}$$



If h fixes endpoints, $\frac{1}{2}\frac{d}{du}E(h_u) = -\int_0^1 \left\langle \frac{\partial h}{\partial u}, \frac{\partial^2 h}{\partial t^2}\right\rangle$

So if γ is a shortest path in $\mathbb{R}^n$ from p to q, then $\frac{\partial^2 h}{\partial t^2} = 0$ – zero accel.

If γ is a shortest path in M from p to q, then $\frac{\partial^2 h}{\partial t^2}$ is normal to M. – no acceleration tangent to M.

How do we apply this in general? We need a notion of "acceleration tangent to M".

If $M \subset \mathbb{R}^n$, "easy" – if ~~V is a vector of~~ γ is a curve, V is a vector field on γ (i.e. $\forall t$, $V(t) \in T_{\gamma(t)}M$), we let ~~DV = projection of $\frac{dV}{dt}$ to TM~~ then we can think of V as a fn to $\mathbb{R}^n$ and let

$$D_t^T V(t_0) = p_{T_{\gamma(t_0)}M}(V'(t_0))$$

Good: If γ is a shortest path, then $D_t^T \gamma' = 0$ ("zero accel").

Bad: ~~Why is this invariant?~~ Not obv. invariant. To prove invariance we generalize to ... break?

Connections: $\nabla : T_pM \times \underbrace{\mathfrak{V}(M)}_{\text{vector field on } M} \to T_pM$

If $X_p \in T_pM$, $Y \in \mathfrak{V}(M)$, $\nabla_{X_p} Y \in T_pM$ is the covariant derivative of Y in direction X_p.

∇ is a connection if
- bilinear in X_p, Y
- $\nabla_{X_p}(fY) = (X_p f)Y_p + f(p)\nabla_{X_p} Y$.
- varies smoothly – i.e. $\nabla : \mathfrak{V}(M) \times \mathfrak{V}(M) \to \mathfrak{V}(M)$ is smooth.

(Ex: Directional derivative in $\mathbb{R}^n$).

But in general, many connections —
if $u^1, \dots, u^n$ are coordinate fns on a patch, w/ corresponding v. fields $\partial_1, \dots, \partial_k$, then the connection is determined by the fields $\nabla_{\partial_i} \partial_j$.

Each of these has form $\nabla_{\partial_i} \partial_j = \sum_k \Gamma^k_{ij} \partial_k$ for smooth fns Γ^k_{ij} (Christoffel symbols of connection), so ∇ is locally determined by n^3 smooth fns ~~locally~~.

If $M \subset \mathbb{R}^n$, we define the tangential connection ∇^T: if $X, Y \in \mathcal{V}(M)$, let $\bar{X}, \bar{Y}$ be smooth extensions. Let $p^T : M \times \mathbb{R}^n \to TM$ be projection, define $\nabla^T_X Y = p^T(\bar{\nabla}_{\bar{X}} \bar{Y})$, where $\bar{\nabla}$ is Euclidean conn. (directional deriv.) This is well-defined.

~~We can use~~ One of the big apps of connections is derivs along curves:
Thm: if ∇ is a connection, $\forall \gamma$, $\exists!\ D_t : \mathcal{V}(\gamma) \to \mathcal{V}(\gamma)$ (covariant derivative along γ)

s.t. $-D_t(V+W) = D_t(V) + D_t(W)$
$-D_t(fV) = \frac{df}{dt} V + f D_t(V)$
$-$ if V is the restriction of $\bar{V} \in \mathcal{V}(M)$, then $D_t V = \nabla_{\frac{d\gamma}{dt}} \bar{V}$.

So we can rewrite geod eq. as $D_t \frac{d\gamma}{dt} = 0$.
Equiv, we say that the velocity field $\frac{d\gamma}{dt}$ is parallel.

Thm: If $c \in T_{\gamma(t_0)} M$, then $\exists!$ parallel v. field V on γ s.t. $V_c(t_0) = c$, $D_t V_c = 0$.

~~We def~~ If γ is a curve, ~~$v \in T_{\gamma(t_0)} M$~~, we define parallel transportation $P_{t_0, t_1} : T_{\gamma(t_0)} M \to T_{\gamma(t_1)} M$ so that $P_{t_0,t_1}(c) = V_c(t_1)$

So far, nothing we've done used the metric on M (well, except when M is a submfld) — connections, parallel trans, etc. are all purely smooth notions. Let's start bringing in the metric — that's ult. where uniqueness will come in.
Thm: The tangential connection is compatible with the induced metric on M. That is, it satisfies the following equiv. conditions:
- ~~If V, W are parallel v. fields on γ, then $\langle V, W \rangle$ is constant~~
- P_{t_0, t_1} is an isometry $\forall t_0, t_1$
- if $V, W \in \mathcal{V}(\gamma)$, then $\frac{d}{dt}\langle V, W\rangle = \langle D_t V, W\rangle + \langle V, D_t W\rangle$
- If $X, Y, Z \in \mathcal{V}(M)$, then $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$.

~~Note:~~
Unique? No: if $X, Y, Z = \partial_1, \dots, \partial_m$, we get m^3 linear equations but swapping Y, Z gives same eq. so we have $\sim \frac{m^3}{2}$ compatible connections.

For ex, not hard to construct "twisted" connections

let $F : \mathbb{R}^n \to O(n)$. ~~so that define~~ $\cancel{\nabla}$ ~~so that~~

~~$P_{ab}(v) = F(\gamma(b))F(\gamma(a))^{-1}v$. can divide $T\mathbb{R}^n$ into~~

{so we ~~have a~~ parallel sections ~~of~~ the ~~tangent bundle~~: $\{\forall v,\ p \mapsto F(p)v\}$ is parallel}.

define ∇ so that the sections $\{(p, F(p)v) \mid p \in \mathbb{R}^n\}$ are parallel. This is compatible with metric b.c.

$$P_{ab}(v) = F(\gamma(b))F(\gamma(a))^{-1}v \text{ is an isometry.}$$

What distinguishes these is torsion

Thm: The tangential connection is torsion-free i.e.

$$\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \equiv 0 \quad \forall X, Y \in \mathcal{V}(M).$$

Fund Lemma of Riem. Geom. Thm: If M is a Riemannian mfld, $\exists!$ connection that is torsion-free, compatible w/ metric.

Pf: Dimension count: τ is antisymmetric, so

$\tau(\partial_i, \partial_j) \equiv 0$ leads to $\frac{m(m-1)}{2}$ vector eqs. $\sim \frac{m^3}{2}$ scalar eqs.

Combining these with the $\sim\frac{m^3}{2}$ eqs from compatibility, we get uniqueness and a formula:

~~(Note: if metric is (g_{ij}), then inverse of metric is (g^{ij}), then $\Gamma^l_{ij} = \sum_k \frac{1}{2}(\partial_i$~~

Let x^i be coordinate fns, $\partial_i = \frac{\partial}{\partial x_i}$ be vectors, $g_{ij} = \langle \partial_i, \partial_j\rangle$. Γ^k_{ij} be s.t.

$$\nabla_{\partial_i}\partial_j = \sum_k \Gamma^k_{ij}\partial_k. \text{ Then, if } (g^{ij}) = (g_{ij})^{-1},$$

$$\Gamma^l_{ij} = \sum_k \tfrac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right) g^{kl}.$$

(or reference WP, textbook)

Curvature of a connection:

Define: $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

Thm: - R is trilinear in X, Y, Z (in fact ptwise trilinear)

- R measures the path-dependence of parallel transport
  Ex: sphere, saddle?
- R satisfies:
  - $R(X,Y)Z + R(X,Y)Z = 0$
  - $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$
  - $\langle R(X,Y)Z, W\rangle + \langle R(X,Y)W, Z\rangle = 0$
  - $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$

Generally, full curvature tensor is too much information to work with conveniently; so we'll define various slices, averages, contractions.
- For surfaces, all the identities mean that the tensor boils down to one number, Gaussian curvature
- In higher dimensions, various invariants derived from tensor: Ricci curv, sectional curv, scalar curv.

Next time: geodesics: when do shortest paths exist, how to find them, etc

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Last time: connections, parallel transport. 2016-02-03

Today: use these to define curvature, geodesics.

Let M be a manifold, ∇ the Riemannian connection on M.

If $X, Y, Z \in \mathcal{V}(M)$, define $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

Then:
- R is trilinear in X, Y, Z
- R is a tensor
- R satisfies various symmetries:

$$R(X,Y)Z + R(Y,X)Z = 0$$
$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$
$$\langle R(X,Y)Z, W\rangle + \langle R(X,Y)W, Z\rangle = 0$$
$$\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$$

- R measures the path-dependence of parallel transport (holonomy).
  - if γ is a path from x to y, let $p_\gamma : T_x M \to T_y(M)$ be parallel transport. If γ is a closed curve, then p_γ is a map from $T_x M$ to itself, called the holonomy of γ.

~~If $p_\gamma = id$ for every null homotopic closed curve γ, then we say M is flat (not hard to show that the~~

Thm: If $X, Y \in T_u M$ and if $f : \mathbb{R}^2 \to M$ is a map s.t. $f(0,0) = u$, $\frac{\partial f}{\partial x} = X$, $\frac{\partial f}{\partial y} = Y$, then

$$R(X,Y)Z = \lim_{s \to 0} \frac{Z - P_{\gamma_s}(Z)}{s^2},$$ where γ_s is the boundary of the $s \times s$ square in $\mathbb{R}^2$.

~~So we can calculate so This is like Stokes' Theorem~~

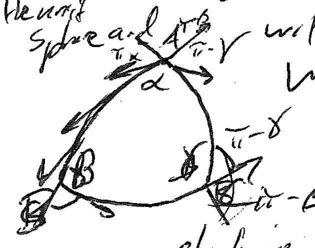
This is like Stokes' Theorem - given a curve  and a surface, we can calculate parallel transport by ~~integration solving equations for the~~ integrating around the boundary or by integrating the curvature on the surface.

Ex: If M is a 2-manifold, the symmetries reduce M to ~~one number~~ ~~R~~ a 2-form; ~~so its characterized entirely by~~ $\langle R(X,Y)Y,X\rangle$, so its characterized by the ratio of the 2-form to the area form. We define the Gaussian curvature of M by $K = \frac{\langle R(X,Y)Y,X\rangle}{\|X\|\|Y\| - \langle X,Y\rangle^2}$.

If M is the unit sphere, we can calculate using spherical geometry.

Fact: If $A,B,C \in S^2$ are vertices of ~~a triangle~~ $\triangle ABC$, with angles α, β, γ, then Area $\triangle ABC = \alpha+\beta+\gamma-\pi$.

S^2 is the unit sphere and ...



What is the holonomy?

Construct a parallel field

Start with direction of the geodesic.

$P_{\partial\Delta}$ = rotation by $\pi-\alpha+\pi-\beta+\pi-\gamma$ clockwise ~~in negative~~ = CCW rot ~~clockwise~~ by $\alpha+\beta+\gamma-\pi$ = Area$(\triangle)$.

~~So if X,Y orthonormal, positive~~ then

$\langle R$ $\square$

In general, $P_{\partial U}$ = CCW rotation by Area(U).

$\Rightarrow K=1$.

In general, if ~~S^2~~ ~~has~~ $M = S^2(r)$, then $K = \frac{1}{r^2}$.

(This is not too far from the Gauss-Bonnet formula: (some of details as exercise)

Thm: If M is a surface, then $\chi(M) = \frac{1}{2\pi}\int K\,dA$

~~In general, on a general surface, we have~~

By usual Stokes args, if $U \subset M$, then $P_{\partial U}$ = CCW rotation by $\int_U K\,dA$.

~~so if you have~~ OTOH, if $\triangle$ is a geodesic triangle ~~polygon~~ with angles α,β,γ, then $P_{\partial\Delta}$ = CCW rotation by $\alpha+\beta+\gamma-\pi$ ~~Σ exterior angles~~.

splice to end

So, ~~if M is a surface w/ a geodesic triangulation, then~~

$\Rightarrow$ for every geodesic triangle in M, $\int_\Delta K\,dA = \alpha+\beta+\gamma-\pi$

So, if M has a geodesic triangulation with V vertices, E edges, F faces, we have $E = \frac{3}{2}F$ and

$$\int_M K\,dA = \sum_\Delta \int_\Delta K\,dA = \sum_\Delta \alpha+\beta+\gamma - \sum_\Delta \pi$$

$$= 2\pi V - \pi F = \cancel{2\pi V - 2\pi E + 2\pi F}\ 2\pi(V-E+F)$$

$$= 2\pi\,\chi(M)$$

Ex: Another characterization of R: it's the deviation from being flat.

We say M is flat if $R=0$ — what does this imply?

Prop: ~~If M is flat and γ is a null-homotopic closed curve in M, then $P_\gamma = id$.~~

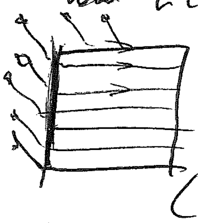
~~Pf: Stokes' thm Break~~ Consider a square

Thm: If M is flat, then M is locally isometric to $\mathbb{R}^n$.

~~Lemma: If M is flat, then M locally has a field of parallel orthonormal frames.~~

~~Pf:~~ Lemma: If M is flat and γ is a null-htpic piecewise smooth closed curve in M, then $P_\gamma = id$.

Pf: ~~Suppos~~ Let $I = [0,1]$, $\alpha : I^2 \to M$ be s.t. $\alpha(\partial I^2) = \gamma$. Let $v \in T_{\alpha(0,0)}M$ and let $X \in \mathcal{V}(\alpha)$ be the field s.t. $X(\alpha(0,0)) = v$ and X is parallel along $0 \times I$
" " " " $I \times t$ $\forall t$



This is smooth (why?) and $\nabla_{\partial_1} X = 0$ where $\partial_i := \frac{\partial \alpha}{\partial x_i}$.

Claim: $\nabla_{\partial_2} X = 0$

Well, $\nabla_{\partial_2}(t,0) = 0$.

Consider $\nabla_{\partial_1} \nabla_{\partial_2} X = \nabla_{\partial_2} \nabla_{\partial_1} X + \nabla_{[\partial_1, \partial_2]} X + R(\partial_1, \partial_2) X$

$= 0 + 0 + 0$

(since $\nabla_{\partial_1} X = 0$) (commuting fields) (flatness)

So $\nabla_{\partial_2} X = 0 \Rightarrow X$ is parallel $\Rightarrow P_\gamma(v) = v$.

Break?

Cor: ~~M has a Locally M has a field of parall~~
$\forall x \in M$ $\exists$ $U \ni x$ and v. fields $X_1, \dots, X_m \in \mathcal{V}(U)$ st. X_i are parallel, orthonormal.

Pf: Let $v_1, \dots, v_m \in T_x M$ be orth. vectors, let $U \ni x$ ~~be a nbhd diffeo~~ be st. $U \cong B_1 \subset \mathbb{R}^n$.
$\forall$ $y \in U$, let γ_y be a path from x to y, define $X_i(y) = P_{\gamma_y}(v_i)$. These are well-defined parallel, orthonormal.

Finally, consider $[X_i, X_j]$. Since ∇ is torsion-free,

$$[X_i, X_j] = \nabla_{X_i} X_j - \nabla_{X_j} X_i = 0.$$

These are commuting v. fields, so $\exists$ a map $\beta : U' \underset{\subset \mathbb{R}^m}{\longrightarrow} U$ s.t. $\beta(0) = x$, and $\frac{\partial \beta}{\partial u_i} = X_i$ $\forall i$.
by Frobenius's Thm.

Break?

So flat $\Rightarrow$ no local holonomy $\Rightarrow$ locally $\mathbb{R}^n$.

Geodesics: $\gamma : I \to M$ is a geodesic if ~~B~~ its velocity field is parallel, i.e. $D_t \frac{\partial \gamma}{\partial t} = 0$. (in particular, γ has constant speed)

~~Thm~~ Then: (assortment of ~~things~~ ~~that~~ basic facts, prob. from last term)

stet – $\forall x \in M$, $\forall v \in T_x M$, $\exists \varepsilon > 0$ s.t. $\exists! \gamma : (-\varepsilon, \varepsilon) \to M$ a geod s.t. $\gamma(0) = x$, $\gamma'(0) = v$. (local existence)

stet – ~~$\forall x \in M$, $\exists$ a nbhd $0 \in U \subset \mathbb{R}^n$~~ ~~Furthermore, we can extend this to~~

– Further, this will depend ~~locally~~ smoothly on x, v:

stet – $\forall p_0 \in M$, $\exists$ a nbhd U of p_0, $\varepsilon > 0$ s.t. $\forall p \in U$, $\forall v \in T_p M$ s.t. $\|v\| < \varepsilon$, $\exists$ a unique geod $\gamma_v : (-2, 2) \to M$ s.t. $\gamma_v(0) = p_0$, $\gamma_v'(0) = v$.

stet – If we define $\exp_p(v) = \gamma_v(1)$, then $\exp_p : W \subset T_p M \to M$ is a smooth map defined on a nbhd of W.

– ~~By IFT,~~ let ~~Furthermore, if you fix p and let v~~ ~~$\ell(p, v) = (p, \exp_p v) \in M \times M$.~~ vary, this ~~is locally~~ invertible ~~because~~ of IFT

~~Then $D\ell(p, 0)$ is inv.~~

– $\exists$ a nbhd W of p s.t. $\forall x, y \in W$, $\exists! \gamma_{x,y}$ a geod s.t. $\ell(\gamma_{x,y}) \leq \varepsilon$ and $\gamma_{x,y}$ ~~is a geod~~ $(0) = x$, $\gamma_{x,y}(1) = y$.

This depends smoothly on x, y.

– $\forall q \in W$, $B_\varepsilon \in T_q W$ is mapped diffeomorphically

These exist locally. Not hard to show following

~~Thm~~: Further, you can use IFT to prove following:

Thm: $\forall p_0 \in M$, $\exists$ a nbhd W of p_0 and an $\varepsilon > 0$ s.t.

- $\forall x, y \in W$, x and y are joined by a unique geod of length $< \varepsilon$.
- this geod depends smoothly on x, y
- $\forall x \in W$, the map $\exp_x$ sends $B_\varepsilon(0)$ diffeomorphically to a nbhd ~~containing~~ $\exp_x B_\varepsilon(0) \supset W$.

splice in Gauss-Bonnet.

What makes these interesting is that they're connected to ~~minimal curves~~ length minimizers.

Jeff said you saw Gauss's Lemma? What version?

optional Lemma: If ε is as above, $r < \varepsilon$, then γ_v is orthogonal to $\exp_q(S_r(0))$ $\forall v \in T_q M$

Lemma: Any path If $a, b < \varepsilon$, then any path from $\exp_q S_a(0)$ to $\exp_q S_b(0)$ has length $\geq |b - a|$, with equality only for geodesics.

Both proved using polar coords.

1.: let $f(r, t) = \exp r\theta(t)$, compute $\frac{\partial}{\partial r}\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \rangle = \langle D_r \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \rangle + \cdots$

2. Let $\gamma = \exp r(t)\theta(t)$, define $f(r, t) = \exp r\theta(t)$, compute $\ell(\gamma) = \int_0^1 \| \frac{d\gamma}{dt} \| = \int_0^1 \| r'(t) \frac{\partial f}{\partial r} + \frac{\partial f}{\partial t} \|$

Cor: Any length-minimizing path is a geodesic
(called a minimal geodesic)

If γ is length-minimizing, consider $p \in \gamma$, and a nbhd W of p as in lemma. ~~Then shortest paths~~ Pick q on one side, r on other

-(rewrite epilogue?)