Cor: Any length-minimizing path is a geodesic (called a minimal geodesic).

If γ is length-minimizing, consider $p \in \gamma$, and a nbhd W of p as in lemma. ~~Then shortest paths~~ Pick q on one side, r on other

-(rewrite epilogue?)

2016-02-10

Recall: $\gamma : [0,1] \to M$ is a geodesic if $D_t \frac{d\gamma}{dt} = 0$

Local existence: $\forall p_0 \in M$, $\exists$ a nbhd W, an $\varepsilon > 0$ s.t.

- any two points in W are joined by a unique geod of length $< \varepsilon$.
- $\forall x \in W$ $\exp_x|_{B_\varepsilon^*}$ is a diffeo from $B_\varepsilon = \{v \in T_xM \mid \|v\| < \varepsilon\}$.

Calculus of variations argument:

γ is a length-minimizer if γ is piecewise smooth and $\ell(\gamma) \le \ell(\gamma')$ for any PS curve γ' with the same endpoints.

Any length-minimizer is a geodesic (a minimal geodesic)

Q: when do length minimizers exist?

First: local existence:

Gauss's Lemma: If ε is as above and $r_0 < \varepsilon$, then $\forall v \in T_xM$, γ_v is orthogonal to $\exp_x S_{r_0}$, where $S_{r_0} = \{v \mid \|v\| = r_0\}$.

Pf: Suppose $v \in S_{r_0}$, $\theta : (-1,1)$ is a curve s.t. $\theta(0) = v$. Claim $\exp_x \theta(t)$ intersects γ_v orthogonally.

Let $f(r,t) = \exp_x r\theta(t)$. ETS that $\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \rangle = 0$.

~~velocity of γ_v~~ velocity of ray through x ~~velocity of $\exp_x \theta(t)$~~ ~~velocity of~~ tangent to sphere

$$\frac{d}{dr}\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle = \underbrace{\cancel{\left\langle D_r \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle}}_{0} + \left\langle \frac{\partial f}{\partial r}, D_r \frac{\partial f}{\partial t} \right\rangle$$

geodesic has parallel velo

$$= \left\langle \frac{\partial f}{\partial r}, D_t \frac{\partial f}{\partial r} \right\rangle \quad \text{(by torsion-freeness applied to } \tfrac{\partial f}{\partial r}, \tfrac{\partial f}{\partial t}\text{)}$$

$$= \frac{1}{2} \frac{d}{dt} \left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial r} \right\rangle = 0 \quad \left(\left\| \frac{\partial f}{\partial r} \right\| = 1 \right)$$

So $\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \rangle$ is constant wrt. r.

But ~~$\frac{\partial f}{\partial t} = 0$ when~~ $f(0,t) = x$ $\forall t \Rightarrow \frac{\partial f}{\partial t}(0,t) = 0$

$\Rightarrow \langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \rangle = 0$ everywhere. //

Cor: ~~Geodesics are locally length minimizing~~ Short -If $a < b < \varepsilon$ and $\gamma(t) = \exp_x r(t)\theta(t)$ is a path from S_a to S_b, then $\ell(\gamma) \ge b - a$, with equality iff θ is constant and r is monotone.

Pf: Let $f(r,t) = \exp_x r\theta(t)$, so $\gamma(t) = f(r(t),t)$.

Then $\frac{d\gamma}{dt} = \underbrace{\frac{\partial f}{\partial r} \cdot r'(t)}_{\text{radial}} + \underbrace{\frac{\partial f}{\partial t}}_{\text{on sphere}}$, $\quad \left\| \frac{d\gamma}{dt} \right\| \ge \left\| \frac{\partial f}{\partial r} \cdot r'(t) \right\| = |r'(t)|$

$\ell(\gamma) = \int_0^1 \|\frac{d\gamma}{dt}\| \geq \int_0^1 |r'(t)| \geq b-a.$

Equality $\Rightarrow$ $r'(t) \geq 0 \ \forall t, \ \frac{d\theta}{dt} = 0 \ \forall t.$ //

Cor: If $\|v\| < \varepsilon$, then $\gamma_v|_{[0,1]}$ is a minimal geodesic.

Longer minimal geodesics?

(Note: The standard way to ~~do this~~ find these would be to take a sequence of shorter and shorter curves, then take a limit. ~~Unfortunately, this doesn't work here here~~ — let $x, y \in M$, ~~Say we're in the plane. Then $x, y \in \mathbb{R}^2$~~. let $L = \inf_{\gamma \text{ from } x \text{ to } y} \ell(\gamma)$, ~~Let $L = \inf \ell(\gamma)$~~ ~~γ from x to y~~ suppose $\gamma_1, \gamma_2, \cdots$ are s.t. $\ell(\gamma_i) \to L$.

Then claim that $\exists$ a subseq. of γ_i ~~s.t.~~ that converges to γ, and $\ell(\gamma) = L$. Doesn't work here ~~yet~~. Why?

This needs the fact that metric balls are compact.

Note: Minimal geodesics need not exist. Ex: $M = \mathbb{R}^2 \setminus \{0\}$.

Then there is no minimal geod from x to $-x$.

So there's an obstruction - ~~if there~~ are missing ~~points~~, then we want a geod in this direction, but there isn't one.

But it's the only obstruction.

Def: M is geodesically complete if $\forall x \in M$ $v \in T_xM$, there is an infinite geodesic ~~through v in the direct~~ through x in direction of v. Local existence means that complete $\Rightarrow$ geodesically complete.

Thm (Hopf-Rinow): If M is geodesically complete, then any $x, y \in M$ are connected by a minimal geod.

So instead we'll use "shooting argument".

Pf: ~~Suppose~~ Let $r = d(x,y)$ $(= \inf_{\gamma \text{ from } x \text{ to } y} \ell(\gamma))$ Let ε be as above. If $\varepsilon > r$, done.

Otherwise, let $U_x = \exp_x B_\varepsilon$, $S = \exp_x S_\delta$, where $\delta < \varepsilon$.

S_δ is compact, so S is compact, so $\exists$ some $x_0 \in S$ s.t. $d(y, x_0)$ is minimal and $x_0 = \exp_x \delta v$ for some unit vector v.

Let $\gamma(t) = \exp_x(tv)$. Claim: $\gamma(r) = y$.

ETS $d(\gamma(t), y) = r - t. \ \forall t \in [\delta, r]$.

$t = \delta$ - ie, claim: $d(x_0, y) = r - \delta$.

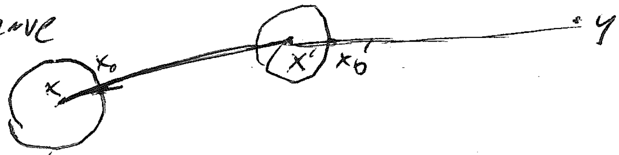
Any path γ' from x to y passes through S so it breaks into two segments

$$d(x,y) = \min_{s \in S} d(x,s) + d(s,y)$$

$$r = \delta + d(x_0, y)$$

Now, let $t_0 = \sup\{t \in [\delta, r] \mid d(\gamma(t), y) = r - t\}$.

If $t_0 = r$, done. Otherwise, we have $x' = \gamma(t_0)$. Repeat the process: let $S' = \exp_{x'} S_{\delta'}$, let $x_0' \in S'$ minimize $d(x_0', y)$. Then $d(x_0', y) = r - t_0$



$d(x_0', y) = r - t_0 - \delta'$, and ~~I~~ we let γ' be geod through x_0'.

By Δ ineq, $d(x, x_0') \geq t_0 + \delta'$. But ~~this is a broken~~ $\gamma([0, t_0]) \cup \gamma'([0, \delta'])$ is a path from x to x_0' of length $t_0 + \delta'$ — i.e., it's a minimal geodesic. ~~The only way that can happen is if~~ ~~it's a~~ So ~~δ' is~~ in fact, $\gamma(t_0 + \delta') = x_0'$. This contradicts the maximality of t_0, so in fact, $t_0 = r$ //

Cor: Geodesically complete $\Rightarrow$ ~~metric balls are compact~~ complete $\Rightarrow$ closed metric balls are compact.

Pf: ~~If $d(x, y) \leq r$, then~~ The metric ball $\overline{B_r(x)} = \{y \in M \mid d(x, y) \leq r\} = \exp_x \overline{B_r(0)}$

But $\overline{B_r(0)}$ is compact, so $\overline{B_r(x)}$ is compact.

~~Q: How to find~~ geodesics? Examples?

1. Explicitly solve geod equation – possible, but tedious.

~~Ex~~ 2. Symmetry: Suppose $I: M \to M$ is an isometry with fixed-point set C. Def: A set S is totally geodesic if every geodesic tangent to S lies in S.

Thm: Every connected component of C is totally geodesic.

Pf: Let $C_0 \subseteq C$ be a cpt, let $x \in C_0$, $v \in T_x C_0$. ~~Then~~ Let γ_v be the unique geod ~~through x tangent to~~ s.t. $\gamma_v(0) = x$, $\gamma_v'(0) = v$.

Then $I \circ \gamma_v(0) = x$, $(I \circ \gamma_v)'(0) = v$, so $\gamma_v = I \circ \gamma_v \Rightarrow \gamma_v$ lies in C_0.

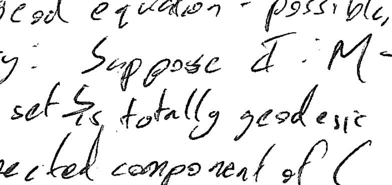
Ex: ~~$\mathbb{R}^2$~~ $\mathbb{R}^2$: reflections through lines fix those lines $\Rightarrow$ lines are geod

S^2: Reflections through planes fix great circles $\Rightarrow$ great circles are geods

And these are all the geodesics, because there's one in each dir through each pt.

Further, ~~we can~~ this tells us about geods in spaces that only have local isometries

Ex: Cylinder



geods determined by local structure, so lines descend to geods

But this is generally not useful – generic mflds have no isometries, in fact, no totally geod submanifolds. (exercise?)

3. Calculus of variations / Morse theory. Basic idea of Morse theory:
Recall: Suppose $f: M \to \mathbb{R}$. How does $M_a = f^{-1}((-\infty, a])$ change as $a \to \infty$?

Def: If $\nabla f(x) = 0$, x is a critical point of f.

~~Lemma~~: Morse theory has two key pieces. 1. What happens away from critical pts?

1. Lemma: If $f^{-1}([a,b])$ is compact and has no critical pts, then M_a is diffeo to M_b. Further, M_a is a def. retract of M_b.

skip

Pf: Consider $V_f = \frac{-\nabla f}{\|\nabla f\|^2}$. This is a smooth v. field on $f^{-1}([a,b])$, and if γ ~~is a~~ ~~satisfies $\gamma'(t) = V_f(\gamma$~~ is a trajectory of V_f (i.e. ~~$\gamma'(t) = V_f(\gamma(t))$~~, $\frac{d\gamma}{dt} = V_f$),

then $\gamma^f(t) = -1$ ~~V_f flows down the gradient lines~~

— V_f flows $f^{-1}(b)$ to $f^{-1}(a)$ in time $b-a$.

~~We can~~ use ~~V_f~~ to define a map $M_b \to M_a$.

Extend V_f to M and ~~de~~ //

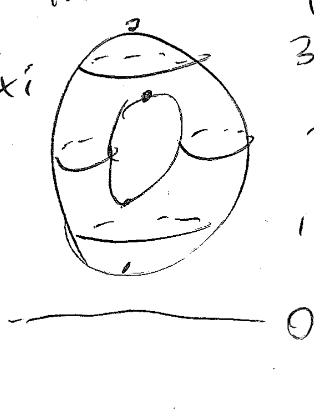
2. What happens at critical points?

Def: If $p \in M$ is a critical point and $(x^1, \dots, x^m)$ is a chart, let
$$H_f(p) = \left(\frac{\partial^2 f}{\partial x^i x^j}\right)$$
If $H_f(p)$ is invertible, we say that p is nondegenerate. (Check: this is indep of chart).

Lem: In this case, $\exists$ a chart $(u^1, \dots, u^m)$ s.t.
$$f = f(p) - \sum_{i=1}^{\lambda} u_i^2 + \sum_{i=\lambda+1}^{m} u_i^2,$$
where $\lambda = \#$ of negative eigvals of $H_f(p)$.
We call λ the index of p.

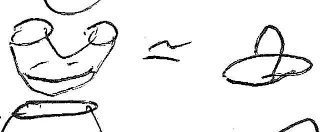
Thm: If p is a nondegenerate crit pt. with index λ, ~~then~~ $c = f(p)$, ~~and~~ ~~then~~ and $\varepsilon > 0$ is s.t. $f^{-1}([c-\varepsilon, c+\varepsilon])$ contains no crit pts other than p, then $M_{c+\varepsilon}$ is homotopic to $M_{c-\varepsilon}$ with a λ-cell attached.

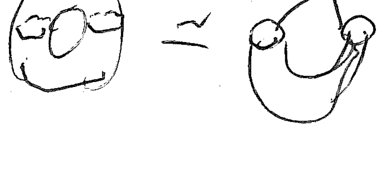
Ex:



$M_0 = \emptyset$

$M_1 =$  $\simeq$.

$M_2 =$  (imagine gradient flow)

$M_3 =$ 

Applications: ~~-~~ exotic 7-spheres
— OTOH, if we can show the existence of crit pts, we get geodesics.

2016-02-18

~~Recall Morse theory~~: Today: Start understanding the space of paths.
– ultimately, Morse theory on space of paths.
Basic idea of Morse theory: let $f: M \to \mathbb{R}$ be a smooth function — then the topology of $M_a = f^{-1}((-\infty, a])$ only changes at critical ~~values~~ points, and the way it changes is determined by the ~~tah~~ index of the critical point:
Def: p is a crit pt if $\nabla f = 0$ at p.
~~Def~~ If p is a critical pt and $u_1, \dots, u_m$ are coords centered at p, define $H(f)(p) = \left(\frac{\partial^2}{\partial u_i \partial u_j}\right)_{i,j=1 \dots m}$.

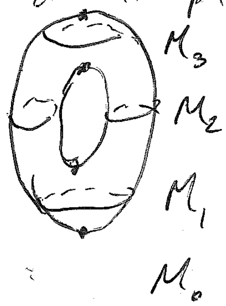
If H is invertible, then p is a nondegenerate critical pt. (and this is independent of choice of u_i.)

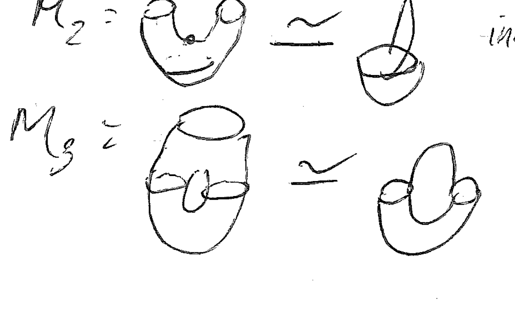
Further, since $H(f)$ is symmetric, it's diagonalizable, and we define
index(p) = # of negative eigenvalues
nullity(p) = # of zero eigenvalues.

Lemma: If p is nondegenerate, then $\exists$ coords $u_1, \dots, u_m$ with index λ, s.t.

$$f = \sum_{i=1}^{\lambda} -u_i^2 + \sum_{i=\lambda+1}^{m} u_i^2$$

So when M is a mfld, we know exactly what happens as we pass through a nondegenerate crit pt: Lemma: if $f^{-1}([a,b])$ is compact, contains exactly one crit pt p, and p is nondegen with index λ, then M_b is htpic to M_a with a λ-cell attached.
Standard example is a surface:





index 0
index 1

~~So. If we apply this~~

~~Morse theory lets us des connects crit pts to topology~~

~~So up to homotopy~~, we can describe a lot about a mfld from Morse theory:

~~Thm Def~~ If we have a fn w/ only nondegen crit pts, we can apply the to a whole mfld:

~~Thm~~ Def: If $f: M \to \mathbb{R}$ has only nondegenerate crit pts and $f^{-1}((-\infty, a])$ is compact $\forall a$, we say f is a <u>Morse function</u>.

Thm: If f is a Morse fn for M, then M is htpy equivalent to a CW complex with ~~one~~ a cell for each crit pt of f with index k.

Very powerful: Beautiful app by Milnor — exotic 7-spheres: ~~If~~ If M is cpt has exactly two crit pts, it's homeo to a sphere. Milnor constructed a particular 7-fold that is not diffeo to a sphere, but it has a Morse fn with 2 smooth crit pts. — homeo to sphere, but not diffeo to a sphere!

We ultimately want to go the other way — ~~show that if~~ apply this to space of paths. The space of paths has known topology, ~~so~~ we want to show that that topology forces critical pts — geodesics

(break?)

~~So, what are our ingredients~~: In order to do that, we need ~~some careful~~ a picture of what ~~the~~ the space of paths looks like near a crit pt. That's what we'll do today.

Notation: If $p, q \in M$, let $\Omega = \Omega_{p,q} = \{$piecewise smooth paths from p to $q\}$

(C_1-topology) Let $E(\gamma) = \int_0^1 \|\frac{d\gamma}{dt}\|^2 dt$.

If $\gamma \in \Omega$, let $T_\gamma \Omega = \{$piecewise-smooth v.fields on $\gamma\}$ with $V(0) = V(1) = 0$

~~Def~~ A <u>variation</u> of γ is a fn $\bar{\alpha}: (-\varepsilon, \varepsilon) \to \Omega$ s.t.

- $\bar{\alpha}(0) = \gamma$
- $\exists\ 0 = t_0 < t_1 < \dots < t_n = 1$ s.t. $\alpha: (u, t) \mapsto \bar{\alpha}(u)(t)$ is smooth on $[-\varepsilon, \varepsilon] \times [t_{i-1}, t_i]$
- $\alpha(u, 0) = p$, $\alpha(u, 1) = q$ $\forall u$.

(Usually, we'll use α to refer to α or $\bar{\alpha}$)

Let $\frac{\partial \alpha}{\partial u}$ be the <u>variation field</u> of α

~~Then~~ $\forall\, U \in T_\gamma \Omega$, $\exists\ \alpha$ st. $\frac{d\alpha}{du} = U$

~~Def~~ γ is a <u>critical point</u> of E if $\frac{dE}{du} = 0$ $\forall$ variations of Ω.

if α is a variation with field W,

Then, 1VF: if $V(t) = \frac{d\gamma}{dt}$, $A(t) = D_t \frac{d\gamma}{dt}$, $\Delta_t V = V(t_+) - V(t_-)$

then $\frac{1}{2}\frac{dE(\alpha)}{du} = -\sum_t \langle W, \Delta_t V\rangle - \int_0^1 \langle W, A\rangle dt$

(~~Let deg~~ ~~second deriv~~ is "derivative" of V)

~~So γ is a crit pt $\Leftrightarrow$ γ is a geodesic~~ $\Delta_t V = 0$, $A = 0 \Leftrightarrow \gamma$ is a geodesic

~~What does E look like near γ?~~

Previously, we used Hessian to describe ~~small~~ crit pt of fns on mflds.
Same idea here, except we need to revise the def a bit.

Thm: If x is a crit pt of f, $v, w \in T_x M$, let $\alpha : (-\varepsilon, \varepsilon)^2 \to M$ ~~be~~ a smooth fn s.t. $\alpha(0,0) = x$, $\frac{\partial \alpha}{\partial u_1}(0,0) = v$, $\frac{\partial \alpha}{\partial u_2}(0,0) = w$, and ~~then~~ define

$$H(f)(v,w) = \frac{\partial^2 f(\alpha)}{\partial u_1 \partial u_2}(0,0).$$ Then $H(f)$ is a well-defined bilinear symmetric form on $T_x M$.

Pf: Exercise.

Same if x ~~$\in\Omega$~~ is a crit pt in Ω.

Thm (Second Variation Formula): If $\gamma \in \Omega$ is a geodesic, $W_1, W_2 \in T_\gamma \Omega$, let α ~~be a 2-param~~ $\alpha : (-\varepsilon, \varepsilon)^2 \to \Omega$ be s.t. $\alpha(0,0) = \gamma$, $\frac{\partial \alpha}{\partial u_i}(0,0) = W_i$.

Then ~~$H(E)(W_1, W_2) =$~~

$$\frac{1}{2} H(E)(W_1, W_2) = \frac{1}{2}\frac{\partial^2 E}{\partial u_1 \partial u_2}(0,0) = -\sum_t \langle W_2(t), \Delta_t D_t W_1\rangle - \int_0^1 \langle W_2, D_t^2 W_1 - R(V, W_1)V\rangle$$

where $\Delta_t D_t W_1(t) = D_t W_1(t_+) - D_t W_1(t_-)$

(Like 1VF - ~~discontinuity term~~ has to do with "second derivatives" of W_1 and curvature) ~~(Consequently~~

Pf: Diff 1VF:

$$\frac{1}{2}\frac{dE}{du_2} = -\sum_t \left\langle \frac{\partial \alpha}{\partial u_2}, \Delta_t \frac{\partial \alpha}{\partial t}\right\rangle - \int_0^1 \left\langle \frac{\partial \alpha}{\partial u_2}, D_t \frac{\partial \alpha}{\partial t}\right\rangle$$

$$\frac{1}{2}\frac{\partial^2 E}{\partial u_1 \partial u_2} = \cancel{-\sum_t \left\langle D_{u_1}\frac{\partial \alpha}{\partial u_2}, \Delta_t \frac{\partial \alpha}{\partial t}\right\rangle} - \sum_t \left\langle \frac{\partial \alpha}{\partial u_2}, \Delta_t D_{u_1}\frac{\partial \alpha}{\partial t}\right\rangle$$

$$\cancel{-\int_0^1 \left\langle D_{u_1}\frac{\partial \alpha}{\partial u_2}, D_t \frac{\partial \alpha}{\partial t}\right\rangle} - \int_0^1 \left\langle \frac{\partial \alpha}{\partial u_2}, D_{u_1} D_t \frac{\partial \alpha}{\partial t}\right\rangle$$

At $(0,0)$, $\Delta_t \frac{\partial \alpha}{\partial t} = D_t \frac{\partial \alpha}{\partial t} = 0$, so

$$\frac{1}{2}\frac{\partial^2 E}{\partial u_1 \partial u_2}(0,0) = -\sum_t \langle W_2, \Delta_t D_{u_1} \tfrac{\partial \alpha}{\partial t}\rangle - \int_0^1 \langle W_2, D_{u_1} D_t \tfrac{\partial \alpha}{\partial t}\rangle$$

swap partials

$$= -\sum_t \left\langle W_2, \Delta_t D_t \frac{\partial \alpha}{\partial u_1}\right\rangle - \int_0^1 \left\langle W_2, D_t D_{u_1}\frac{\partial \alpha}{\partial t} - R\left(\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial u_1}\right)\frac{\partial \alpha}{\partial t}\right\rangle$$

$$= -\sum \langle W_2, \Delta_t D_t W_1\rangle - \int_0^1 \langle W_2, D_t^2 W_1 - R(V, W_1)V\rangle \quad \blacksquare$$

Note: Sometimes you'll see this stated in terms of second derivs, likes a - same thing

Note: This is symmetric bilinear form - not written symmetrically, but $\frac{\partial^2 E}{\partial u_1 \partial u_2} = \frac{\partial^2 E}{\partial u_2 \partial u_1}$.

- ~~Just like in $\mathbb{R}^n$ we can use this for a Taylor expansion~~ – if $\beta : (-\varepsilon,\varepsilon) \to \Omega$ is a variation of γ, then $E(\beta) = E(\gamma) + \frac{1}{2} H(E)\left(\frac{\partial \beta}{\partial u}, \frac{\partial \beta}{\partial u}\right) u^2 + O(u^3)$
- ~~The corresp quadratic form $W \to H(E)(W,W)$ describes E to second order near γ: if $\beta : (-\varepsilon,\varepsilon) \to \Omega$ is a variation,~~ with $\frac{\partial \beta}{\partial u} = W$ then $\frac{dE(\beta)}{dt} = 0$, $\frac{dE(\beta)}{dt^2} = H(E)(W,W)$

~~What else can we do? Jacobi fields~~ what we'll do next:

The reason we write this in this asymmetric form is we can pull out the nullspace. ~~Recall that for~~ ~~we were~~ Recall the nullity of a symmetric bilinear form was the # of zero eigenvalues. Corresponds to the nullspace – set of W_1's that force the form to be zero. ~~Call the null-space the Jacobi fields.~~

Jacobi fields

In this case, the nullspace is the fields that satisfy $W(0)=W(1)=0$ and Jacobi eq.

Def: If $W \in T_\gamma(\gamma)$, we say it is a Jacobi field if it is twice differentiable and

$$D_t^2 W = R(V,W)V. \quad \text{(Jacobi equation)}$$

If so, then

$$\frac{1}{2} H(E)(W,W') = -\sum_t \langle W', \Delta_t D_t W\rangle - \int_0^1 \langle W', D_t^2 W - R(V,W)V\rangle$$

$$= 0$$

(Conversely, if $H(E)(W,W') = 0 \ \forall W'$, then W is a Jacobi field)

(Did you do these?) Facts, stop me if you've heard these:
- If $\alpha : (-\varepsilon,\varepsilon) \to \Omega$ is a variation through geodesics, then $\frac{\partial \alpha}{\partial u}$ is a Jacobi field.
- If W is a Jacobi field that is normal to γ at two points, it is normal everywhere. (in fact, $\frac{d^2}{dt^2}\langle V,W\rangle = 0$.)
- There are $2m$ lin ind Jacobi fields on γ.
  (Often break into tangential Jacobi fields $(W = (at+b)V)$ and normal fields $\langle V, W\rangle = 0$)
- Every Jacobi field arises from a variation through geodesics.
  ~~So we can use Jacobi fields to measure the geodesics~~

Ex: S^2, compare to $\mathbb{R}^2$: Let V = tangent to gc, N = normal, $W = fV + gN$, calculate.

Next time: Conjugate pts:
When is the nullspace nontrivial – when $W(0) = W(1) = 0$.
These are special – conjugate pts.
Next time, study them

2016-02-24

Jacobi fields:

~~Last time:~~ 2VF: If γ is ~~a~~ ~~crit~~ geod, $W_1, W_2 \in T_\gamma\Omega$

$$\frac{1}{2}H(E)(W_1,W_2) = \frac{1}{2}\frac{\partial^2 E}{\partial u_1 \partial u_2} = -\sum \langle W_2, \Delta_t D_t W_1\rangle - \int_0^1 \langle W_2, D_t^2 W_1 - R(V,W)V\rangle$$

~~Nullspace of $H(E)$:~~ Def: ~~A field~~ $W \in T_\gamma\Omega$ is a Jacobi field if W is smooth and $D_t^2 W = R(V,W)V$

Thm: W is in the nullspace of $H(E)$ $\iff$ W is a Jacobi field.
($H(E)(W,X) = 0$ $\forall X \in T_\gamma\Omega$)

But there's another way to get this eq:
Suppose $\alpha : (-\varepsilon,\varepsilon)\times[0,1] \to M$ is a variation through geodesics
– i.e. $\alpha(u)$ is a geod $\forall u \in (-\varepsilon,\varepsilon)$.
~~Then~~ Then: $\frac{\partial\alpha}{\partial u}$ is a Jacobi field.

Pf:
$$D_t^2 \frac{\partial\alpha}{\partial u} = D_t D_u \frac{\partial\alpha}{\partial t} = \cancel{D_u D_t \frac{\partial\alpha}{\partial t}} + R(V, \tfrac{\partial\alpha}{\partial u})V.$$

And conversely:
Lem: Any Jacobi field is the variation field of a variation through geodesics.

Pf: ~~There~~ Jacobi eq is second order vector eq, so there ~~are $2m$ lin ind Jacobi fields~~ (det by $W(0)$, $D_t W(0)$)
Can we construct ~~$2m$~~ dimension of variations = dim{Jacobi fields} = $2m$?

Let γ be a geod, let U be a nbhd of $\gamma(0)$ s.t. any two points p,q in U are connected by a unique smoothly varying geod $\gamma_{p,q}$.
Let ε be small enough that $\gamma(\varepsilon)\in U$, then for λ_1, λ_2 ~~curves thru~~ s.t. $\lambda_1(0)=\gamma(0)$ $\lambda_2(0)=\gamma(\varepsilon)$, $\alpha(u) = \gamma_{\lambda_1(u)\lambda_2(u)}$ is a variation through geods and $\frac{\partial\alpha}{\partial u}(0) = \lambda_1'(0)$ $\frac{\partial\alpha}{\partial u}(\varepsilon) = \lambda_2'(0)$. ~~The dimens~~
~~This gives rise to $2m$ dimensions~~ ~~Sol~~
This gives a map $T_{\gamma(0)}M \times T_{\gamma(\varepsilon)}M \to$ Jacobi fields on $\gamma_{[0,\varepsilon]}$.
– injective – dimensions equal – isomorphism,
and the fields on $\gamma_{[0,\varepsilon]}$ extend uniquely to γ.
$\downarrow$ Jacobi fields on γ

What else? ~~Tangential component is easy~~ Sometimes, break into tangential, normal
Lem: If W is a Jacobi field that is normal to γ at two points, it's normal everywhere.

Pf: In fact, $\frac{d^2}{dt^2}\langle W, V\rangle = \langle \cancel{D_t^2 V}, W\rangle + 2\langle \cancel{D_t V}, D_t W\rangle + \langle V, D_t^2 W\rangle$
$$= \langle V, R(V,W)V\rangle = 0.$$
$$\Rightarrow \langle V, W\rangle = at + b$$

Cor: If W is Jacobi,

$W = W^{\perp} + (at+b)V$, where $a,b \in \mathbb{R}$, $\langle W^{\perp}, V\rangle = 0$, $W^{\perp}$ is Jacobi normal Jacobi field.

Jacobi fields describe normal coordinates.

If $p \in M$, then $\exp_p$ is a ~~local~~ diffeo on a small ball (normal coordinates). Jacobi describes this diffeo:

let $p \in M$, γ a geod s.t. $\gamma(0) = p$, $\gamma(1) = q$.

let ~~v~~ $\vec{v} = \frac{d\gamma}{dt}(0)$, so $\exp_p(\vec{v}) = q$.

If $w \in T_pM$ then $\alpha : u \longmapsto \gamma_{\vec{v}+u\vec{w}}$ is a variation through geods

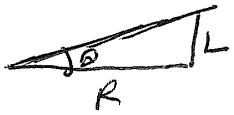
and $W = \frac{\partial \alpha}{\partial u}$ has $W(0) = 0$, $W(1) = (D\exp_p)_{\vec{v}}(\vec{w})$.

$D_t W(0) = \vec{w}$

So we can calculate $D\exp_p$ by solving Jacobi eq.

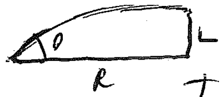
Ex: Understand curv by asking ~~what~~ how light travels in curved spaces.

In $\mathbb{R}^2$:



$\theta \approx \frac{L}{R}$.

In S^2



~~If~~ $\theta > \frac{L}{R}$

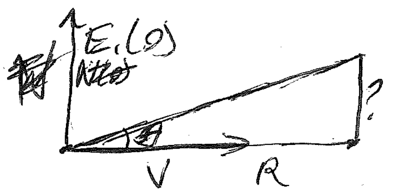
~~If $R = \pi$, $\theta = \infty$.~~

$\theta \approx \frac{L}{\sin R}$

In H^2:

$\theta < \frac{L}{R}$.

More generally, Recall that if $V, W \in T_pM$, then $K(V,W) = \frac{\langle R(V,W)W, V\rangle}{\|V \wedge W\|^2}$ $= $ sectional curvature. (area of parallelogram)



~~Suppose V, N unit vectors, $\langle V, N\rangle = 0$~~

~~Let $N(0)$ be direction of line,~~

~~N be parallel field normal so $\langle V, N\rangle = 0$~~

Find ~~a Jacobi field W s.t. $W(0) = 0$, $D_t W(0) = N$.~~

~~Et~~ Choose an parallel orthonormal field frames on γ, $E_0, \dots, E_{m-1}$,

$E_0 = V$

$E_1 =$ direction of line

Find a Jacobi field $W = \sum f_i E_i$ s.t. $W(0) = 0$, $D_t W(0) = \sum f_i'(0) E_i(0) = E_1$.

$f_i(0) = 0$ $\quad$ $f_i'(0) = 0 \quad \forall i \neq 1$

$f_1'(0) = 1$ $\quad$ $\langle E_i, D_t^2 W\rangle = \langle R(V,W)V, E_i\rangle$

$f_i''(t) = \langle R(V,W)V, E_i\rangle$

$f_i''(0) = 0$.

$f_i'''(t) = \frac{d}{dt}\langle R(V,W)V, E_i\rangle = \langle D_t R(V,W)V, E_i\rangle$ Because $W(0)=0$

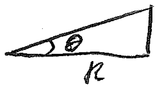
$f_i'''(0) = \langle R(V, D_t W)V, E_i\rangle = \langle R(V,E_i)V, E_i\rangle$.

$f_1'''(0) = -K(V,E_1) \Rightarrow f_1(t) = t - \frac{K}{6}t^3 + O(t^4)$

$f_i(t) = O(t^3)$

$\|W(t)\|^2 = \sum f_i(t)^2 + \ldots = (t - \frac{K}{6}t^3)^2 + O(t^4)$.

So,



$L \approx \theta(R - \frac{K}{6}R^3)$ If $L' = \theta R$ (apparent length locally), then $L \approx L'(1 - \frac{K}{6}R^2)$

Ex: ~~Constructing Metrics with constant sectional curvature~~

Sphere: E_1, E_0

Suppose W is a normal Jacobi field:

$W = f_1 E_1$ Jacobi

$f_1'' = \langle R(E_0, fW)E_0, E_1\rangle = -f$

~~$= \langle R(E_0, E_0)E_0, E_1\rangle f_0 + \langle R(E_0, E_1)E_0, E_1\rangle f_1$~~

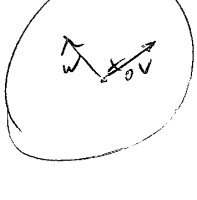
~~$= -f$~~ $f_0'' = 0$

$f_1'' = -f_1$

~~If W is norm~~ $f(t) = a\cos t + b\sin t$. If $f(0)=0$, $f = b\sin t$.

Polar coords on metric: $x = \exp_{x_0} r(\cos\theta\, \vec{v} + \sin\theta\, \vec{w})$

$dg^2 = dr^2 + d\theta^2 \sin r$



Likewise, can construct a surface with constant negative sectional curvature:

$f'' = f$

$f(0) = 0 \Rightarrow f = c\sinh t$.

$dg^2 = dr^2 + d\theta^2 \sinh r$ — more on this later.

Note: Something happens in the sphere when $r = \pi$ — there's a singularity in the exponential map because there's a Jacobi field with zeroes at 0 and r.

Def: If γ is a geod from p to q, then p is conjugate to q along γ if there is a Jacobi field W on γ s.t. $W(p) = W(q) = 0$

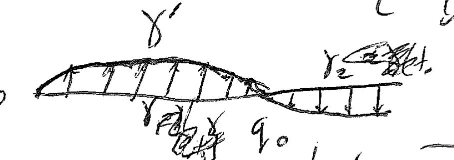
Then:
- If p, q are not conjugate, then W is determined by $W(p), W(q)$.
- Conjugate points correspond to singular points of ~~exponential~~ $\exp_p$
- Conjugate pts are rare - by Sard, ~~they correspond to~~ only a measure-zero set of the pairs in M are conjugate.
- Conjugate pts are closely related to whether a geod is minimizing.

The order of conjugacy $= \dim\{W \mid W \text{ Jacobi}, W(p) = W(q) = 0\}$.

Thm: If γ is a geodesic from p to q and if γ contains a conjugate pt. of p, then γ is not minimal.

Pf: Suffices to find W s.t. $H(E)(W,W) < 0$. Let $q_0 = \gamma(t_0)$ be conjugate to p. Let X be a J. field s.t. $X(p) = X(q_0) = 0$.

Let $Y(t) = \begin{cases} X(t) & \text{if } t < t_0 \\ 0 & \text{if } t > t_0 \end{cases}$



Then $D_t Y$ has a discontinuity at t_0 (broken Jacobi field)

Let $Z \in T_\gamma \Omega$ be a smooth field s.t. $Z(t_0) = \Delta_t D_t Y(t_0)$.

and consider

$$\tfrac{1}{2} H(E)(Y+\varepsilon Z, Y+\varepsilon Z) = \underbrace{\cancel{H(E)(Y,Y)}}_{0} + 2\varepsilon H(E)(Y,Z) + \varepsilon^2 H(E)(Z,Z)$$

$$\tfrac{1}{2} H(E)(Y,Z) = \underbrace{-\langle Z(t_0), \Delta_t D_t Y(t_0)\rangle}_{<0} - \int_0^1 \langle Z, \underbrace{D_t^2 Y - R(V,Y)V}_{0} \rangle$$

So if ε is small, then $H(E)(Y+\varepsilon Z, Y+\varepsilon Z) < 0 \Rightarrow \gamma$ not minimal.

~~Rem Fact: The~~ Converse?

Thm: If γ is not minimizing, then γ contains a conjugate pt of $\gamma(0)$.

In fact, can quantify.

~~Thm (Morse) If γ is a geod from p to q, then~~

If ~~F is a bilinear fn~~ $F: V \times V \to \mathbb{R}$ is bilinear,

$\text{index}(F) = \max\{\dim(N) \mid N \subset V, F \text{ is negative definite on } N \ (F(n,n) < 0 \ \forall n \in N, n \neq 0)\}$

- If 0 is a minimum of $F(x,x)$, then $\text{index } F = 0$
- If ~~V is finite di~~ $\dim(V) < \infty$, then $\text{index}(F) = \#$ of negative eigs
- ~~Measures how close 0 is to being~~

Thm (Morse): If γ is a geod from p to q, then

$$\text{index}(\gamma) = \text{index}(H(E)_\gamma) = \sum_{0<t<1} \text{order}(\gamma(t))$$

where $\text{order}(\gamma(t))$ = order of conjugacy of $\gamma(t)$

This index is always finite.

2016-03-02

Morse Index Theorem

Def: If $F: V\times V \to \mathbb{R}$ is a symmetric bilinear form,
$\text{index}(F) = \max\{\dim W \mid W\subset V,\ F \text{ is negative definite on } W\}$
$= \max\{\dim W \mid F(w,w)<0\ \forall w\in W, w\neq 0\}$.
$= \max\{\dim W \mid F|_W < 0\}$

If $V=\mathbb{R}^m$, let $e_1,\dots,e_m$ be standard basis, ~~Then~~ let $M=(F(e_i,e_j))$
Then $F(v,w)=v^TMw$, $\text{index}(F) = \#$ of negative eigs of M.
In fact, decompose

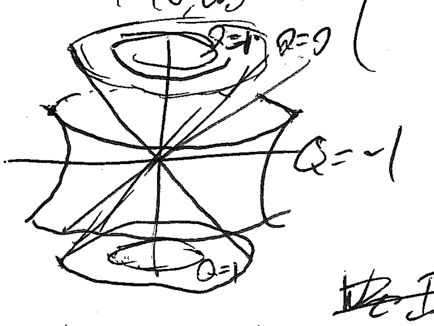
$$V = \underbrace{V_-}_{\text{negative eigspaces}} \oplus \underbrace{V_0}_{\text{nullspace of } M} \oplus V_+ \quad \text{- pos. eigspaces}$$

$\dim V_- = \text{index}(F)$ $\quad F(V_-,V_+) = F(V_-,V_0) = F(V_0,V_+) = 0.$
$\dim V_0 = \text{nullity}(F)$

~~Ex~~ (But generally there are many ways to decompose ~~like this~~ into orthog. subspaces.

Ex.

$$F(v,w) = v^T\begin{pmatrix} -1 & & \\ & -1 & \\ & & 1\end{pmatrix} w \qquad \underbrace{F((x,y,z),(x,y,z))}_{Q(x,y,z)} = -x^2-y^2+z^2$$



Pick v s.t. $Q(v)>0$.
Then $v^\perp$ is 2-dim, ~~neg def~~ $F|_{v^\perp}<0$

~~We If W∈V~~ (and generally this is how we work with index when V is ∞-dim - find a subspace where $F<0$ for a lower bound, find a complementary subspace on which $F>0$)

Thm (Morse Index Theorem). If $\gamma:[0,1]\to X$ is a geod from p to q, then
$$\text{index}(\gamma) = \text{index } H(E)_\gamma = \sum_{t\in(0,1)} \text{order}_p(\gamma(t))$$
~~total order of the points~~

where $\text{order}_p(\gamma(t))$ is the order of conjugacy of p and $\gamma(t)$.
Furthermore, $\text{index}(\gamma)<\infty$.

Pf: Step 1: $\text{index}(\gamma)<\infty$ (Find ~~a positive-def~~ $W\subset T_\gamma\Omega$ s.t. $F|_W>0$, $\text{codim } W<\infty$)

Cover γ by nbhds U_i s.t. if $x,y\in U_i$ then $\exists!$ minimal geod from x to y, varying smoothly. Let $0=t_0<\dots<t_k=1$ be a partition s.t.
$\gamma([t_i,t_{i+1}])\subset U_j$ for some j and $\gamma|_{[t_i,t_{i+1}]}$ is minimal.
(in particular, $\gamma(t_i)$ and $\gamma(t_j)$ are non-conj.

Let $J = J(t_0,\dots,t_k) = \{\text{broken Jacobi fields}\}$
$= \{W\in T_\gamma\Omega \mid W(0)=W(1)=0,\ W|_{[t_i,t_{i+1}]} \text{ is Jacobi}\}$

*Then ~~is~~ $J \cong T_{\gamma(t_1)}M \oplus \cdots \oplus T_{\gamma(t_{k-1})}M\}$. ~~and~~

~~Claim~~ ~~Claim~~ ~~Let~~ $J^\perp = \{W \in T_\gamma\Omega \mid W(t_i) = 0\}$

Check: - $H(E)(J, J^\perp) = 0$ Write $H = H(E)$

~~J has discs at t_i~~ - $J \cap J^\perp = 0$

- $T_\gamma \Omega = J \oplus J^\perp$

Furthermore, $H(E)|_{J^\perp} > 0$ - if $W \in J^\perp$, ~~then~~ let $W_i = W|_{[t_i, t_{i+1}]}$.

~~$H(E)(W, W') = \sum_i H(E)(W|_{[t_i}$~~

Then $H(E)(W, W) = \sum_i H(E)(W_i, W_i)$
≥ 0 by minimality of γ.

If $H(E)(W_i, W_i) = 0$ then $W_i \perp J$.

~~and W_i~~ ~~If $H(E)(W, W') = 0$, then~~ $W \in J$ ~~and~~

If $W' \in J^\perp$, ~~then~~ $H(E)(W, W') = \frac{d}{dt}\frac{1}{2}$ ~~$H(E)(W + tW', W + tW')$~~

consider ~~$\frac{1}{2}\frac{d}{dt}$~~ $H(W,W) +$

$\frac{1}{2}H(W + tW', W + tW') = \cancel{\frac{1}{2}H(W,W)}_{=0} + tH(W, W') + t^2 H(W', W')$

Lemma: if $F \geq 0$ and nullity$(F) = 0$ then $F > 0$.

$\frac{d}{dt}\frac{1}{2}H(W + tW', W + tW')\Big|_{t=0}^{\geq 0} = H(W, W) = 0$

$\Rightarrow W \perp J^\perp \Rightarrow W \in$ null$(H) \cap J^\perp \Rightarrow W = 0$.

Step 2: ~~Claim~~ index H = index $H|_J$

Let $p: T_\gamma\Omega \to J$, $p^\perp: T_\gamma \Omega \to J^\perp$ be projections.

Then $\forall W$, $H(W, W) \geq H(p(W), p(W))$

$\Rightarrow$ If $N \in T_\gamma \Omega$, $H|_N < 0 \Rightarrow$ ~~$p(H)$~~ $H|_{p(N)} < 0$, and $H \cap J^\perp = 0$
$\Rightarrow \dim p(N) = N$.

Likewise, nullity(H) = nullity$(H|_{J})$.

Let ~~γ~~ ~~$\lambda(\tau) =$ index~~ $\gamma_\tau = \gamma|_{[0,\tau]}$. $\lambda(t) =$ index γ_τ, $H_\tau = H(E)|_{\gamma_\tau}$

Step 3: Analyze λ.

- λ is monotone in τ
  - Any v.field on γ_τ extends to a v.field on γ.
  - $\exists\ \varepsilon > 0$ s.t. $\lambda(\tau) = 0\ \forall\ \tau \leq \varepsilon$.
    (take ε s.t. $\gamma|_{[0,\varepsilon]}$ is minimal.)

Q: When does λ change?

(Idea: ~~Only changes when theres~~ In order to change, a vector passes from positive to negative — so there's only a change when there's a nullspace — a conj pt.)

Step 3.1: $\forall \tau$, $\lambda(\tau - \varepsilon) = \lambda(\tau)$ ~~the~~ if ε is suff small.

Suppose $\tau \in [t_i, t_{i+1}]$. Then $\lambda(\tau) = \text{index } H_\tau|_{J(t_0, \cdots, t_i, \tau)}$.

$J(t_0, \cdots, \tau) \cong T_{\gamma(t_1)} M \oplus \cdots \oplus T_{\gamma(t_i)} M = \Sigma$
~~and if zero~~ so we think of H_τ as continuously varying form on Σ. Then if $N \subset \Sigma$, $H|_N < 0$,
then $H_{\tau-\varepsilon}|_N < 0$ for suff. small ε.
(cts from left) So $H(\tau-\varepsilon) \geq H(\tau)$ $H(\tau-\varepsilon) \leq H(\tau)$ by mont.

Step 3.2: If $n = \text{nullity}(H_\tau)$, then $\forall$ small $\varepsilon > 0$, $\lambda(\tau+\varepsilon) = \lambda(\tau) + n$.
Let Σ as above, let
$\Sigma = \Sigma_+ \oplus \Sigma_0 \oplus \Sigma_-$ where ~~dim~~ $H|_{\Sigma_+} > 0$
$H|_{\Sigma_-} < 0$
$\Sigma_0 = \text{nullspace}$
$\dim \Sigma_0 = n$.

If ε is small, then $H_{\tau+\varepsilon}|_{\Sigma_+} > 0$, so

$$\lambda(\tau+\varepsilon) \leq \dim \Sigma - \dim \Sigma_+ = \lambda(\tau) + n$$

Second claim $\lambda(\tau+\varepsilon) \geq \lambda(\tau) + n$
let $W_1, \cdots, W_{\lambda(\tau)} \in T_{\gamma_\tau} \Omega$ span a neg-def subspace,
let $J_1, \cdots, J_n$ be lin. ind Jacobi fields on γ_τ.

~~Goal: extend these~~ Extend these to $T_{\gamma_{\tau+\varepsilon}} \Omega$.
Claim: $S = \langle W_1, \cdots, W_{\lambda(\tau)}, J_1, \cdots, J_n \rangle$ can be perturbed to be neg. def.

In this basis,
$$H|_S = \begin{array}{c|cc} & W & J \\ \hline W & \begin{pmatrix} M < 0 & 0 \\ 0 & 0 \end{pmatrix} & \\ J & & \end{array}$$

Perturb: The J_i's have disconts at τ:
- $\Delta_+ D_+ J_i(\tau)$ are linearly indep.
- Choose duals $y_i \in T_{\gamma(\tau)} M$ s.t.

$$\langle \Delta_+ D_+ J_i(\tau), y_j \rangle = -\delta_{ij},$$

let $Y_i \in T_{\gamma_{\tau+\varepsilon}} \Omega$ be s.t. $Y_i(\tau) = y_i$ —

then $H(J_i, Y_j) = -2\delta_{ij}$. Let $S' = \langle W_1, \dots W_\lambda, c^{-1}J_1 + cY_1, \cdots, c^{-1}J_n + cY_n \rangle$,

c small. Then

$$H|_{S'} = \begin{pmatrix} M < 0 & cA \\ cA^T & -4I + c^2 B \end{pmatrix}$$ where $B = (H(Y_i, Y_j))$

If c is small, this is negative definite. //

Applications:

Def: Let $Ric(U_1, U_2) = Tr(W \to R(U_1, W)U_2)$

~~Then~~ This is symmetric, bilinear form.

If $U_1, \cdots, U_m$ is an orth. basis, then

$$Ric(U_i, U_i) = \sum_{j \neq i} K(U_i, U_j).$$

Thm (Myers): If $Ric(U, U) \geq \frac{m-1}{r^2}$ $\forall x \in M$, $U \in T_x M$, $\|U\| = 1$, then any geod of length $> \pi r$ contains conjugate points.

Pf: Let γ be a geod, $L = \ell(\gamma)$, $E_0, \cdots, E_{m-1}$ orth parallel frame, $E_0 = \frac{d\gamma}{dt} / L$.

~~Claim~~ Let $W_i = \sin(\pi t) E_i$. Claim that $H(W_i, W_i) < 0$ for some i.

Pf: $\frac{1}{2} H(W_i, W_i) = -\int_0^1 \langle W_i, D_t^2 W_i - R(V, W_i)V \rangle$

$D_t^2 W_i = -\pi^2 \sin(\pi t) E_i$

$$= -\int_0^1 -\pi^2 \sin^2(\pi t) - \sin^2(\pi t) \cdot \langle R(LE_0, E_i)LE_0, E_i \rangle$$

$$= \int_0^1 \pi^2 \sin^2(\pi t) - \sin^2(\pi t) L^2 K(E_0, E_i)$$

$$\sum_{i=1}^{m-1} \frac{1}{2} H(W_i, W_i) = \int_0^1 \sin^2(\pi t)\left(\pi^2 (m-1) - L^2 Ric(E_0, E_0)\right)$$

But $L^2 Ric(E_0, E_0) > \pi^2 (m-1) \Rightarrow \sum_{i=1}^{m-1}$ negative.

So $\exists i$ s.t. $H(W_i, W_i) < 0$ $\Rightarrow$ index $\gamma > 0$
$\Rightarrow \exists$ a conj. pt on γ //

Cor: If M is complete and $Ric(U, U) \geq \frac{(n-1)}{r^2}$, $\forall \|U\| = 1$, then $diam(M) \leq \pi r$

Cor: If M is as above, then $\pi_1(M)$ is finite.

Morse theory on $\Omega = \Omega(p,q)$ 2016-03-09

Goal: Describe Ω in terms of the geodesics from p to q.

Topology: If $\omega_1, \omega_2 \in \Omega$, let $d(\omega_1, \omega_2) = \max [d(\omega_1(t), \omega_2(t))] + \left[\int_0^1 \left(\|\frac{d\omega_1}{dt}(t)\| - \|\frac{d\omega_2}{dt}(t)\|\right)^2 dt\right]^{1/2}$

(uniform convergence + L^2 convergence of speed). Then E is cts wrt this metric. ~~we~~ ~~could~~

Let $\Omega^a = E^{-1}([0,a])$ $\operatorname{int}\Omega^a = E^{-1}([0,a))$.

Goal: Retract Ω^a to a CW-complex.

First: Retract to ~~strings~~ a fin-dim mfld.

Idea: like one of the problems in PS#1: approx by broken geods.

If $0 = t_0 < t_1 < \cdots < t_k = 1$, let $\Omega(t_0, \dots, t_k) = \{$broken geods$\}$ $= \{\omega \mid \omega|_{[t_i, t_{i+1}]}$ is geod $\forall i\}$. $\omega(0)=p$, $\omega(1)=q$.

Let $\Omega^a(t_0, \dots, t_k) = \Omega(t_0, \dots, t_k) \cap \Omega^a$

$\operatorname{Int}\Omega^a$ " " $=$ " " $\cap \operatorname{Int}\Omega^a$.

~~Then the~~ If $t_0, \dots, t_k$ is fine enough, this energy bound is v. coercive. ~~recall that for all s,t~~

$E(\omega|_{[s,t]}) = \int_s^t \|\dot\omega\|^2 dt$, so $L(\omega|_{[s,t]}) \le \sqrt{(t-s)E(\omega|_{[s,t]})}$

$E \geq (t-s)\left(\frac{L(\omega)}{t-s}\right)^2 = \frac{L^2}{t-s}$

Prop: If M is complete, then $\forall c>0$, $\exists t_0, \dots, t_k$ s.t. $\operatorname{Int}\Omega^c(t_0, \dots, t_k)$ is a fin-dim mfld.

Pf. Choose $\varepsilon > 0$ ~~small~~ so that small, choose t_i so that $\max(t_{i+1} - t_i) < \frac{\varepsilon^2}{c}$.

~~Then d~~ Then ~~w is determined~~

$\omega \in \operatorname{Int}\Omega^c(t_0, \dots, t_k)$ is determined by $\omega(t_0), \dots, \omega(t_k)$,

and varies smoothly with the $\omega(t_i)$.

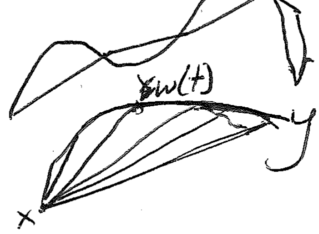
Therefore $\operatorname{Int}\Omega^c(t_0, \dots, t_k) \cong U \subset M^{k-1}$ //

~~Let B is a diffeo~~ Let $B = \Omega^c(t_0, \dots, t_k)$, let $E' = E|_B$.

Then:

- $E': B \to \mathbb{R}$ is smooth.
- $\forall a < c$, $B^a = (E')^{-1}([0,a])$ is compact.
- B^a is a def. retract of Ω^a.
- The crit pts of E on Ω^a are the same as the crit pts of E' on B^a
- $\forall$ geod $\gamma \in B^a$, $\operatorname{index}_E(\gamma) = \operatorname{index}_{E'}(\gamma)$, $\operatorname{nullity}_E(\gamma) = \operatorname{nullity}_{E'}(\gamma)$.

(only tricky one is def retract - pic



~~On each segment,~~

$$w_t = \gamma_{x,w(t)}\, w|_{[t,1]}$$

~~Thm (Fund Thm of Morse Theory)~~

Therefore, Thm (Fund Thm of Morse Theory):
If M is a complete ~~Riemannian manifold~~ and $p,q \in M$ are two ~~pts that are~~ not conjugate along any geod of length $\leq \sqrt{a}$, then Ω^a has the htpy type of a finite CW complex with one cell of dimension λ for each geod in Ω^a with index λ.

In fact, Thm: If p,q are nonconjugate, then Ω has the htpy type of a CW complex with one cell of dim λ for each geod with index λ

Applications: (Ω: fund gp? homology groups? higher homotopy gps? univ. cover? htpy type?)

Thm (Cartan-Hadamard): Suppose that M is complete, s. conn. $K(U,V) \leq 0$ $\forall\, U,V \in TM$. (we say M has nonpositive sectional curvature) Then $M \cong \mathbb{R}^n$.

Lemma: M has no conjugate pts.
Pf: Let γ be a geod, W a Jacobi field on γ.
~~Then $D_t^2 W = R(V,W)V$~~
Consider $\frac{d}{dt}\langle D_t W, W\rangle$.

$$\frac{d}{dt}\langle D_t V, W\rangle = \langle D_t^2 W, W\rangle + \langle D_t W, D_t W\rangle$$
$$= \langle R(V,W)V, W\rangle + \|D_t W\|^2$$
$$= -K(V,W)\cdot\|V\wedge W\|^2 + \|D_t W\|^2$$
$$\geq \|D_t W\|^2.$$

So $\langle D_t V, W\rangle$ is nondecreasing. If $W(0) = W(1) = 0$, then $\langle D_t W, W\rangle \equiv 0$
$\Rightarrow \|D_t W\|^2 \equiv 0 \Rightarrow W \equiv 0$ //

Pf of Thm: $\forall\, p,q \in M$, $\forall$ geod γ from p to q, index$(\gamma) = 0$
So
$$\Omega_{p,q} \simeq \text{union of 0-cells}$$
— i.e., $\Omega_{p,q}$ is a disjoint union of contractible components. (mean curvature flow decreases energy, is continuous, terminates at minimal geodesic)
But since M is s.c, $\Omega_{p,q}$ is connected
$\Rightarrow \exists!$ geod from p to q.

Therefore, $\exp_p$ is 1-to-1 and onto, and it has no critical pts.
$\Rightarrow$ exp is a diffeomorphism //

Def: If M is complete, $K \le 0$, we say M is a Hadamard mfld.

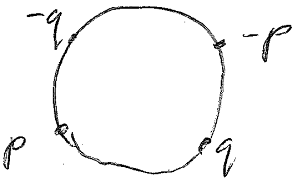
Cor: If M is Hadamard, then $\forall p,q \in M$, $\exists!$ geod from p to q in each htpy class.

Pf: Lift to univ cover.

Geom/top of spheres: let's apply this to the sphere:

Let $p, q \in S^m$ ~~be~~ be s.t. $p \ne \pm q$.

We can draw



- ~~the geod~~ conjugate pts of p are $\pm p$, both with index ~~m~~ along any geod.

~~Geods from p to q run clockwise~~

$\forall n \ge 0$, $\exists!$ a geod γ_n from p to q with ~~a~~ index $n(m-1)$

$\Omega \simeq X$ (CW complex with one ~~0~~ cell, $n(m-1)$ cell for $n \ge 0$. ~~Suppose M≃?~~

~~$H_i(\Omega) = \mathbb{Z}$ if $i = n(m-1)$, 0 otherwise?~~

~~Let $M \cong S^m$. Geods of M? # of cells not invariant, but homology of ΩM is. A different metric~~ leads to a different Morse fn, ~~but~~ so ~~what~~ can we ~~bring~~ from one to the other? ~~# of~~ cells ~~not~~ invariant, but ~~the homology~~ is —

~~$\forall n$, E_M has ≥ 1 crit pt of index $n(m-1)$.~~ (homology?)

Thm: If $M \cong S^m$, $m > 2$, and p, q are non-conjugate, then $\exists$ inf'ly many geodesics from p to q.

~~Let $M \cong$~~
~~Consider~~ We have $\Omega M \cong \Omega S^m$. Let ~~$E_M: \Omega M \to \mathbb{R}$~~.

~~# of cells not invariant, but homology is:~~

~~Let~~ $\Omega M \simeq X_M$ a CW complex with one cell for each geod.
$\simeq X$ but # of cells is not an invariant

~~Consider~~ Homology is — $H_i(X) \cong \begin{cases} \mathbb{Z} & \text{if } i = n(m-1) \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow H_i(X_M) \ne 0$ for inf'ly many i $\Rightarrow X_M$ has cells of inf'ly many dimensions.

(Q: Higher homotopy groups?)

Thm (Freudenthal) If $k < 2m-1$, then $\pi_k(S^m) \cong \pi_{k+1}(S^{m+1})$

Pf: $\Omega S^m \simeq X = \{pt\} \cup D^{m-1} \cup D^{2(m-1)} \cup \dots$

$X^{(2m-3)} \cong S^{m-1}$

$\pi_k(\Omega S^m) \cong \pi_k(S^{m-1})$ if $k < 2m-3$

$\pi_k(\Omega S^m) \cong \pi_{k+1}(S^m)$

Shifting indices, thm follows.

Closed geodesics:

Def: ~~A closed geod is a map~~ $\gamma: S^1 \to M$ ~~satisfying geod. equation~~

Thm (Lyusternik-Fet): ~~equiv a geod~~ $\gamma: [0,1]$ ~~s.t.~~ $\cancel{\gamma(0)=\gamma(1)}$

If M is a closed Riem mfld, it contains a closed geod. $\cancel{\gamma'(0)=\gamma'(1)}$

From exercise, if M is not simply connected, it has a closed geod. If M is simply connected?

Some of the same ideas apply:

Let $\Lambda M = \{f: S^1 \to M\}$, define $E: \Lambda M \to \mathbb{R}$.

As before, ΛM is approximated by $\Lambda M(t_0, \dots, t_k)$ for a fine partition — can we find critical points of E on ΛM?

Problem — too many crit pts: every constant path is a crit pt.

We need to find crit pts with positive energy

- but typically, these will be unstable crit pts (Ex. S^2)

How to find unstable crit pts?  (balls flow toward minima. Repel

~~Minimax technique~~

Thm: Suppose M is a mfld. $f: M \to \mathbb{R}$ ~~a Morse fn.~~ a smooth fn s.t. $M^a = f^{-1}((-\infty, a])$ is cp

~~$(\subset f^{-1}(I))$~~ If $\alpha: S^k \to M$, then

$$y = \inf_{\substack{\beta: S^k \to M \\ \beta \sim \alpha}} \max f(\beta(x))$$ is a critical ~~point~~ value of f.

Pf: If not, let $\varepsilon > 0$ be s.t. f has no critical values in $[y-\varepsilon, y+\varepsilon]$.

Let $\beta_0: S^k \to M$ be s.t. $\beta_0 \sim \alpha$ and $f(\beta_0(x)) \in M_{y+\varepsilon}$.

Then there is a def retract $r: M_{y+\varepsilon} \to M_{y-\varepsilon}$.

$r \circ \beta_0 \sim \alpha$, and $\max_x f(r(\beta_0(x))) < y$. $\times$.

Pf of L-F:

Topological fact: If M is a closed, simply-connected mfld, then $\pi_k(\Lambda M) \cong \pi_{k+1}(M) \oplus \pi_k(M)$ ~~and~~ constant paths

~~Let a minimal $k > 1$ s.t.~~ π_k — sweepouts

Let $k > 1$ be minimal s.t. $\pi_k(M) \neq 0$.

Let $T\alpha \in \pi_{k-1}(M)$ be the corresponding element

By minimax, $y = \inf_{\beta \sim T\alpha} \max E(\beta(x))$ is a crit pt.

Claim $y > 0$.

But if $y = 0$, then $T\alpha \sim \beta$ can be represented by curves of length $< \varepsilon$

~~All of these~~ in β. If ε is suff. small, we can simultaneously contract these to their basepoints, so $\beta \sim \beta_0: S^{k-1} \to M \subset \Lambda M$ (constant paths).

But $\pi_{k-1}(M) \cong 0 \Rightarrow T\alpha \sim 0$ $\times$.