Problem Set 1 (due Feb. 17)

February 10, 2016

1. Show that the covariant derivative can be expressed in terms of parallel transport in the following sense.

   Let $\gamma : [0,1] \to M$ be a smooth curve in M and let $p_t : T_{\gamma(0)}M \to T_{\gamma(t)}M$ be the parallel transport maps along γ. Show that if $X \in \mathbf{V}(\gamma)$, then

   $$D_t X(0) = \frac{d}{dt} p_t^{-1}(X(t)).$$

   (It may help to construct a frame of parallel vector fields on γ.)

2. In class, we used the fact that the curvature tensor can be expressed in terms of parallel transport. That is, we claimed that if $q \in M$ and if $\alpha : \mathbb{R}^2 \to M$ is a smooth map such that $\frac{\partial \alpha}{\partial u_1} = X$, $\frac{\partial \alpha}{\partial u_2} = Y$, then for all $Z \in T_q M$, we have

   $$R(X,Y)Z = \lim_{s \to 0} \frac{Z - p_{\gamma_s}(Z)}{s^2},$$

   where $\gamma_s : [0,4] \to M$ is the image under α of the boundary of an $s \times s$ square, i.e.,

   $$\gamma_s(t) = \begin{cases} \alpha(st, 0) & t \in [0,1] \\ \alpha(s, s(t-1)) & t \in [1,2] \\ \alpha(s(3-t), s) & t \in [2,3] \\ \alpha(0, s(4-t)) & t \in [3,4]. \end{cases}$$

   Prove this fact.

   (Hint: Construct a frame of vector fields $V_1, \dots, V_m \in \mathbf{V}(\alpha)$ such that $\nabla_X V_i = 0$ and $\nabla_Y V_i(u_1, 0) = 0$. Any vector field W along γ_s can be expressed as a linear combination of the V_i — when is W parallel?)

3. Suppose M is a homogeneous Riemannian manifold, that is, that for any $x, y \in M$, there is an isometry of M that sends x to y. Show that M is geodesically complete.

4. Suppose that $f : M \to \mathbb{R}$ is a function such that $\|\operatorname{grad} f\| = 1$ everywhere. Prove that if γ is an integral curve of $\operatorname{grad} f$ (i.e., $\gamma'(t) = \operatorname{grad} f(\gamma(t))$), then γ is a geodesic.

5. Suppose that M is a compact Riemannian manifold.

   (a) Prove that there is an $\epsilon > 0$ such that every closed curve of length $< \epsilon$ is null-homotopic.

   (b) Suppose that $\gamma : S^1 \to M$ is a closed curve. Prove that there is a piecewise-smooth curve γ' that is homotopic to γ and such that $\ell(\gamma') \le \ell(\gamma)$.

   (c) If $\gamma : S^1 \to M$ is a closed curve, the free homotopy class of γ is the set of curves that are homotopic to γ. Show that this set has a minimal-length element and that this element is a closed geodesic.