Problem Set 2 (due Mar. 2)

February 17, 2016

1. Suppose that $f : M \to \mathbb{R}$ is a smooth function and that $p \in M$ is a critical point of f. If $V, W \in T_pM$, let $\alpha : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to M$ be a smooth map such that $\alpha(0,0) = p$, $\frac{\partial \alpha}{\partial u_1} = V$, and $\frac{\partial \alpha}{\partial u_2} = W$. Define

$$H(f)(V,W) = \frac{\partial^2}{\partial u_1 \partial u_2} f(\alpha(u_1, u_2)).$$

   Prove that $H(f)$ is a well-defined symmetric bilinear form. What if p is not a critical point of f?

2. Let M be a connected, not necessarily complete, Riemannian manifold and let $p \in M$. Show that if $f, g : M \to M$ are isometries such that $f(p) = g(p)$ and $f_* = g_*$ at p, then $f = g$.

3. Suppose that M is a two-dimensional Riemannian manifold with curvature tensor R.

   (a) Use the symmetries of the curvature tensor to show that if $p \in M$, $V, W \in T_pM$, then

$$K = \frac{\langle R(V,W)W, V \rangle}{\|V\|^2 \|W\|^2 - 2\langle V, W \rangle^2}$$

   is independent of V and W.

   (b) Prove that if K is as above, then

$$R(X,Y)Z = K(\langle Y, Z \rangle X - \langle X, Z \rangle Y)$$

   for all $X, Y, Z \in T_pM$.

4. Let $a < b$ and let $M \subset \mathbb{R}^3$ be the surface of revolution obtained by rotating the curve

$$(x-b)^2 + z^2 = a^2, y = 0$$

   around the z–axis (a torus with major radius b, minor radius a).

   Show that the "equator" $\gamma(t) = ((a+b)\cos t, (a+b)\sin t, 0)$ is a geodesic. When ϵ is small, describe the geodesics that intersect γ at angle ϵ.