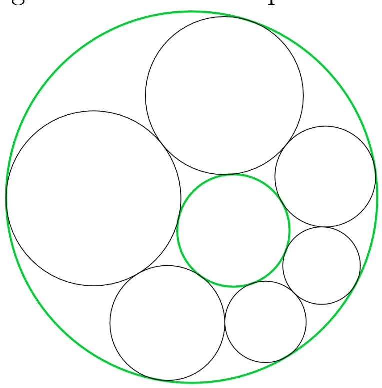
PROBLEM SET 3 (DUE APR. 13)

(1) Suppose that $p \in \mathbb{H}^n$ and that $v \in T_p\mathbb{H}^n$ is a vector of unit length, where we view $\mathbb{H}^n$ as a hyperboloid in $\mathbb{R}^{n,1}$. Find a parameterization of the geodesic through p with velocity v.

(2) Suppose that C and C' are two circles in the plane that do not intersect, one inside the other. A *Steiner chain* is a sequence $D_1, \dots, D_k$ of distinct discs such that each disc lies in the annulus between C and C' and is tangent to C, C', D_{i-1}, and D_{i+1} (treating indices cyclically). Use an inversion to show that if C and C' have a Steiner chain, then any disc between C and C' and tangent to C and C' is part of a Steiner chain.



(3) Put a metric on the infinite strip $[0, \pi] \times \mathbb{R}$ that makes it isometric to the hyperbolic plane by a conformal map (a map that preserves angles). Some complex analysis may help here.

(4) We say that a manifold X has *convex distance function* if for any two geodesics $\gamma_1, \gamma_2 \colon \mathbb{R} \to X$, the function $d(\gamma_1(t), \gamma_2(t))$ is a convex function on $\mathbb{R}$. (In class, we showed/will show that CAT(0) spaces have convex distance functions.) Show that if X is a complete manifold with a convex distance function, then:
- There is a unique geodesic between any two points in X.
- X is contractible (i.e., the identity map is homotopic to a constant map)

(The advantage of convexity is that one can define convexity for any geodesic metric space, not just Riemannian manifolds, and the properties above still hold.)

(5) The *ping-pong lemma* states the following:

Suppose that X is a set and that $\lambda_1, \lambda_2 \in \operatorname{Aut}(X)$ are invertible maps. Suppose that $U_1^\pm, U_2^\pm \subset X$ are disjoint, nonempty sets such that $\lambda_i(X \setminus U_i^-) \subset U_i^+$ and $\lambda_i^{-1}(X \setminus U_i^+) \subset U_i^-$ for $i = 1, 2$. Then the subgroup of $\operatorname{Aut}(X)$ generated by λ_1 and λ_2 is a free group of rank 2. (That is to say, if

$$w = \lambda_{a_1}^{e_1} \dots \lambda_{a_k}^{e_k} = \mathrm{id}_X,$$

for some $e_i \in \{-1, 1\}$ and $a_i \in \{1, 2\}$, then w contains a substring of the form $\lambda_i^{\pm 1}\lambda_i^{\mp 1}$.)

Let $\lambda_1, \lambda_2 \in \operatorname{Isom}(\mathbb{H}^2)$ be hyperbolic isometries with axes γ_1 and γ_2. Suppose that the endpoints of γ_1 and γ_2 are all distinct. Use the ping-pong lemma to show that if m_1, m_2 are sufficiently large, then $\lambda_1^{m_1}$ and $\lambda_2^{m_2}$ generate a free group $\Lambda_{m_1,m_2} = \langle \lambda_1^{m_1}, \lambda_2^{m_2} \rangle$ inside $\operatorname{Isom}(\mathbb{H}^2)$.

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