Problem Set 4 (due May 4)

April 20, 2016

1. Show that CAT(0) spaces have the approximate midpoint property. That is, if X is CAT(0) and $x, y \in X$ are points such that $r = d(x, y) > 0$, then for any ϵ, there is a δ such that if $d(z, x) \leq \frac{r}{2} + \delta$ and $d(z, y) \leq \frac{r}{2} + \delta$, then $d(z, m) \leq \epsilon$, where m is the midpoint of x and y.

2. Let X be a complete CAT(0) space and let $y_1, \dots, y_n \in X$. Show that the function $\sigma(x) = \sum_i d(x, y_i)^2$ has a unique minimum. (This is called the *barycenter* of $\{y_1, \dots, y_n\}$, and it can be used to provide another proof that any isometry with a finite orbit has a fixed point.)

3. Suppose that M is a compact, complete manifold with *negative* sectional curvature.

    - Let $\tilde{M}$ be the universal cover of M. Show that every element of $\pi_1(M)$ acts on $\tilde{M}$ by a hyperbolic isometry.
    - Show that there is a *unique* closed geodesic (up to reparametrization) in each nontrivial free homotopy class of M.
    - Show that for all $L > 0$, there are only finitely many closed geodesics in M with length less than L.

4. Prove that any complete CAT(0) Riemannian manifold is a simply-connected manifold with nonpositive sectional curvature.