NOTES ON ASYMPTOTIC CONES

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1. Introduction

The asymptotic cone of a space is a metric space representing the space viewed from infinity. Since small-scale geometry disappears in this limit, it makes a good tool for studying geometric properties of groups. Here, we'll look at some asymptotic cones and some applications to filling.

2. Definitions

Let X be a metric space, $\{x_i\}_{i\in\mathbb{N}}$ a sequence of points in X (thought of as a sequence of centers), and $\{d_i\}_{i\in\mathbb{N}}$ a sequence of scaling factors such that $d_i \to \infty$, and an ultrafilter ω.

I'm not going to get into the details of ultrafilters here, but the main property of an ultrafilter is that it lets you define the ultralimit $\lim_\omega a_i$ of any bounded sequence a_i. Ultralimits use the axiom of choice to extend the notion of limit to arbitrary bounded sequences; they are linear, and the ultralimit of a sequence is a limit point of the sequence.

We use these to construct an asymptotic cone. Let $X_b^{\mathbb{N}}$ be the set of sequences $\{y_i\}_{i\in\mathbb{N}}$ such that $d(x_i, y_i)/d_i$ is bounded. This looks huge, but it's like defining real numbers by Cauchy sequences – a lot of sequences have the same limit. Define $d_\omega(\{x_i\},\{y_i\}\}) = \lim_\omega d(x_i, y_i)/d_i$, and define

$$\operatorname{Con}_\omega X = X_b^{\mathbb{N}}/\sim$$

$$\{x_i\} \sim \{y_i\} \text{ iff } d_\omega(\{x_i\},\{y_i\}) = 0.$$

This is a metric space; we call it an asymptotic cone for X. Denote the equivalence class of $\{y_i\}_{i\in\mathbb{N}}$ by $[y_i]$. Alternatively, this is the Gromov-Hausdorff limit of a sequence of scalings of X with basepoint x_i.

3. Examples

- $X = \mathbb{R}^n$:

  X is already scale-invariant, so $\operatorname{Con}_\omega \mathbb{R}^n$ is just $\mathbb{R}^n$. It's instructive to look at the maps between the two. For simplicity, take $x_i = 0$ for all i. Then the maps

  $$f : \mathbb{R}^n \to \operatorname{Con}_\omega \mathbb{R}^n$$
  $$f(x) = [d_i x]$$
  $$g : \operatorname{Con}_\omega \mathbb{R}^n \to \mathbb{R}^n$$
  $$g([a_i]) = \lim_\omega a_i/d_i$$

  are isometries.

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- $X = \mathbb{H}^n$:

  Embedded geodesics in X correspond to geodesics in $\operatorname{Con}_\omega X$, so we can build the asymptotic cone of $\mathbb{H}^n$ by looking at geodesics in the hyperbolic plane. Let $x_i = x_0$ for all i. There are uncountably many geodesics through x_0, and each one corresponds to a geodesic in $\operatorname{Con}_\omega \mathbb{H}^n$ going through the point $x = [x_0]$. Now consider another point $y = [y_i] \neq x$ in $\operatorname{Con}_\omega \mathbb{H}^n$. The limit of the geodesics connecting x_0 and y_i is a geodesic between x and y, and there are uncountably many other geodesics through y. As i increases, the y_i get further from x_0, and fewer of the geodesics through y intersect the geodesics through x. In the asymptotic cone, none of the geodesics through x intersect geodesics through y except for ones that travel along the geodesic segment connecting x and y. The same is true for any point in the asymptotic cone. In fact, the asymptotic cone of $\mathbb{H}^n$ is an $\mathbb{R}$-tree where every point has uncountable degree. Any hyperbolic group has such an asymptotic cone.
- X a symmetric space:

  Like the geodesics in the previous example, the flats in X correspond to copies of $\mathbb{R}^k$ in $\operatorname{Con}_\omega X$. In a symmetric space, the flats intersect in more complicated ways. Kleiner and Leeb proved that the asymptotic cone of a symmetric space is a building [4].

Remarks:

- Asymptotic cones have a lot of symmetry. If G acts on X discretely and cocompactly, then $\operatorname{Con}_\omega$ doesn't depend on the sequence of scaling centers and has a transitive group of isometries coming from limits of isometries of X. If the asymptotic cone is independent of the choice of x_i and d_i, it has a family of scaling maps, but this isn't true in general.
- If X and Y are quasi-isometric, then every asymptotic cone of X is Lipschitz equivalent to an asymptotic cone of Y.
- Sequences of Lipschitz maps to X pass to $\operatorname{Con}_\omega$. If $\{f_i\}$ is a sequence of maps $f_i : K \to X$ and f_i is cd_i-Lipschitz, then $[f_i](x) = [f_i(x)]$ is c-Lipschitz. Going the other way is tougher. A map $f : K \to \operatorname{Con}_\omega$ corresponds to a sequence of maps $f_i : K \to X$, but one can only control the behavior of f_i on a finite subsets of K.
- Generally, a space can have uncountably many asymptotic cones, depending on the choices of x_i, d_i, and ω. Take a space consisting of circles of length r_i joined at a basepoint x_0. Any asymptotic cone also consists of circles joined at a point, but the circles that are "visible" in the limit depends on the scaling factors d_i and the ultrafilter.

4. Applications to filling

We'd like to see how properties of the space are reflected in its asymptotic cone, in particular, whether filling functions of the space can be determined by its asymptotic cone. The first difficulty is to define an appropriate filling function for $\operatorname{Con} X$. Given a partition τ of a disc into polygons ρ_i and a map f from the vertices of the partition to a geodesic metric space K, define

$$\operatorname{Mesh}(f, \tau) = \max_i \operatorname{perim}(f(\rho_i)),$$

where if $\rho = (v_1, \ldots, v_n)$, then

$$\mathrm{perim}(f(\rho)) = d(f(v_1), f(v_2)) + \cdots + d(f(v_n), f(v_1)).$$

If $\alpha : S^1 \to K$, define

$$\mathrm{FA}^K_\delta(\alpha) = \min_{\mathrm{Mesh}(f,\tau)\leq\delta} \#\tau$$

where $\#\tau$ is the number of polygons in τ and the minimum is taken over maps f which agree with α on the boundary of the disc. You can break up α into $\mathrm{FA}^K_\delta(\alpha)$ curves of length δ.

We can then define a filling function that makes sense for asymptotic cones:

$$\mathrm{FA}^K_\delta(n) = \max_{l(\alpha)<n} \mathrm{FA}^K_\delta(\alpha).$$

This turns out to descend fairly well from an asymptotic cone to the original space. The main theorem is due to Papasoglu, and can be found in [1].

Theorem 1. *If X is a s.c. geodesic metric space with bounded local geometry (e.g., X is a manifold on which a group acts discretely and cocompactly), and there is an $a > 0$ and $p > 0$ such that in each asymptotic cone of X we have*

$$\mathrm{FA}^{\mathrm{Con}\,X}_1(n) \leq an^p$$

for all $n > 1$, then

$$\delta_X(n) \lesssim n^{p+\epsilon}$$

for all $\epsilon > 0$.

This theorem refines a theorem of Gromov, a detailed proof of which can be found in [2].

Theorem 2. *If X is a s.c. geodesic metric space with bounded local geometry and every asymptotic cone of X is s.c., then there is a p such that $\delta_X(n) \lesssim n^p$.*

The proof shows some of the techniques for going back and forth between a space and its asymptotic cone.

Proof. We will show that

$$\mathrm{FA}^X_{n/2}(n) \leq c$$

for $n > l_0$. Then any curve with length $l(\alpha) > l_0$ can be broken into c curves of length $l(\alpha)/2$. We can break a curve α into roughly

$$c^{\log_2 l(\alpha)} = l(\alpha)^{\log_2 c}$$

curves of length $< l_0$ by repeating this procedure. Since the area of each of these curves is bounded, this results in a polynomial isoperimetric function.

We proceed by contradiction. Assume $\mathrm{FA}^X_{n/2}(n) \to \infty$. Then we can find a sequence of curves α_i such that

$$\mathrm{FA}^X_{l(\alpha_i)/2}(\alpha_i) \to \infty$$

Take a sequence of basepoints lying on α_i and let $d_i = l(\alpha_i)$; this gives an asymptotic cone $\mathrm{Con}\,X$ and the α_i correspond to a curve $\alpha : S^1 \to \mathrm{Con}\,X$ of length 1. Since $\mathrm{Con}\,X$ is s.c., there is a disc filling α. A sufficiently fine partition of this disc gives a filling of α with $\mathrm{Mesh}(f, \tau) < 1/3$. But f is the ultralimit of a sequence of maps $f_i : \tau^{(0)} \to K$, each of which agrees with α_i on the boundary of τ. We must have $\mathrm{Mesh}(f_i, \tau) < d_i/2$ for some i, which is a contradiction. $\square$

Papasoglu's theorem is proved by similar methods; instead of bounding $\mathrm{FA}^X_{n/2}(n)$, he uses the bound on $\mathrm{FA}_1^{\mathrm{Con}\,X}(n)$ to find bounds on $\mathrm{FA}^X_{n/k}(n)$ for large k.

5. Higher dimensions?

Extending this sort of result to higher dimensions is tough. The problem is that a sequence of curves is a sequence of Lipschitz maps and gives a curve in the asymptotic cone. A sequence of spheres is less controlled, so you need an extra step, namely, breaking up an arbitrary sphere into spheres with some sort of controlled geometry.

This is tricky, and there's still a lot of work to be done. One of the biggest successes is in spaces with cone-type inequalities (e.g., non-positively curved spaces). In these spaces, one can cut a map into "round" pieces whose diameters correspond to their volumes [3, 7], and use this decomposition to prove euclidean filling inequalities in all dimensions. Papasoglu used a related technique to show that under certain conditions, there is a "gap" in the possible exponents of second-order Dehn functions [5].

In nilpotent groups, one can use scaling and approximation techniques to decompose a sphere into a collection of simplices [8]. This operates differently from the previous method; instead of looking for small separating cycles, one constructs triangulations of the nilpotent group at all scales and approximates a map in those triangulations. As such, it isn't clear how to apply this method to groups with more complicated asymptotic cones.

Once you've done this decomposition, asymptotic cone methods can be applied to get bounds on spheres with controlled geometry. Riley [6], for instance, showed that if all asymptotic cones of a group are highly connected, then spheres with volume $< n$ and diameter $< l$ can be filled polynomially in n and l. Similarly, it may be much easier to prove bounds on filling maps with Lipschitz constant $< n$ or some other measure of bounded complexity than it is to prove bounds on maps with bounded volume.

References

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