Quantifying simple connectivity: an introduction to the Dehn function

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  A: The Dehn function

Measuring simple connectivity: The Dehn function

Let X be a simply-connected simplicial complex or manifold and let $\alpha : S^1 \to X$ be a closed curve. Define

$$\delta(\alpha) = \inf_{\substack{\beta : D^2 \to X \\ \beta|_{S^1} = \alpha}} \operatorname{area} \beta.$$

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In the case of $\mathbb{R}^2$, the circle has maximal area for a given perimeter, so $\delta_{\mathbb{R}^2}(2\pi r) = \pi r^2$.

The word problem: how do you recognize the identity?

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The word problem: If w is a product of generators (a word), how can we tell if it represents the identity?

Reducing using relations

Any two words representing the same group element can be transformed into each other by:

- Application of a relation:

$$wr_i^{\pm 1}w' \leftrightarrow ww'$$

- Free insertion/reduction:

$$wg_i^{\pm 1}g_i^{\mp 1}w' \leftrightarrow ww'$$

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Q: How many steps does this take?

The Dehn function of a group

If w represents the identity, define

$$\delta(w) = \# \text{ of applications of relations to reduce } w$$

and

$$\delta_G(n) = \max_{\substack{\ell(w) \leq n \\ w =_G 1}} \delta(w).$$

Example: $\mathbb{Z}^2$

Let $\mathbb{Z}^2 = \langle x, y \mid [x, y] \rangle$. Going from xy to yx takes one application of the relation:

$$xy \to (yxy^{-1}x^{-1})xy \to yx.$$

So if $w = x^2y^2x^{-2}y^{-2}$, then w represents the identity and $\delta(w) = 4$.
Similarly, $\delta(x^ny^nx^{-n}y^{-n}) = n^2$.

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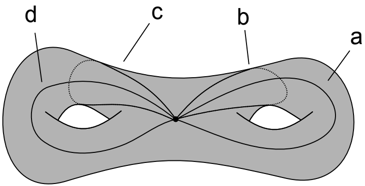
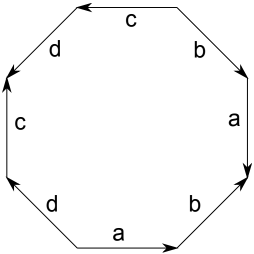
Theorem (Gromov)

When G acts geometrically (properly discontinuously, cocompactly, by isometries) on a space X, the Dehn function of G and of X are the same up to constants.

Fundamental groups of surfaces





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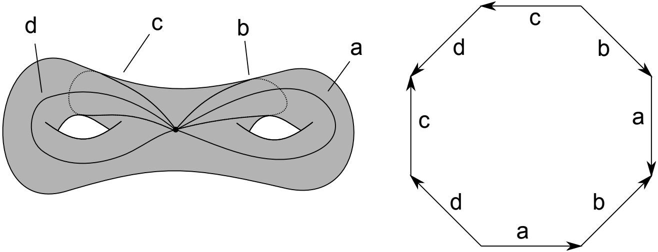


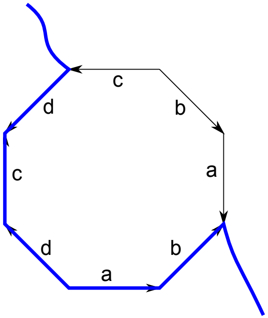
$$G = \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$$

Dehn's algorithm for the word problem

Let w be a word.

1. Look for a subword that consists of more than half of the octagon

$$w = \ldots dc^{-1}d^{-1}ab\ldots$$



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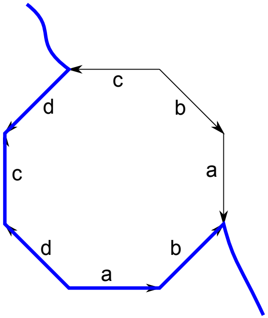

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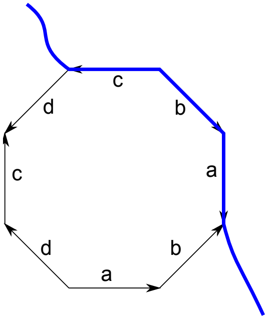

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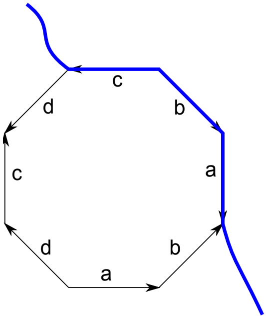

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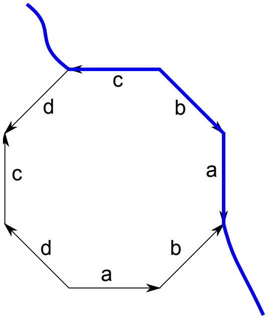
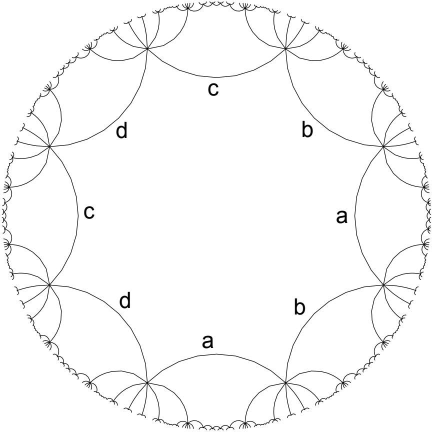
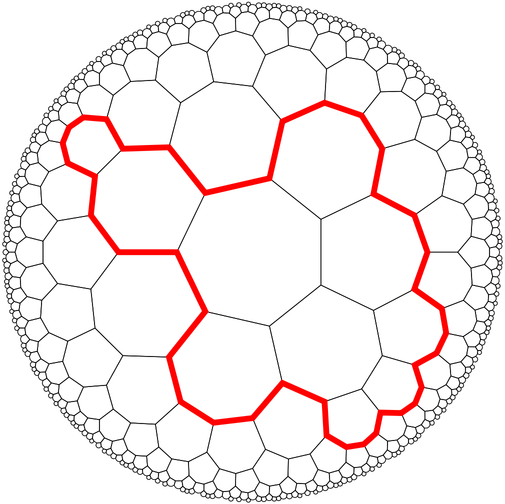
If this reduces w to the trivial word, it represents the identity; otherwise, it doesn't.

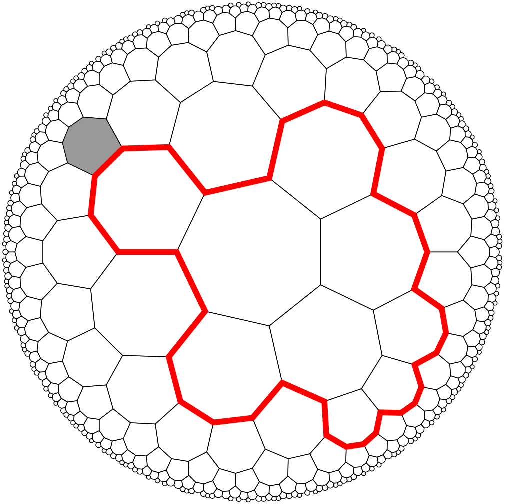


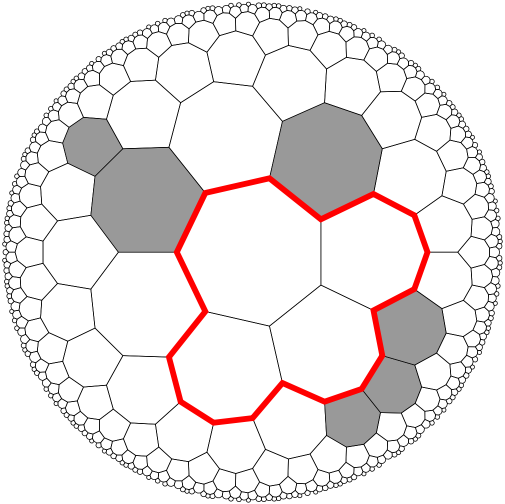
The universal cover is the hyperbolic plane

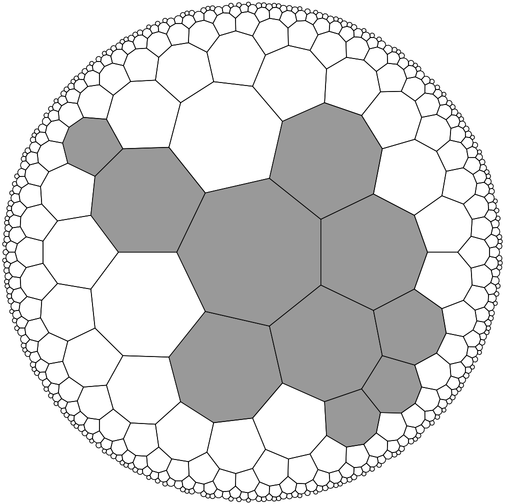


Any closed path of edges (and thus any word that represents the identity) must contain most of an octagon.









Linear Dehn functions correspond to negative curvature

Theorem (Gromov, Lysenok, Cannon)

If G is a finitely presented group, the following are equivalent:

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- *Several other definitions*

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- There are groups with two generators and one relation which have Dehn function larger than any tower of exponentials.
- If G has unsolvable word problem, then δ_G is larger than any computable function.

$$\mathsf{Sol}_3 = \left\{ \begin{pmatrix} e^t & 0 & x \\ 0 & e^{-t} & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, t \in \mathbb{R} \right\}$$

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$$\mathrm{Sol}_3 \subset \left\{ \begin{pmatrix} e^a & 0 & x \\ 0 & e^b & y \\ 0 & 0 & 1 \end{pmatrix} \right\} \cong \left\{ \begin{pmatrix} e^a & x \\ 0 & 1 \end{pmatrix} \right\} \times \left\{ \begin{pmatrix} e^b & y \\ 0 & 1 \end{pmatrix} \right\} = \mathrm{Hyp}^2 \times \mathrm{Hyp}^2$$

$$\mathrm{Sol}_5 = \left\{ \begin{pmatrix} e^{t_1} & 0 & 0 & x \\ 0 & e^{t_2} & 0 & y \\ 0 & 0 & e^{t_3} & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \sum t_i = 0 \right\}$$

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But has spheres which are exponentially difficult to fill!

Open questions

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- Filling spheres and cycles rather than curves?
- Other groups?