Filling functions and nonpositive curvature

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Fillings in lattices in semisimple groups

Conjecture (Gromov, Bestvina-Eskin-Wortman, Leuzinger-Pittet)

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- Why should we believe this conjecture?
- Why is it harder to fill spheres than to fill curves?
- How can we find fillings of spheres in solvable groups?

The Dehn function: Measuring simple connectivity

Let X be a simply-connected simplicial complex or manifold and let $\alpha : S^1 \to X$ be a closed curve. Define

$$\delta(\alpha) = \inf_{\substack{\beta: D^2 \to X \\ \beta|_{S^1} = \alpha}} \operatorname{area} \beta.$$

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In the case of $\mathbb{R}^2$, the circle has maximal area for a given perimeter, so $\delta_{\mathbb{R}^2}(2\pi r) = \pi r^2$.

The word problem: how do you recognize the identity?

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The word problem: If w is a product of generators (a word), how can we tell if it represents the identity?

Reducing using relations

Any two words representing the same group element can be transformed into each other by:

- Application of a relation:

$$wr_i^{\pm 1}w' \leftrightarrow ww'$$

- Free insertion/reduction:

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Q: How many steps does this take?

The Dehn function of a group

If w represents the identity, define

$$\delta(w) = \# \text{ of applications of relations to reduce } w$$

and

$$\delta_G(n) = \max_{\substack{\ell(w) \leq n \\ w =_G 1}} \delta(w).$$

Example: $\mathbb{Z}^2$

Let $\mathbb{Z}^2 = \langle x, y \mid [x, y] \rangle$. Going from xy to yx takes one application of the relation:

$$xy \to (yxy^{-1}x^{-1})xy \to yx.$$

So if $w = x^2y^2x^{-2}y^{-2}$, then w represents the identity and $\delta(w) = 4$.
Similarly, $\delta(x^n y^n x^{-n} y^{-n}) = n^2$.

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Theorem (Gromov)

When G acts geometrically (properly discontinuously, cocompactly, by isometries) on a space X, the Dehn function of G and of X are the same up to constants.

Dehn function examples – Combinatorial

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- If G has unsolvable word problem, then δ_G is larger than any computable function.
- If G is automatic, then $\delta_G(n) \precsim n^2$.

Dehn function examples – Geometric

The smallest Dehn functions are equivalent to negative curvature.

- If X has pinched negative curvature, then we can fill curves using geodesics. These discs have area linear in the length of their boundary, so $\delta_X(n) \sim n$.
- In fact, G is a group with sub-quadratic Dehn function ($\precnsim n^2$) if and only if G is δ-hyperbolic (Gromov).

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- If X has non-positive curvature, we can fill curves with geodesics, but the discs may have quadratically large area.
- But the class of groups with quadratic Dehn functions is extremely rich; it includes Thompson's group (Guba), many solvable groups (Leuzinger-Pittet, de Cornulier-Tessera), some nilpotent groups (Gromov, Sapir-Ol'shanskii, others), $\mathrm{SL}(n;\mathbb{Z})$ for large n (Y.), and many more.

Sol_3 and Sol_5

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But Sol_5 has spheres which are exponentially difficult to fill!

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- i.e., Sol_{2n-1} is a subset of a symmetric space of rank n
- So Sol_{2n-1} contains lots of $(n-1)$-dimensional surfaces (intersections with flats), but no n-dimensional surfaces.
- It should be easy to find fillings of spheres with dimension $\leq n-2$, but there are $(n-1)$-dimensional spheres which are hard to fill.

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Low dimensions

Constructing short curves in rank 2:

- (Lubotzky-Mozes-Raghunathan) If Γ is an irreducible lattice in a semisimple group G of rank ≥ 2, then $d_\Gamma(x, y) \sim d_G(x, y)$ for all $x, y \in \Gamma$.

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- (Leuzinger-Pittet) If Γ is an irreducible lattice in a semisimple group G of rank 2, then it has exponential Dehn function.

Higher dimensions

Fillings by balls (large k):

- (Bestvina-Eskin-Wortman) If Γ is an irreducible lattice in a semisimple group G which is a product of n simple groups and $k < n$, then the $(k-1)$st Dehn function of Γ is bounded by a polynomial.

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- (Y.) If $k \leq n-2$, any k-sphere of volume r^k in Sol_{2n-1} has a filling of volume $\sim r^{k+1}$.

Filling curves in Sol_5

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- Most flats in $\mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^2$ intersect Sol_5 in an octahedron.
- Curves in this octahedron are (quadratically) easy to fill.
- So we'll fill arbitrary curves by breaking them down into *simple* curves that lie in finitely many flats.

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An arbitrary curve can be broken into triangles. Adding up the filling areas of these curves gives us $\delta_{\mathsf{Sol}_5}(\ell) \leq \ell^2$.

Filling in higher dimensions

If $n > k$, then a $k-1$-sphere in a finite union of flats in Sol_{2n-1} can be filled with a k-disc in a (possibly larger) finite union of flats.

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How do we break down a $k-1$-sphere into "simple spheres"?

From Lipschitz spheres to arbitrary cycles

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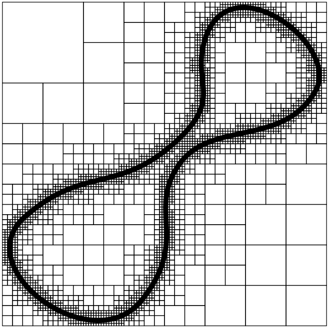
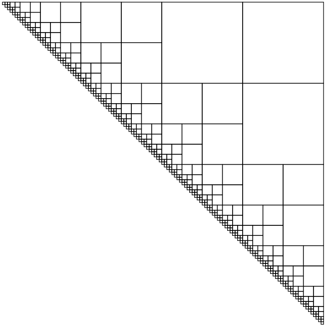
Solution: In $\mathbb{R}^n$, we can use Whitney decompositions:



From Lipschitz spheres to arbitrary cycles

In general, there's a nice analogue of this construction due to Lang and Schlichenmaier – $(H_2)^n$ breaks down into nice pieces and we can use that to break a filling of α into nice simplices.



Generalizing further

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- One project: Apply this to lattices in semisimple groups.
- Question: Which other spaces does this work for?