THE GEOMETRY OF SURFACES AND 3-MANIFOLDS

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Note: Most of the illustrations in these notes are omitted. Please draw your own!

1. Surfaces

A surface is a space where every point has a neighborhood which is (topologically) an open disc.

Example 1.1. The plane is a surface; every point has a neighborhood which is a disc.

Example 1.2. Spheres are surfaces. On a sphere of radius 1, any neighborhood of radius at most π is (topologically) a disc.

Experimental confirmation that spheres are surfaces comes from the Age of Exploration – as long as you don't go too far away from one point, the Earth is mostly indistinguishable from a plane.

Example 1.3. Other examples of surfaces: the torus (the surface of a donut), an infinite cylinder, the torus with n handles, etc.

Example 1.4. Open discs (in fact, any open subset of a surface) are surfaces. Closed discs aren't, because points on the boundary don't have disc neighborhoods.

Each of these examples is embedded in Euclidean space, but surfaces don't have to be embedded: another way to construct surfaces is to glue polygons together, for example, the torus (a square with its edges glued), a Klein bottle (a square with one pair of edges glued and the other pair glued with a twist), or a projective plane (a bigon with its edges glued by a twist).

1.1. **Geometry on surfaces.** In order to do geometry on surfaces, we need to have notions of lines and distances and angles. If a surface is smoothly embedded in Euclidean space (no corners, no self-intersections), then there are nice ways to define these. The distance between two points on an embedded surface is just the length of the shortest path between them. The angle between two paths is just the angle between the paths in 3-space.

Lines are a little more complicated. Lines in the plane have two properties – a line is the shortest path between two points, and a line is "straight"; it doesn't turn. What does it mean to go in a straight line on a curving surface? You need to make some turns just to stay on the surface, so a "straight" line should be the path that has the minimal amount of turning to stay on the surface. This is called a *geodesic*.

So, for example, let's take a cylinder.

- Show that lines, circles, and helices are geodesics, and that these are all of them.

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Or a sphere – the normal vector to the sphere always points toward/away from the center, and a path on the sphere which only accelerates toward the center is a great circle.

So that's one definition of a "line" – a curve that goes "straight" (or at least as straight as possible). It is a remarkable theorem that this agrees with the other definition of a "line": any length-minimizing curve is in fact a geodesic, and any geodesic is *locally length-minimizing*. That is, modifying the curve in a small neighborhood only increases the length of the curve.

In particular, the geodesics just depend on the length structure of the surface, not on how the surface is embedded in $\mathbb{R}^3$! If you bend a surface without stretching it, the geodesics don't change!

For example, a circular cylinder can be squashed into a cylinder over an ellipse, or a cylinder with a square cross section, and when you do that, the geodesics bend along with it – the geodesics are all still either lines, "circles", or "helices" (but now the circles and helices are bent a bit). We can even cut the cylinder along a line and unroll to get a rectangle. The rectangle is part of the plane, so we know the geodesics on the rectangle – they're just straight lines. So any straight line on the rectangle is a geodesic on the cylinder when you roll it back up, and any geodesic on the cylinder corresponds to a straight line on the rectangle (except that it might wrap around the cut line.)

This lets us calculate distances between points exactly! In general, this is difficult – if I have some arbitrary bumpy surface, it's tricky to find the shortest curve between two points, but here, it's easy. First, the shortest curve has to be a geodesic, so on the rectangle, it's a straight line that wraps around. So here's an example (figure omitted):

But that might not be the shortest way to get there, because the shortest way might wrap around the cylinder like so:

How do we figure out the geodesic here? Well, there are a couple ways: we could set up an equation for where the curve hits the boundary, or we could cut the rectangle here and reglue – that gives us the same cylinder, just drawn a different way. But the other thing we can do is this: take a copy of the rectangle and glue it on here. In fact, take a lot of copies, all glued together. Then, instead of wrapping around, we can just go to the next rectangle over – we just have to remember that really, all the copies of a point are "the same". So if we want to construct geodesics from u to v, we can look at all the different copies of v in this picture and draw lines from u to all of them. Then, in the original cylinder, we have geodesics from u to v that go straight along the cylinder, around the cylinder once, twice, n times, etc. This is all the geodesics, and if we want the shortest path from u to v, it has to be one of these. In fact, it has to be the line from u to the closest copy of v.

1.2. **Flat surfaces.** What if we want to do the same thing on the torus?

There's a problem – the torus isn't nearly as symmetric as a cylinder, so apart from a few examples, it's not clear what the geodesics should be (Examples: meridians, some longitudes).

But there is a very symmetric metric on the torus; take a square and glue the edges to form a torus. We can check that every point on this torus is surrounded by a neighborhood isometric to part of the plane (check the corners and edges).

So, on *this* torus, we know exactly what the geodesics are – they're straight lines! Furthermore, we can do the same trick as before to calculate distances:

- Example: distances and geodesics on the flat torus

This is very nice – if we have a torus like this (a flat torus), then geometry on the torus is a lot easier, and a big part of that is that every point looks like part of the plane. So, the big question:

What surfaces are like this?

If every point on a surface has a neighborhood which is isometric to part of the plane, we call it a *flat surface*. It turns out that having a flat metric is a pretty restrictive condition – spheres and genus-g surfaces don't have any flat metrics.
Why?

1.2.1. *Argument 1: Euler characteristic.*

Any compact manifold can be broken into triangles. If F is the number of faces, E the number of edges, and V the number of vertices, then the Euler characteristic, $F - E + V$, depends only on the manifold. Since all the faces are triangles, we know that $E = 3F/2$, because each triangle has three edges, each of which is in two triangles. If the manifold was flat, then all the vertices are surrounded by angle 2π, so $2\pi V = \pi F$, because we can count the sum of the angles in two ways – the angles of each triangle add up to π, and the angle around each vertex add up to 2π.

Together, these imply that $F - E + V = 0$, so (if you've taken "Classification of surfaces") a compact manifold with a flat metric must be a torus or a Klein bottle.

1.2.2. *Argument 2: Tilings of the plane.*

Suppose Σ is a flat surface.

Take a triangulation of Σ. We can form a tessellation of the plane by "unwrapping" the triangulation of Σ: start with a vertex, add triangles around that, pick another vertex, add more triangles, etc. We always have angle 2π around each vertex, so we don't get any overlaps around a vertex. (figure omitted). (In fundamental group terms, this is the universal cover of the surface.)

If the surface has holes (as opposed to handles), though, we can get overlaps when we go around a hole. (figure omitted) A surface is *complete* if it's complete as a metric space. Complete surfaces have no holes, so a triangulation of a complete surface leads to a tessellation of the plane.

Furthermore, every triangle shows up in many places, all basically the same, so the tessellation has a lot of symmetries. So, there's one copy of each polygon for each element of the fundamental group. And in fact, the symmetry group of the tiling contains the fundamental group. (In terms of the fundamental group, the deck transformations are symmetries of the tiling.)

This gives us a group of symmetries of the plane; each element of the fundamental group corresponds to an isometry of the plane.

What about the other direction? Say we have a group of symmetries. Then we can choose a *fundamental domain* for the tiling. That is, a set whose images tile the plane. Some of the boundary points on this fundamental domain are identified, and if we glue, we (sometimes) get a surface. This is the *quotient* of the plane by the action.

Ex: Torus. (figure omitted)

Warning: These aren't always flat surfaces! Ex: rotation by $\pi/3$ (cone). Group of translations along the x axis (line) (figures omitted)

If the orbit of a point is discrete (the orbit of any point is a discrete set) and if the action is free (none of the symmetries have a fixed point), then the quotient is a complete flat surface.

This is a one-to-one correspondence: every complete flat surface leads to a discrete, fixed-point free group of symmetries of the plane, and every discrete, fixed-point free group of symmetries has a quotient which is a complete flat surface.

So if we want to list the complete flat surfaces, we can list symmetry groups of the plane. Any symmetry of the plane is either

- a rotation
- a reflection
- a translation
- a glide reflection (a reflection followed by a translation along the axis)

If it has no fixed points, it must be a translation or a glide reflection. All the axes of the glide reflections have to be parallel, because otherwise, we could combine two glide reflections to get a rotation. So "half" the elements of the group have to be translations. (That is, the subgroup of translations is either the whole group or an index 2 subgroup.)

If you have three independent translations, the group can't be discrete, so we can break it down by looking at the number of independent translations.

- 0 translations:

    In this case, the group has to be trivial. The quotient of the plane by a trivial group is the plane.
- 1 translation:

    If the group only has translations, the surface is a cylinder. Otherwise, the quotient is a Möbius strip.
- 2 translations:

    If the group only has translations, the surface is a torus. Otherwise, it's a Klein bottle.

1.3. **Exercises.**

(1) A closed geodesic is a loop which is locally length-minimizing, so any segment is a geodesic. For example, every geodesic on the sphere is closed, and circles on a cylinder are closed geodesics.

    Let T be the torus formed by gluing the edges of the unit square. What are the geodesics on T? How can you decide which ones are closed? What do the closed geodesics look like? What do the non-closed geodesics look like?

(2) If you glue the edges of a sector in the plane with central angle θ, you get a cone. Describe the geodesics on a cone that avoid the vertex. How many geodesics are there that connect a point to itself? (Note that these aren't closed geodesics – they return to their starting point, but not in their starting direction.)

(3) Consider the surface of the unit cube. Each point has a neighborhood which is isometric to part of the plane except the eight corners.

    (a) Find the shortest path on the cube between the midpoint of one of the vertical edges and the center of the top face.

    (b) Find some closed geodesics on the cube. Can you find a closed geodesic that passes through all six faces? Can you find a closed geodesic that intersects itself?

    (c) Show that the shortest path on the cube between any two non-corner points never goes through a corner.