Homological and homotopical filling functions

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The isoperimetric problem

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If $\gamma : S^1 \to \mathbb{R}^2$, let $F(\gamma)$ be the "area" of γ. Then we want to calculate:

$$i(n) = \sup_{\substack{\alpha : S^1 \to \mathbb{R}^2 \\ \ell(\alpha) \leq n}} f(\alpha).$$

Generalizing filling area

Q: Given a curve $\alpha : S^1 \to X$, what's its filling area?

The geometric group theory perspective

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So by studying discs in X, we can get invariants related to the combinatorial group theory of G!

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So we can find minimal currents filling a curve by taking limits of surfaces whose area approaches the infimum!

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We call δ_X the homotopical Dehn function and FA_X the homological Dehn function.

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Theorem (Abrams, Brady, Dani, Guralnik, Lee, Y.)

Yes.

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(Equivalently, G has a $K(G,1)$ with finite n-skeleton.)

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So δ and FA are quantitative versions of $\mathcal{F}_2$ and FP_2.

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- K_Y *is finitely generated if and only if* Y *is connected*
- K_Y *is finitely presented if and only if* Y *is simply connected*
- K_Y *is* $\mathcal{F}_n$ *if and only if* Y *is* $n-1$*-connected*
- K_Y *is* FP_n *if and only if* Y *is homologically* $n-1$*-connected*

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Then if we glue infinitely many scaled copies of Y to X, the result should have small homological Dehn function!

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- Furthermore, there is a copy of $F_2 \times F_2$ in A_Y corresponding to that square. Let $E = K_Y \cap F_2 \times F_2$.
- Then E acts on a subset $L_E \subset L_Y$ made up of copies of γ.

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- So if we can find a copy of E in some other group D, we can amalgamate D and K_Y together along E.
- There are semidirect products $D = F_n \rtimes_{(\phi,\phi)} F_2$ which contain copies of E and have large Dehn functions.
- So we can construct an amalgam of D with several copies of K_Y, glued along E. This is a group with large homotopical Dehn function, but small homological Dehn function.

Theorem (ABDGLY)

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Open question: Is there a finitely presented group with $\delta \lnsim \mathrm{FA}$?

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