Homological and homotopical filling functions

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The isoperimetric problem

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Q: Given a curve of length $l$ in the plane, what’s the maximum area it can enclose? If $\gamma : S^1 \to \mathbb{R}^2$, let $F(\gamma)$ be the “area” of $\gamma$. Then we want to calculate:

$$i(n) = \sup_{\alpha : S^1 \to \mathbb{R}^2} f(\alpha).$$

$$\ell(\alpha) \leq n$$
Q: Given a curve $\alpha : S^1 \to X$, what’s its filling area?
Suppose that $X$ is a simply connected complex and $G$ acts geometrically (cocompactly, properly discontinuously, and by isometries) on $X$. Then:

- Paths in $X$ correspond to words in $G$.
- Loops in $X$ correspond to words in $G$ that represent the identity.
- Discs in $X$ correspond to proofs that a word represents the identity.

So by studying discs in $X$, we can get invariants related to the combinatorial group theory of $G$!
The geometric group theory perspective

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So we can find minimal currents filling a curve by taking limits of surfaces whose area approaches the infimum!
Two filling area functions

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$$\delta(\alpha) = \inf_{\beta : D^2 \rightarrow X} \text{area } \beta.$$ 

$$\text{with } \beta|_{S^1} = \alpha.$$
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Let $a$ be a 1-cycle.

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$$\text{FA}(a) = \inf_{\beta \text{ a 2-chain}} \text{area } \beta.$$  
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\delta_X(n) = \sup_{\alpha : S^1 \to X} \delta(\alpha).
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Question: Can we find nice spaces (say, spaces with a geometric action by some $G$) where these are different?
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**Theorem (Abrams, Brady, Dani, Guralnik, Lee, Y.)**

Yes.
Finitely generated and finitely presented are part of a spectrum of properties:
Homotopical finiteness properties

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- Any discrete group acts geometrically on some discrete metric space.
- $G$ acts geometrically on a connected complex $\iff G$ is finitely generated.
- $G$ acts geometrically on a simply-connected complex $\iff G$ is finitely presented.
- $G$ acts geometrically on a $n-1$-connected complex $\iff G$ is $F_n$.
  (Equivalently, $G$ has a $K(G,1)$ with finite $n$-skeleton.)
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Homological finiteness properties

$G$ is FP$_n$ if $\mathbb{Z}$ admits a projective resolution as a $\mathbb{Z}G$-module which is finitely generated in dimensions $\leq n$. 

In particular, if $G$ is FP$_n$, we can take a $K(G,1)$ with finite $n$-skeleton and consider its simplicial chain complex. Or if $G$ acts geometrically on some homologically $n$-connected space (i.e., with trivial $\tilde{H}_i(X;\mathbb{Z})$ for $i \leq n$).

$\delta$ and FA are quantitative versions of FP$_2$ and FP$_2$. 
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So $\delta$ and FA are quantitative versions of $\mathcal{F}_2$ and FP$_2$. 
Theorem (Bestvina-Brady)

*Given a flag complex $Y$, there is a group $K_Y$ such that $K_Y$ acts geometrically on a space consisting of infinitely many scaled copies of $Y$. Indeed, this space is homotopy equivalent to an infinite wedge product of $Y$'s.*
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- $K_Y$ is finitely generated if and only if $Y$ is connected

- $K_Y$ is finitely presented if and only if $Y$ is simply connected

- $K_Y$ is $F_n$ if and only if $Y$ is $n-1$-connected

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First version of construction

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- $\pi_1(Y)$ is nontrivial,
- $\pi_1(Y)$ is generated by conjugates of $\gamma$. 
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1. $H_1(Y)$ is trivial,
2. $\pi_1(Y)$ is nontrivial,
3. $\pi_1(Y)$ is generated by conjugates of $\gamma$.

Then if we glue infinitely many scaled copies of $Y$ to $X$, the result should have small homological Dehn function!
Let $Y$ be a flag complex with trivial $H_1$ and nontrivial $\pi_1$, normally generated by a single element $\gamma$. Say $\gamma$ is a path of length 4 in $Y$. Then, by Bestvina-Brady, the level set $L_Y$ is acted on by a subgroup $K_Y$, and $L_Y$ is made up of copies of $Y$. Furthermore, there is a copy of $F_2 \times F_2$ in $A_Y$ corresponding to that square. Let $E = K_Y \cap F_2 \times F_2$. Then $E$ acts on a subset $L_E \subset L_Y$ made up of copies of $\gamma$. 
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Modifying the construction for groups
So if we can find a copy of $E$ in some other group $D$, we can amalgamate $D$ and $K_Y$ together along $E$. 
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There are semidirect products $D = F_n \rtimes_{(\phi,\phi)} F_2$ which contain copies of $E$ and have large Dehn functions.

So we can construct an amalgam of $D$ with several copies of $K_Y$, glued along $E$. This is a group with large homotopical Dehn function, but small homological Dehn function.
Theorem (ABDGLY)

There is a subgroup of a CAT(0) group which has $\text{FA}(n) \lesssim n^5$ but $\delta(n) \gtrsim n^d$ for any $d$, or even $\delta(n) \gtrsim e^n$. 

In fact, if $\delta_k(n)$ is the $k$-th order homotopical Dehn function and $\text{FA}_k$ is the corresponding homological Dehn function, then:

Yes! (ABDGLY)

$\delta_2 \prec \text{FA}_2 \delta_2 \succ \text{FA}_2$

Yes (Y)
No (Gromov, White)

$\delta_3^{+} \prec \text{FA}_3^{+} \delta_3^{+} \succ \text{FA}_3^{+}$

No (Brady-Bridson-Forester-Shankar)
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Theorem (ABDGLY)

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\begin{align*}
\delta &\preceq \text{FA} \\
\delta^2 &\preceq \text{FA}^2 \\
\delta^3+ &\preceq \text{FA}^{3+}
\end{align*}

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\[ \delta \prec \text{FA} \quad \quad \delta \succ \text{FA} \]

\[ \delta^2 \prec \text{FA}^2 \quad \quad \delta^2 \succ \text{FA}^2 \]

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Yes!(ABDGLY) Yes!(Y) No(Gromov, White) No(Gromov, White) No(Brady-Bridson-Forester-Shankar) No(Gromov, White)
Open question: Is there a finitely presented group with $\delta \prec FA$?
Open question: Is there a finitely presented group with $\delta \not\preccurlyeq FA$? This would have to be a group where it’s harder to fill two curves of length $n/2$ than to fill any curve of length $n$. 