CONFORMAL LOOP ENSEMBLES

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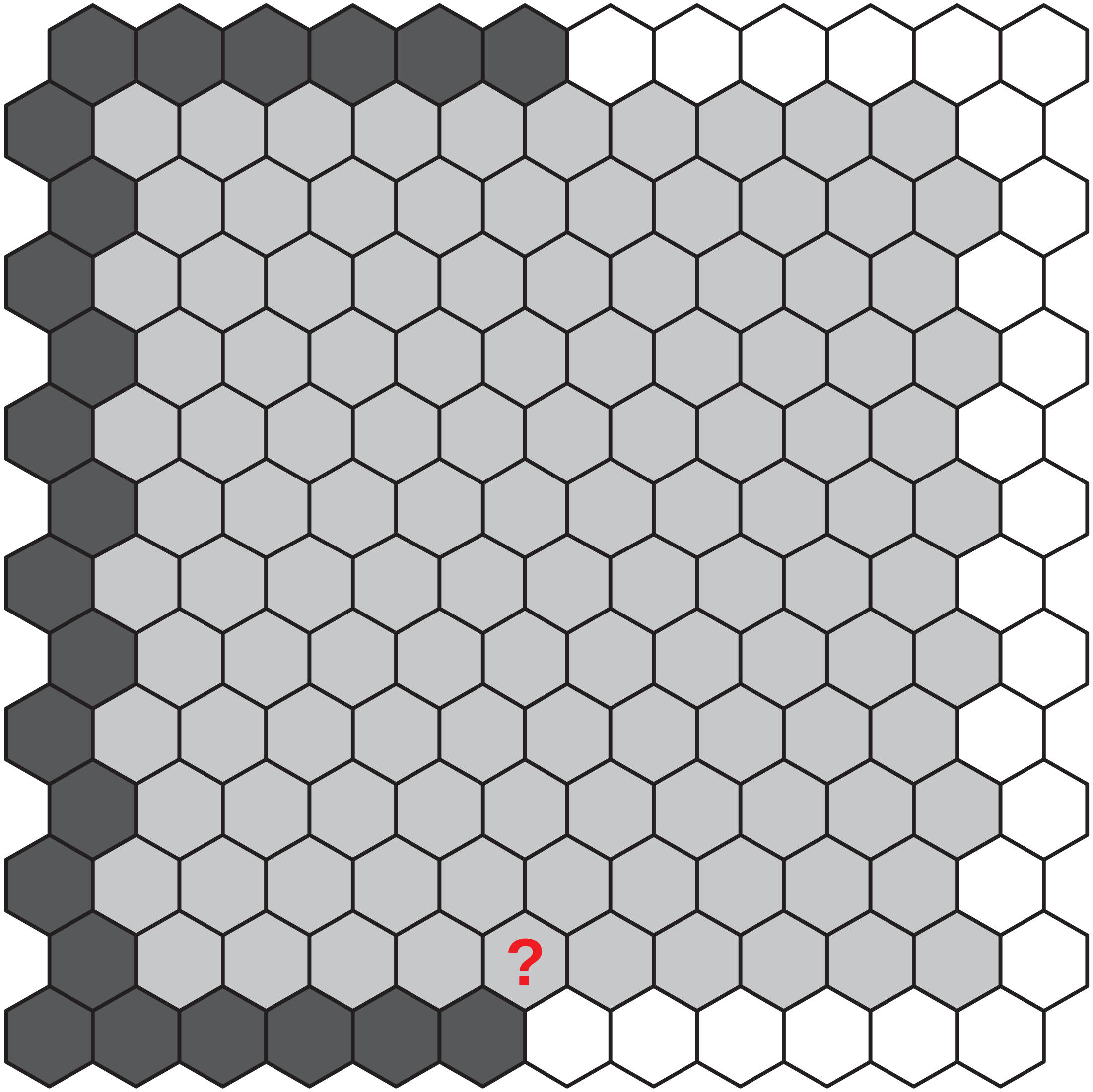
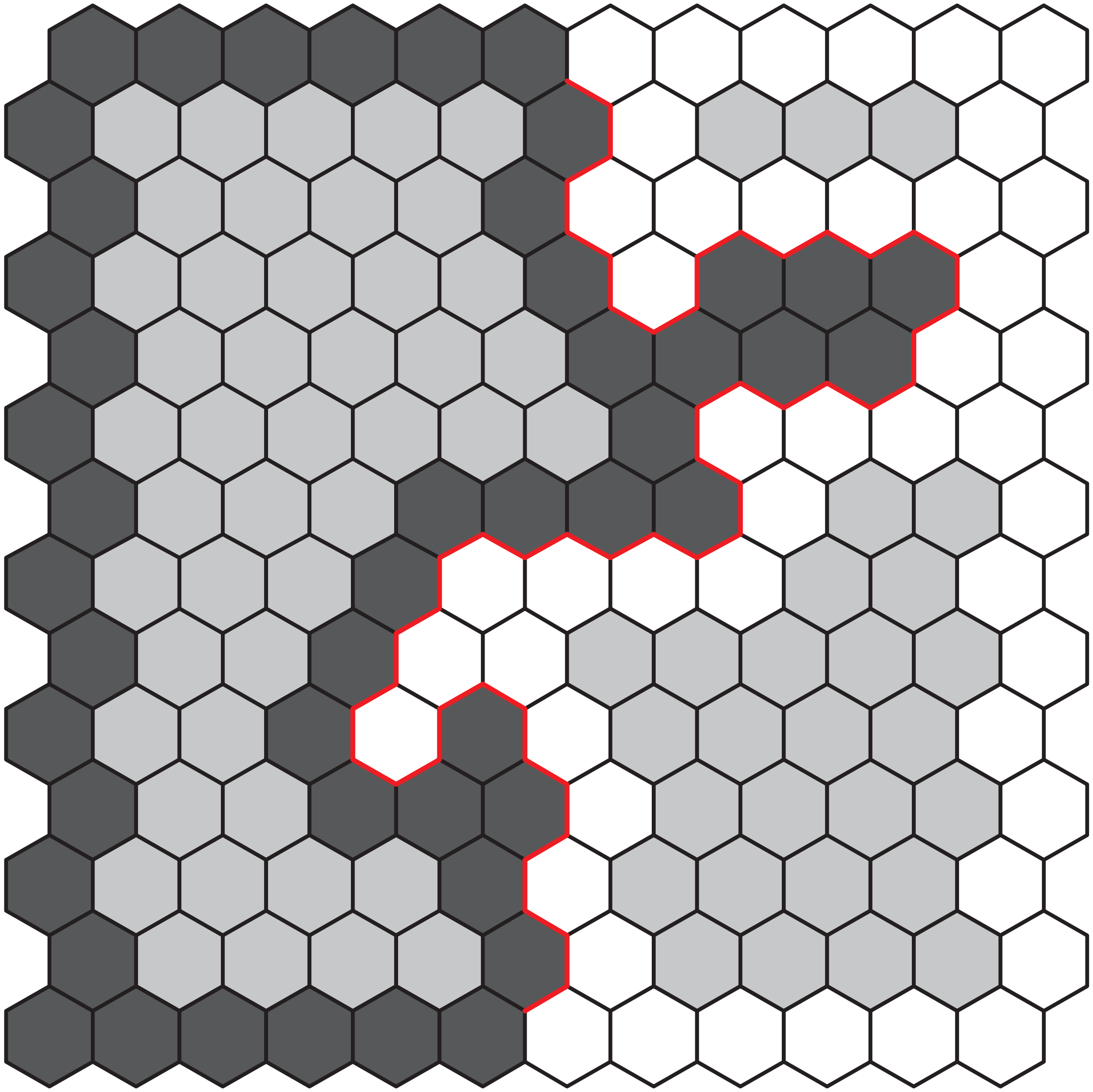
Black white boundary conditions



Consider this random interface



Scaling limits

THEOREM (Schramm; Smirnov; Camia and Newman): If hexagons are chosen by independent Bernoulli percolation, then as the mesh size tends to zero, the scaling limit of the above interface is SLE_6.

QUESTION: What if instead of Bernoulli percolation, we use the critical Ising model?

QUESTION: If we start with all black boundary conditions, what should we expect the scaling limit of the set of all loops to be?

Conformal invariance suggests that we may as well limit attention to outermost loops.

Conformal loop ensemble definition

Fix a bounded planar domain $D \subset \mathbb{R}^2$. Let Ω be the set of all countable collections of disjoint, non-nested simple closed loops in D. It has a natural σ-algebra $\mathcal{F}$, generated by event that two points belong to the same loop.

A *simple* **conformal loop ensemble** (CLE) in D is a random collection $\mathcal{L}$ of countably many pairwise disjoint simple closed loops in D with the following **conformal Markov property**: if A is any fixed simply connected subset of D, then *conditioned* on the random set A' of points that are not on or surrounded by a loop intersecting A^c, the law of the loops strictly contained in A is given by a product of independent loop ensembles—one for each component of A'—whose laws are the same as the original law of Ł conformally mapped to that component.

Conformal loop ensemble

THEOREM: [Sheffield, W.] There exists only a one-parameter family of simple CLEs. The loops look locally like SLE_κ, where $8/3 \leq \kappa \leq 4$.

We'll sketch the proof under an additional (unnecessary but also physically uncontroversial) assumption: that for each ϵ, the total number of loops with diameter greater than ϵ is almost surely finite.

Preliminary observations

Assume $\mathcal{L}$ is a CLE on D.

1. Each point $z \in D$ is almost surely surrounded by a loop in $\mathcal{L}$.
2. Every finite set of points has a positive probability of being entirely surrounded by the same loop in $\mathcal{L}$.
3. Fix two disjoint closed hulls A_1 and A_2 connected to boundary of D; then there is a positive probability that there is a single loop that intersects both of them. Same holds if there are three or more A_i.

Conditioning loop to hit boundary

1. As A_1 and A_2 tend to fixed points z_1 and z_2 on the boundary of D, the law of this loop (conditioned on there being one loop intersecting A_1 and A_2) has a weak limit. A similar statement holds for three points.
2. Observe a particular Markov property of this weak limit.
3. Make the connection with SLE_κ.
4. Deduce that loops have same law as set of excursion paths of $\text{SLE}_{\kappa,\kappa-6}$ for $8/3 \leq \kappa \leq 4$.

Loop soups and existence of CLEs

Consider set Λ of bounded closed simply connected subsets of $\mathbb{R}^2$ endowed with the σ-algebra $\mathcal{F}_L$ generated by the Hausdorff metric.
FACT: There is a unique measure ν^L on Λ satisfying the following:

1. **Conformal invariance:** If D_1 and D_2 are open subsets of $\mathbb{R}^2$ and g a conformal map between them, and $S \in \mathcal{F}_L$ is a subset of the set of loops contained in D_1, then $\nu^L(g(S)) = \nu^L(S)$.
2. **Normalization:** The ν^L measure of the set of loops encircling the unit disc and contained in the disc of radius e is 1.

We can restrict this measure on Λ to set of loops in D. Loop soup is Poisson point process derived from this measure with intensity c.

Closures of loop soup clusters are non-nested CLE for small c

When c is below a critical value c_0, the boundaries of the loop soup clusters are non-self intersecting simple loops almost surely. It is easy to see that they are non-nested CLEs, hence they correspond to the excursion loops of $\mathrm{SLE}_{\kappa,\kappa-6}$ for some κ

The expectation of the conformal radius of the origin-containing loop in an $\mathrm{SLE}_{\kappa,\kappa-6}$ is clearly strictly increasing in κ, tending to zero as $\kappa \to 8/3$. The expectation of the conformal radius of the origin containing loop in a loop soup cluster is also clearly strictly increasing in c, tending to zero as $c \to 0$.

We conclude that the $c \in [0, c_0)$ are in one-to-one correspondence with $\kappa \in [8/3, \kappa_0)$, and that the loop soup boundaries have the same law as the excursion loops in this range.

Adding CLEs: proving $\kappa_0 = 4$ is critical

Adding CLEs is very natural from the loop soup perspective. If Ł_1 and Ł_2 are sampled from distinct CLEs, we can take their union and look at the loops forming the outer boundaries of the clusters in this union (call that Ł_3). If these loops exist (i.e., there is not just one cluster in the union), then it is clear that they are also a CLE. We claim that if that c is small enough so that $\kappa \neq 4$, then we can add a very sparse loop soup (c very small) without making the cluster entirely connected.

Start with a set B. We can construct the cluster of loops of $\text{Ł}_1 + \text{Ł}_2$ intersecting B as follows. First let B_1 be the union of the filled loops of L_1 that intersect B. Let B_2 be the union of the filled loops of L_2 that intersect B_1. Let B_3 be the union of the filled loops of L_1 that intersect B_1, and so forth.

Does the capacity of the sequence B_i tend to infinity, or does it converge to a constant? Use SLE tools to show adding small c loop ensemble to CLE leaves capacity finite when $\kappa < 4$.

Conclusion

1. Uniqueness: if Ł is a non-nested CLE, then it has the same law as the excursion loops of branching $\mathrm{SLE}_{\kappa,\kappa-6}$, for some $8/3 \le \kappa \le 4$.
2. Existence: the cluster boundaries of the loop soup with intensity c have the same law as the excursion loops of branching $\mathrm{SLE}_{\kappa,\kappa-6}$ when $c = (3\kappa - 8)(6 - \kappa)/2\kappa$.

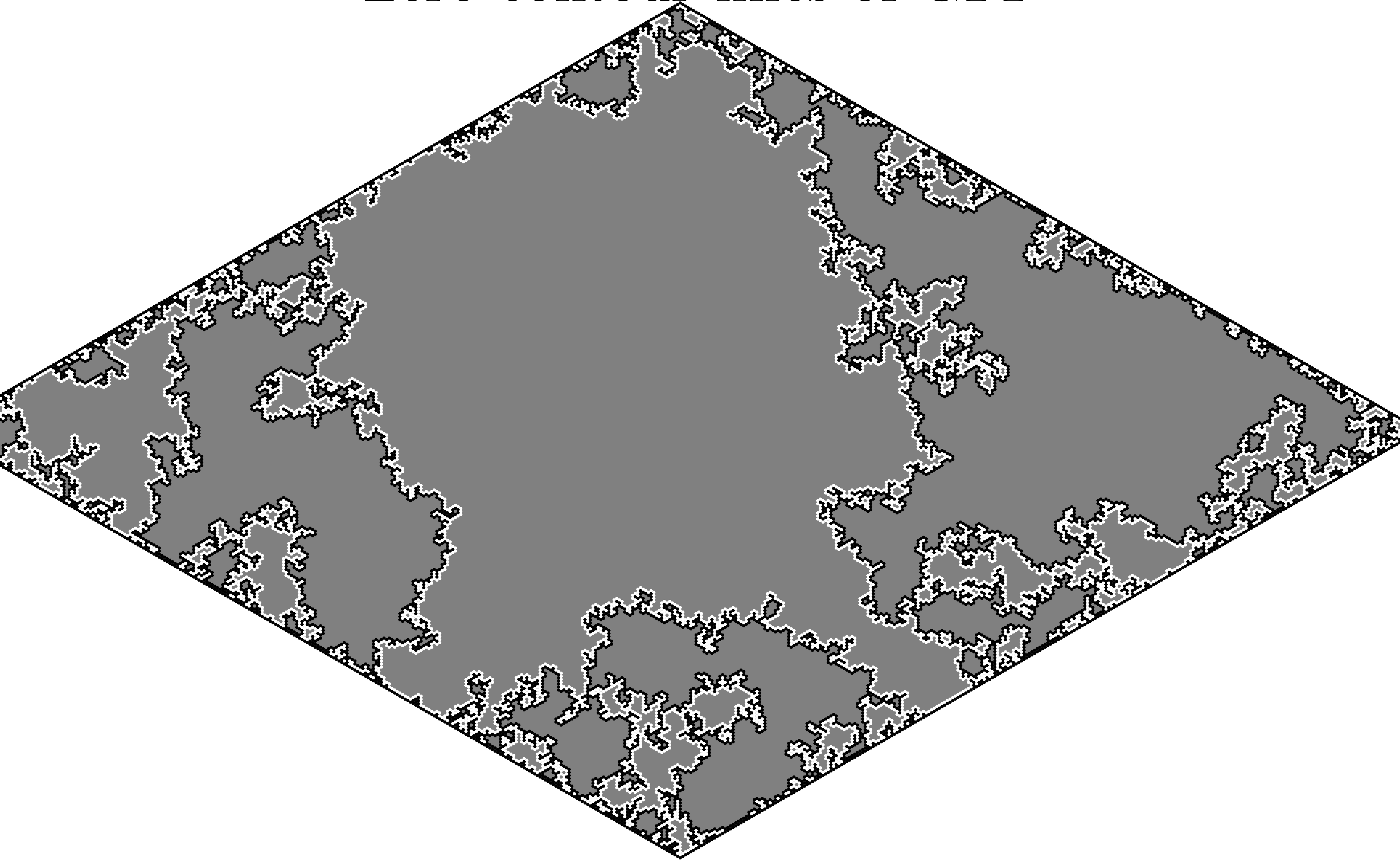
Gasket dimension

Consider set of all points of CLE_κ which are not in the interior of one of the loops.
Theorem (Schramm, Sheffield, Wilson): The “expectation dimension” of the CLE gasket is: $2 - \frac{(8-\kappa)(3\kappa-8)}{32\kappa}$.
This formula was predicted by Duplantier in 1990.
Theorem (Schramm, Sheffield, Wilson): The probability density for the conformal radius of the largest loop surrounding a point is given by $\frac{(\kappa/4-1)\cot(\pi(1-4/\kappa))}{\pi}$.
This formula was predicted by Cardy and Ziff in 2002.

Conformal loop ensembles CLE_κ, defined for $8/3 \leq \kappa \leq 8$, can be constructed as

1. the outer boundaries of the loop soup clusters ($8/3 \leq \kappa \leq 4$).
2. the excursion loops of branching $\mathrm{SLE}_{\kappa,\kappa-6}$ ($8/3 \leq \kappa \leq 8$).

Zero contour lines of GFF



Contour loop ensembles

Consider a GFF with zero boundary conditions in a planar domain D. Using iterated branching $\mathrm{SLE}_{4,a,b}$ it is possible to construct a "contour map" containing all of the contour lines in D whose heights belong to an arithmetic progression (e.g., all lines of integer height).

Theorem (Schramm, Sheffield): Given zero boundary conditions, the set of contour loops with heights that are odd multiples of $\sqrt{\pi/8}$ has the same law as CLE_4.

Conjecture: Scaling limit of double dimer model is CLE_4.

$O(n)$ models and conjectures

Let each coloring have probability proportional to $n^N x^L$ where N is the number of black-white boundary loops and L is their total combined length. Specifically, we conjecture (following conjecture written for SLE by Neinhuis and Kager, which in turn follows earlier work by Duplantier and Neinhuis and others) we conjecture that the scaling limit of the discrete loop ensemble is

1. CLE_κ, where $n = -2\cos(4\pi/\kappa)$ and $4 \leq \kappa \leq 8$, if $0 \leq n \leq 2$ and $x > x_c$.
2. CLE_κ, where $n = -2\cos(4\pi/\kappa)$ and $8/3 \leq \kappa \leq 4$, if $0 \leq n \leq 2$ and $x = x_c$.
3. trivial if either $x < x_c$ or $n \notin [0,2]$.
   where $x_c = [2 + (2-n)^{1/2}]^{-1/2}$. Note: x_c increases monotonically from .29 to .71 as n increases from 0 to 2, and for each $n \in (0,2)$, the equation $n = -2\cos(4\pi/\kappa)$ has two solutions, one in $(8/3, 4)$ and one in $(4, 8)$.

Special cases

1. $n = 1$, $x = 1$ (Bernoulli percolation, $p = 1/2$): CLE$_6$ [Smirnov; Schramm; Camia and Newman].
2. $n = 1$, $x > x_c = 1/\sqrt{3}$ (supercritical Ising model): CLE$_6$.
3. $n = 1$, $x = x_c = 1/\sqrt{3}$ (critical Ising model): CLE$_3$.
4. "$n = 0$" (i.e., condition on having just one loop), $x > x_c$: CLE$_8$ (also, outer boundary of UST, [Lawler, Schramm, Werner])