Spring 2016: Advanced Topics in Numerical Analysis:
Computational and Variational Methods for Inverse Problems
Assignment 1 (due Feb. 25, 2016)

We consider a linear, finite-dimensional inverse problem in $\mathbb{R}^N$ ($N \in \mathbb{N}$):

$$\boldsymbol{d} = K\boldsymbol{p} + \boldsymbol{n},$$

with $\mathbb{R}^N$-vectors $\boldsymbol{d}$ (data/measurements), $\boldsymbol{p}$ (parameters/image) and $\boldsymbol{n}$ (noise/errors). The matrix $K \in \mathbb{R}^{N \times N}$ is symmetric and positive definite. The eigenvalue decomposition of K shows that there is an orthonormal matrix $U \in \mathbb{R}^{N \times N}$ with columns $\boldsymbol{u}_i$ and eigenvalues λ_i, with $\lambda_1 \geq \lambda_2 \geq \ldots > \lambda_N \geq 0$ such that

$$K = U\text{diag}(\lambda_i)U^T.$$

Note that the orthonormality of U implies that $U^T = U^{-1}$. MATLAB examples shown in the class as well as files required for this assignement can be downloaded from http://math.nyu.edu/~stadler/inv16/assignment1/.

1. **[Properties of SVD-based filtering]**

   (a) Show that, for $\alpha > 0$ the Tiknonov-filtered reconstruction of the parameter field, i.e.,

   $$\boldsymbol{p}_\alpha = \sum_{i=1}^{N} \frac{\lambda_i}{\lambda_i^2 + \alpha}(\boldsymbol{u}_i^T \boldsymbol{d})\boldsymbol{u}_i$$

   can also be derived as

   $$\boldsymbol{p}_\alpha = (K^T K + \alpha I)^{-1} K^T \boldsymbol{d}$$

   Why is $K^T K + \alpha I$ invertible?

   (b) Show that $\boldsymbol{p}_\alpha$ is also the solution of the minimization problem

   $$\min_{\boldsymbol{p} \in \mathbb{R}^N} \frac{1}{2}\|K\boldsymbol{p} - \boldsymbol{d}\|^2 + \frac{\alpha}{2}\|\boldsymbol{p}\|^2.$$

   (c) Let us verify a property we used to estimate the reconstruction error due to noise in the data as the noise level tends to zero. Show that for both, the TSVD and Tikhonov filtering functions $\omega(\lambda^2)$ holds

   $$\omega_\alpha(\lambda^2)\lambda^{-1} \leq \frac{1}{\sqrt{\alpha}},$$

   where $\lambda, \alpha > 0$.

2. Discretize the blurring operator[1]

   $$d(x) = \int_0^1 k(x - x')p(x')\,dx' \quad \text{for } 0 < x < 1.$$

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[1]Compare with the MATLAB code `deconv1D.m`.

with 200 discretization points, with $k(x) = c^{-2} \max(0, c - |x|)$ with $c = 0.2$. Use the resulting matrix K to blur the function

$$\boldsymbol{p}_{\text{true}}(x) = \begin{cases} 1 & 0.1 < x < 0.25 \\ 0.25 & 0.25 < x < 0.4 \\ \sin^4(2\pi x) & 0.5 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Add normally distributed noise $\boldsymbol{n}$ with zero mean and variance $\sigma^2 = 0.05$. The resulting blurred and noisy data is $\boldsymbol{d} = K\boldsymbol{p}_{\text{true}} + \boldsymbol{n}$.

(a) Use the truncated singular value decomposition[2] filter (TSVD) with $\alpha = 0.0001, 0.001, 0.01, 0.1$ to compute the TSVD-regularized reconstructions $\boldsymbol{p}_\alpha$.

(b) Use the Tikhonov filter with the same values for α for the reconstruction.

(c) Compute the (approximate) optimal value for the regularization parameter α in the Tikhonov regularization using the L-curve method.

(d) Compute the (approximate) optimal value for the regularization parameter α in the Tikhonov regularization using Morozov's discrepancy criterion

$$D(\alpha) := \|K\boldsymbol{p}_\alpha - \boldsymbol{d}\| = \delta$$

with $\delta = \|n\|$ and $\boldsymbol{p}_\alpha$ is the solution of the Tikhonov problem corresponding to the regularization parameter α.

(e) Plot $\|\boldsymbol{p}_{\text{true}} - \boldsymbol{p}_\alpha\|$ as a function of α, where $\boldsymbol{p}_\alpha$ is the Tiknonov regularized solution. Which value of α (approximately) minimizes the norm[3] of this difference? How does that value compare with the estimates found through the L-curve and the discrepancy criterion?

3. We use Tikhonov regularization to reconstruct the blurred and noisy 2D image shown in Figure 1. A MATLAB code to create the data and to compute the Tikhonov reconstruction is provided (function conv2D.m)[4]. Note that the linear system in the Tikhonov regularization is solved iteratively with the conjugate gradient (CG) method[5] (rather than with a direct solver) since the 2D-blurring operator matrix K is not explicity available (but we can apply it using 1D-blurring operators in x- and y-direction). In the solution of inverse problems with PDEs, one often encounters the case that a linear operator/matrix is only available through its action on vectors.

(a) What would be the size of the matrix K if we followed the same approach as in the previous example?

(b) Compute a (near-)optimal regularization parameter for Tikhonov regularization using the L-curve criterion.

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[2]MATLAB provides the function svd to compute the singular value decomposition of a matrix.

[3]Obviously, the resulting criterion to find a good regularization parameter can only be used if the true solution is known. Hence, this allows to test the performance of the other criteria but cannot be used in practice.

[4]Before running this function, an image has to be imported as matrix into MATLAB's workspace: Choose File→Import Data and select the image coto.png. You are welcome to use any other image you like, but will have to adjust the image size in conv2D.m if you choose a different image.

[5]If you have not heard about this method yet, don't worry—we will cover it in class. The CG method is simply an (approximative) way to solve linear systems without computing matrix factorizations.
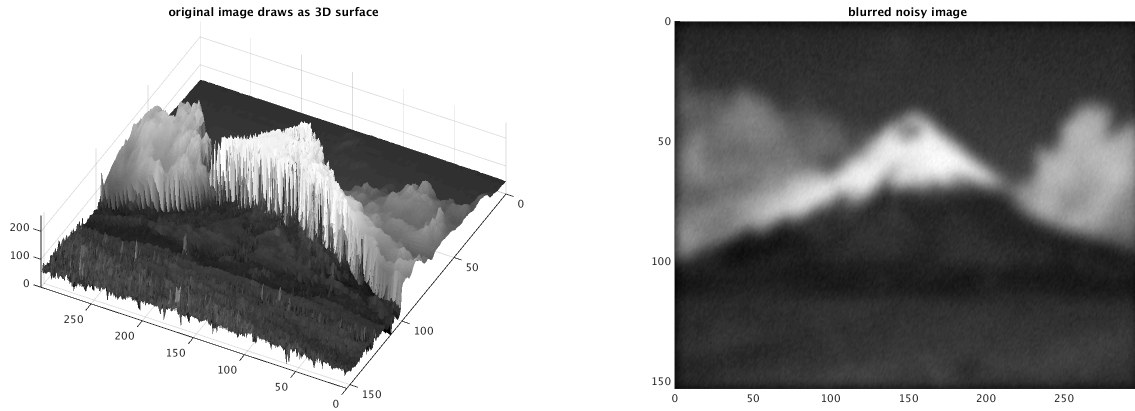
Figure 1: In MATLAB (and in image processing in general), images are represented as matrices, in which each entry corresponds to the gray value of a pixel. Left: Original image plotted as surface seen from the side. The image only has one channel (gray values), which is plotted as z-component. Right: Blurred and noisy image, from which we attempt to reconstruct the original image.

    (c) Which value of α minimizes the norm of the difference between the real image and the Tikhonov reconstruction?