Spring 2016: Advanced Topics in Numerical Analysis: Computational and Variational Methods for Inverse Problems Assignment 4 (due April 18, 2016)

1. **Frequency-domain inverse wave propagation problem.** We formulate and solve an inverse acoustic wave propagation problem in the frequency domain, which is governed by the Helmholtz equation. Let $\Omega \subset \mathbb{R}^n$ be bounded ($n \in \{2,3\}$). The idea is to propagate harmonic waves from multiple sources $f_j(x)$, $1 \le j \le N_s$ at multiple frequencies ω_i, $1 \le i \le N_f$ into a medium and measure the amplitude $u_{ij}(x)$ of the reflected wave fields at points x_k, $1 \le k \le N_d$ for each source and frequency in order to infer the soundspeed $c(x)$ of the medium. The inverse problem of estimating the locally varying slowness (i.e., the inverse of the wave speed) is as follows:

$$\min_m J(m) := \frac{1}{2}\sum_i^{N_f}\sum_j^{N_s}\sum_k^{N_d}\int_\Omega (u_{ij} - u_{ij}^{obs})^2 \delta(x - x_k)\,dx + \frac{\alpha}{2}\int_\Omega \nabla m \cdot \nabla m\,dx,$$

where u_{ij} depends on the medium parameter $m(x)$, which equals $1/c(x)^2$, through the solution of the Helmholtz problems

$$\begin{aligned} -\Delta u_{ij}(x) - k_i^2 m(x) u_{ij}(x) = f_j(x) \text{ in } \Omega, \quad i = 1,\dots,N_f, \quad j = 1,\dots,N_s,\\ u_{ij} = 0 \text{ on } \partial\Omega. \end{aligned}$$

In the above problem, $u_{ij}^{obs}(x)$ denotes given measurements (for frequency i and source j), u_{ij} is the amplitude of the acoustic wave field, and $\alpha > 0$ is the regularization parameter. Moreover, $\delta(x - x_k)$ is the Dirac δ-distribution, i.e., $\int_\Omega f(x)\delta(x - x_k)\,dx = f(x_k)$ for a continuous function f.[1]

   (a) Compute the gradient of J using the Lagrangian method for a single source and frequency, i.e., for $N_f = N_s = 1$.

   (b) Compute the application of the Hessian to a vector in the single source and frequency case.

   (c) Compute the gradient for an arbitrary number of sources and frequencies[2]. How many state and adjoint equations have to be solved for a single gradient computation?

2. **Inverse elliptic advection-diffusion problem.** We would like to solve the inverse problem for the advection-diffusion equation on $\Omega = [0,1] \times [0,1]$:

$$\min_a J(m) := \frac{1}{2}\int_\Omega (u - u^{obs})^2\,dx + \frac{\gamma}{2}\int_\Omega \nabla m \cdot \nabla m\,dx, \tag{1}$$

where the field $u(x)$ depends on the diffusivity $m(x)$ through the solution of

$$\begin{aligned} -\nabla \cdot (m\nabla u) + v \cdot \nabla u = f \text{ in } \Omega,\\ u = 0 \text{ on } \partial\Omega, \end{aligned} \tag{2}$$

with the advective velocity $v = (v_1, v_2)$, regularization parameter $\gamma > 0$ and $f(x)$ is a source term. We synthesize the measurement data $u^{obs}(x)$, by solving the forward advection-diffusion

[1] Note that we assume that the solutions of the Helmholtz equation are continuous such that this point evaluation is well defined. The δ-Dirac function is then just a notation for point evaluation.

[2] Every state equation with solution u_{ij} has a corresponding adjoint variable p_{ij}. The Lagrangian functional involves the sum over all state equations.

equation with $m(x, y) = 4$ for $(x - 0.5)^2 + (y - 0.5)^2 \leq 0.2^2$ and $m(x, y) = 8$, and a source $f \equiv 1$. Noise is added to this "data" to simulate actual instrumental noise.

We provide an implementation `PoissonDeterministic-SD.py`, which implements a steepest descent method[3] for this problem with advective velocity $v = 0$ (i.e., a pure diffusion problem).

(a) Report the solution of the inverse problem and the number of required iterations for the following cases:
   - Noise level of 0.01 (roughly 1% noise), and regularization $\gamma = 5 \cdot 10^{-10}$.
   - Same, but with $\gamma = 0$, i.e., no regularization[4].
   - No noise, and use $m \equiv 4$ and $m \equiv 8$ as initialization for the paramter. Do you find the same solution? Try to explain the behavior.

(b) Add the advective term with $v = (30, 0)$ to the implementation and plot the resulting reconstruction of m for a noise level of 0.01 and for a reasonably chosen regularization parameter.

(c) Since the coefficient m is discontinuous, a better choice is to use total variation regularization rather than Tikhonov regularization to prevent an overly smooth reconstruction. Modify the implementation and plot the result for a reasonably chosen regularization parameter. Use $\varepsilon = 0.1$ to regularize the non-differentiable TV regularization term (ε has the same meaning as in the previous assignment). Do not forget to adjust the cost function, which is used in the line search, properly.

[3]Note that we do not want to advocate the use of the steepest descent method for variational inverse problem since its convergence can be very slow. For all tests, please use a maximum number of 300 iterations—you will need it! In the next assignment, we will solve this problem using an inexact Newton-CG method, which is more efficient.

[4]Inverse problems can also be "iteratively regularized", which means that iterative methods are terminated early in an iterative process. This can allow a reasonable solution also without regularization. The basic idea is that iterative methods such as the conjugate gradient method captures the most important modes, which are determined by the data in its initial iterations. A critical issue in iterative regularization is to find appropriate termination criteria such that the important information is found while the iterative method has not targeted the noisy modes yet. See for instance the book by Kaipio and Somersalo for more on iterative regularization methods.