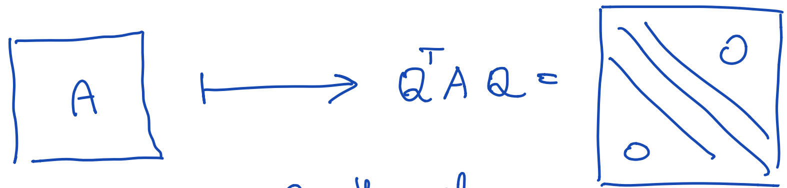
Now, we aim at methods to compute <u>all</u> eigenvalues of a matrix. We'll use 2 steps:

<u>Step 1:</u> Transform the matrix A, $A^T = A$ to a tridiagonal matrix without changing its eigenvalues



Q orthogonal matrix

<u>Step 2:</u> Find eigenvalues of tridiagonal matrices iteratively $\longrightarrow$ qr-method

Today we'll discuss Householder transformation, which is also an orthogonalization method (i.e. computes orthogonal bases), other ways are Givens rotations (plane rotations) or Gram-Schmidt method (which is not stable w.r.t. roundoff errors).

§5.5 Householder's method (for tridiagonalization)

<u>Goal:</u> Reduce a matrix to tri-diagonal form using orthogonal transformations

<u>Def:</u> For $v \in \mathbb{R}^n$, $v \neq 0$ define

$$H = H(v) = I - \frac{2}{v^T v} \cdot v v^T \in \mathbb{R}^{n \times n}$$

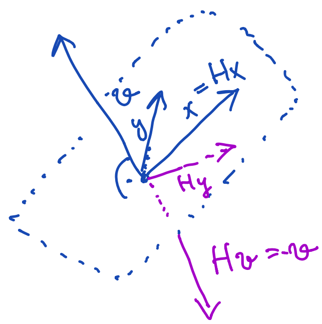
"Householder matrix"

$x \in \mathbb{R}^n$: $\quad Hx = x - \frac{2}{v^T v} v(v^T x) = x - \frac{2}{v^T v}(v^T x)\, v$

$\Longrightarrow$ $Hx, x, v \in \mathbb{R}^n$ are in the same (hyper) plane

$$Hv = v - \frac{2}{v^T v}(v^T v)\, v = -v$$

$$x \perp v \longrightarrow Hx = x - \frac{2}{v^T v} \underbrace{(v^T x)}_{=0} v = x$$



Lemma: Householder reflections are symmetric and orthogonal.

Proof: Symm: $H = I - \frac{2}{v^T v} v v^T$ ✓

orthogonal:

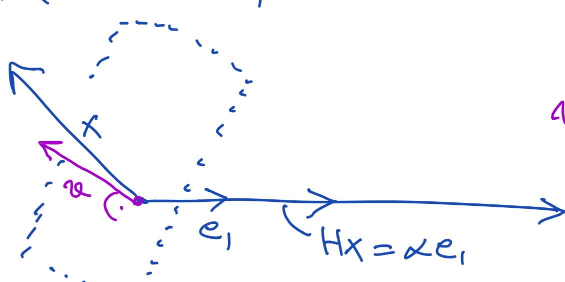
$$H^T H = H^2 = \left(I - \frac{2}{v^T v} v v^T\right)^2$$

$$= I - \frac{4}{v^T v} v v^T + \frac{4}{(v^T v)^{\cancel{2}}} \underbrace{(v v^T)(v v^T)}$$

$$= v(v^T v) v^T = \cancel{(v^T v)}\, v v^T$$

$$= I \quad ✓$$

Lemma: Let $x \in \mathbb{R}^n$, $x \neq 0$. Then there exists a Householder matrix H such that all but the first element in Hx are zero, i.e. $Hx = \alpha e_1$, $\alpha \neq 0$.



$$v = x + c e_1$$

Want: $Hx = \alpha e_1$, $H = H(v) = I - \frac{2}{v^Tv} vv^T$

$$v^Tx = (x + ce_1)^T x = x^Tx + c\underbrace{e_1^Tx}_{\beta}$$

$$v^Tv = (x+ce_1)^T(x+ce_1) = x^Tx + 2c\beta + c^2$$

$$Hx = x - \frac{2}{v^Tv}(v^Tx)v = x - \frac{2(x^Tx + c\beta)(x+ce_1)}{x^Tx + 2c\beta + c^2}$$

$$= \frac{(\cancel{x^Tx} + \cancel{2c\beta} + c^2 - \cancel{2}x^Tx - \cancel{2c\beta})x - 2c(x^Tx + c\beta)e_1}{x^Tx + 2c\beta + c^2}$$

$$= \frac{(c^2 - x^Tx)x - 2c(x^Tx + c\beta)e_1}{x^Tx + 2c\beta + c^2}$$

$\Longrightarrow$ Choose c such that $c^2 - x^Tx = 0$ and $x^Tx + 2c\beta + c^2 \neq 0$

$$c = \begin{cases} \operatorname{sgn}\beta \sqrt{x^Tx} & \beta \neq 0 \\ \sqrt{x^Tx} & \text{if } \beta = 0 \end{cases}$$

$\Longrightarrow$ $Hx = -ce_1$ as required

(i.e. $\alpha = -c$) $\square$

Example:

$$x = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} \in \mathbb{R}^4 \qquad v = x + \operatorname{sgn}(1)\sqrt{10}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \sqrt{10} \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

$\Longrightarrow$ $H = I - \frac{2}{v^Tv} vv^T$

Theorem: Given $A \in \mathbb{R}^{n\times n}$, $A^T = A$, $n \geq 3$. Then there exists $Q_n \in \mathbb{R}^{n\times n}$, product of $n-2$ Householder matrices

$$Q_n = H_{(n,n-1)} \cdot H_{(n,n-2)} \cdot \ldots \cdot H_{(n,2)}$$

such that $Q_n^T A Q_n = T_n$ ... tridiagonal.

Proof: (Sketch)

$$A = \begin{bmatrix} \alpha & - & b^T & - \\ | & & & \\ b & & C & \\ | & & & \end{bmatrix} \xrightarrow[\text{from left}]{H_{(n,n-1)}}$$

$$H_{(n,n-1)} = \begin{bmatrix} 1 & 0 & - & 0 \\ 0 & & & \\ | & & H_{n-1} & \\ 0 & & & \end{bmatrix}$$

H_{n-1} is a Householder matrix that maps b to a multiple of e_1 in $\mathbb{R}^{n-1}$.

$$\longrightarrow \begin{bmatrix} \alpha & - & b^T & - \\ * & & & \\ 0 & & C' & \\ \vdots & & & \\ 0 & & & \end{bmatrix} \xrightarrow[\text{from right}]{H_{(n,n-1)}} \begin{bmatrix} \alpha & * & 0 & - & 0 \\ * & & & & \\ 0 & & C'' & & \\ | & & & & \\ 0 & & & & \end{bmatrix}$$

$$\xrightarrow[\text{with } C'']{\text{repeat}} \ldots \longrightarrow \begin{bmatrix} \alpha & * & & 0 \\ * & * & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \end{bmatrix}.$$

Example:

$$A = \begin{pmatrix} 4 & 1 & 2 & 1 & 2 \\ 1 & 3 & 0 & -3 & 4 \\ 2 & 0 & 1 & 2 & 2 \\ 1 & -3 & 2 & 4 & 1 \\ 2 & 4 & 2 & 1 & 1 \end{pmatrix}$$

1.) Find H s.t. $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$ becomes mult. of e_1

$$H = \begin{bmatrix} 1 & 0 & - & 0 \\ 0 & & & \\ | & & H_{n-1} & \\ 0 & & & \end{bmatrix}$$

$$H_{n-1} = I - \frac{2}{v^Tv} v v^T$$

$$v = \begin{pmatrix} 1+\sqrt{10} \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

$\Longleftrightarrow$ use Householder in $\mathbb{R}^5$ with

$$w = \begin{pmatrix} 0 \\ 1+\sqrt{10} \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

FLOPS: $\frac{1}{3} n^3$ for tridiagonalization.