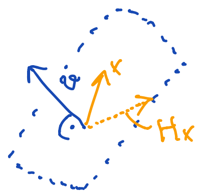
Householder matrices, $v \in \mathbb{R}^n$

$$H = H(v) = I - \frac{2}{v^T v}\, v v^T \in \mathbb{R}^{n \times n}$$

is a reflection on the hyperplane $\perp v$



$x \in \mathbb{R}^n$ map to αe_1 mult. of unit vector

$$v = x - \overset{\mathrm{sgn}(\beta)}{\sqrt{x^T x}}\; e_1$$

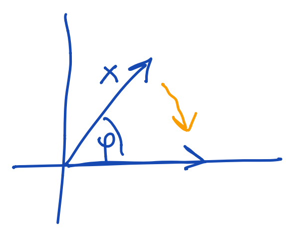
$$\overset{A}{\begin{bmatrix} * & \cdots & * \\ * & & \\ \vdots & & \\ * & \cdots & * \end{bmatrix}} \xrightarrow{HAH^T} \begin{bmatrix} * & * & 0 & \cdots & 0 \\ * & & & & \\ 0 & & * & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}$$

Plane rotations, Givens rotations (§5.3)

Besides reflections, rotations are also orthogonal transformations

In 2D, a rotation has the form

$$R(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$



$$c^2 + s^2 = 1$$

Properties: $R(\varphi)^T = R(-\varphi)$

$$R(\varphi) R(-\varphi) = I$$

Plane rotations in $\mathbb{R}^n$:

$$R^{kl} = \begin{array}{c} \\ k \\ l \end{array} \overset{\qquad\qquad k \qquad\quad l}{\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & c & & s & \\ & & & & 1 \ddots 1 & & \\ & & & -s & & c & \\ & & & & & & 1 \ddots 1 \end{pmatrix}} \in \mathbb{R}^{n \times n}$$

$$R^{kl} x = \begin{array}{cc} \begin{pmatrix} x_1 \\ \vdots \\ x_{k-1} \\ c x_k + s x_l \\ \vdots \\ -s x_k + c x_l \\ \vdots \\ x_n \end{pmatrix} & \begin{array}{c} \\ \\ \\ k \\ \\ l \\ \\ \\ \end{array} \end{array}$$

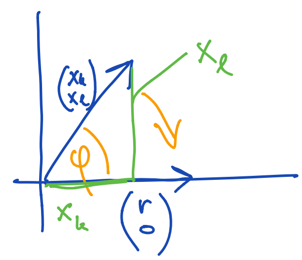
Can we choose φ (and thus c, s) such that $-s x_k + c x_l = 0$

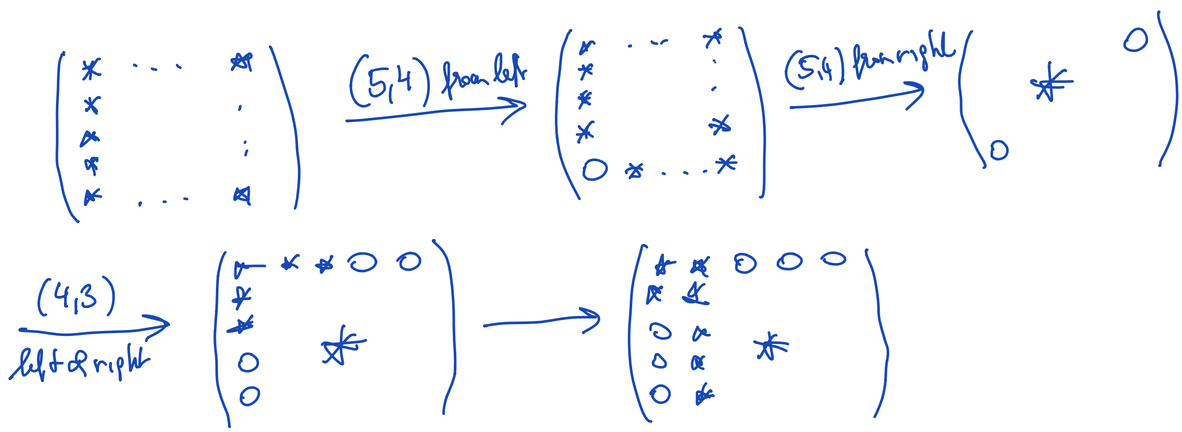
$$\begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} x_k \\ x_l \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix}$$

$$r = \sqrt{x_k^2 + x_l^2}$$

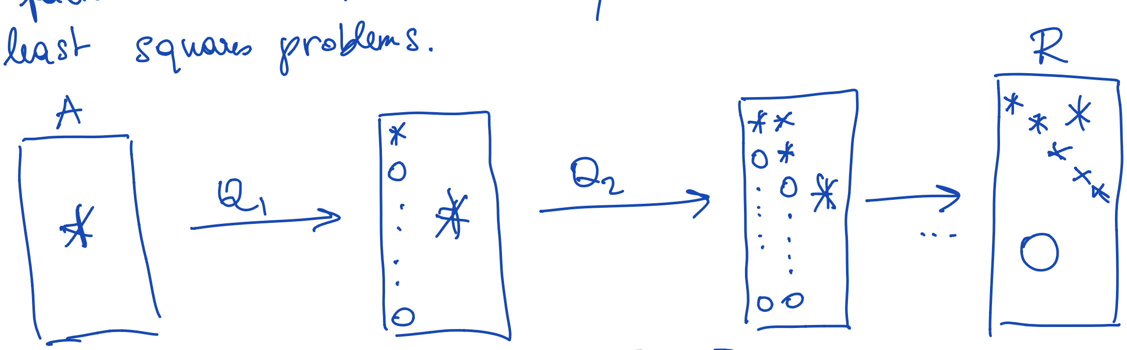
$$c = \cos(\varphi) = \frac{x_k}{r}$$

$$s = \sin(\varphi) = \frac{x_l}{r}$$





Both, Householder & Givens can be used to compute QR-factorizations of $A \in \mathbb{R}^{m \times n}$, $m \geq n$ as occurring in least squares problems.



$$Q_n \cdots Q_2 Q_1 A = R \Longrightarrow A = \underbrace{Q_1^T \cdots Q_n^T}_{Q} R = QR$$

The QR algorithm for eigenvalues of tridiagonal matrices (§5.7)

$$A = \begin{bmatrix} \ddots & \ddots & & 0 \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix}$$

$A \in \mathbb{R}^{n\times n}$, symmetric, tri-diagonal

The QR algorithm computes matrices $A^{(k)}$, $k = 0, 1, 2, \dots$ starting from $A^{(0)} = A$:

for $k = 0, 1, 2, \dots$

- compute QR decomposition of $A^{(k)}$, $A^{(k)} = QR$
- $A^{(k+1)} = RQ$

end

This algorithm converges to a diagonal matrix that contains the eigenvalues of A

First: Eigenvalues of $A = A^{(0)}, A^{(1)}, \dots$ are the same because:

$$A^{(k)} = QR \Longrightarrow Q^T A^{(k)} = R$$

$$\begin{aligned} A^{(k+1)} &= RQ \\ &= Q^T A^{(k)} Q \end{aligned}$$

$\Longrightarrow$ so they have the same eigenvalues as this is only a similarity transform.