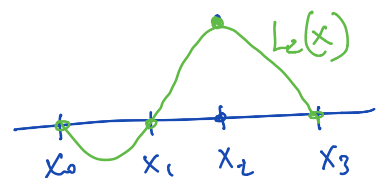
Lagrange interpolation

Int'l problem
$$\left[\begin{array}{l} x_0,\dots,x_n \in \mathbb{R}, \quad y_0,\dots,y_n \in \mathbb{R}, \quad x_i \neq x_j \text{ for } j \neq i \\ \text{Find } p_n \in P_n \text{ such that } p_n(x_i) = y_i \quad i=0,\dots,n \end{array}\right.$$

Lagrange polynomials $\quad L_k(x) = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}$

$$L_k(x_i) = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}$$



$$p_n(x) = \sum_{k=0}^{n} L_k(x)\, y_k \in P_n$$

space of polynomials of degree $\leq n$

<u>Lemma</u>: $n \geq 1$, For given distinct points $x_0,\dots,x_n$, $y_0,\dots,y_n \in \mathbb{R}$ there exist a unique $p_n \in P$ that satisfies $p_n(x_i) = y_i \quad i=0,\dots,n$.

<u>Proof</u>: <u>Existence</u> ✓

<u>Unique</u>: Let $p_n, q_n \in P_n$ be interpolating polynomials

$\Longrightarrow p_n(x_i) - q_n(x_i) = y_i - y_i = 0 \quad i=0,\dots,n$

$p_n - q_n \in P_n$ with $(n+1)$ roots

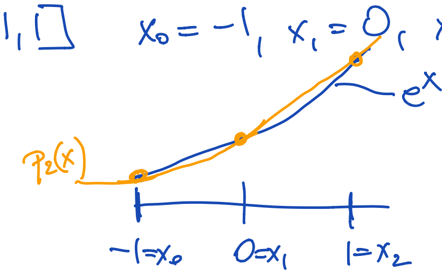
$\Longrightarrow p_n - q_n = 0 \Longrightarrow p_n = q_n \quad \square$

The $\boxed{p_n = \sum_{k=0}^{n} L_k(x)\, y_k}$ is the unique Lagrange interpolating polynomial

Given $f : \mathbb{R} \to \mathbb{R}$, then

$p_n = \sum_{k=0}^{n} L_k(x)\, f(x_k)$ is the unique Lagrange polynomial that interpolates f.

Example: $f : x \mapsto e^x$ on $[-1,1]$, $x_0 = -1$, $x_1 = 0$, $x_2 = 1$



$$L_0(x) = \prod_{\substack{k=0 \\ k\neq 0}}^{2} \frac{(x-x_i)}{(x_0-x_i)} =$$

$$= \frac{(x-0)(x-1)}{(-1-0)(-1-1)} = \frac{1}{2}x(x-1)$$

$$L_1(x) = 1-x^2, \quad L_2(x) = \frac{1}{2}x(x+1)$$

$$\Longrightarrow \quad p_2(x) = \frac{1}{2}x(x-1)e^{-1} + (1-x^2)\cdot 1 + \frac{1}{2}x(x+1)e^{1}$$

$$= 1 + x \sinh 1 + x^2(\cosh 1 - 1)$$

Theorem: $n \geq 0$, $f : [a,b] \to \mathbb{R}$, $(n+1)$st derivative of f exists and is continuous. Then for $x \in [a,b]$ exists a $\xi \in (a,b)$ such that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,\pi_{n+1}(x)$$

$$\pi_{n+1}(x) = (x-x_0)\cdot(x-x_1)\cdot \ldots \cdot(x-x_n)$$

and:

$$|f(x) - p_n(x)| \leq \frac{M_{n+1}}{(n+1)!}\,|\pi_{n+1}(x)|$$

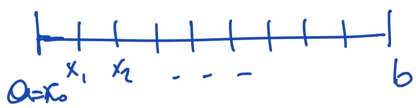
$$M_{n+1} = \max_{x\in[a,b]} \left|f^{(n+1)}(x)\right|$$

§ 6.3 Convergence

Does $p_n(x)$ converge to f as $n \to \infty$, and in what sense.

Answer: Not always, in particular this depends on the choice of $x_0, x_1, \ldots, x_n$

Assume equally spaced points $x_j = a + j\dfrac{(b-a)}{n}$ $\quad j = 0, \dots, n$



What happens to $\dfrac{M_{n+1}}{(n+1)!} \max\limits_{x \in [a,b]} |\pi_{n+1}(x)|$

What can happen is that this does not go to zero as $M_{n+1} \max\limits_{x \in [a,b]} |\pi_{n+1}(x)|$ goes to ∞ faster than $(n+1)!$

§ 6.4 Hermite interpolation

$x_0, \dots, x_n \in \mathbb{R}$, $\quad x_i \neq x_j \quad i \neq j$

$y_0, \dots, y_n \in \mathbb{R}$

$z_0, \dots, z_n \in \mathbb{R}$

Find $p_{2n+1} \in P_{2n+1}$ such that

$$p_{2n+1}(x_i) = y_i \qquad i = 0, \dots, n$$
$$p'_{2n+1}(x_i) = z_i$$

Theorem: (Hermite interpolation)

Consider

$$\boxed{H_k(x) = (L_k(x))^2 \left(1 - 2L'_k(x_k)(x - x_k)\right)}$$

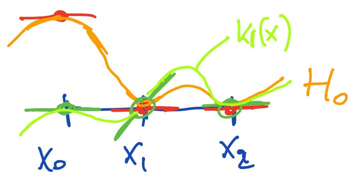
$$\boxed{K_k(x) = (L_k(x))^2 (x - x_k)}$$

$L_k(x) = \prod\limits_{\substack{i=0 \\ i \neq k}}^{n} \dfrac{x - x_i}{x_k - x_i}$ $\quad \Big| \quad H_k, K_k \in P_{2n+1}$

H_k, K_k satisfy:

$$H_k(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad H'_k(x_i) = 0$$

$$K_k(x_i) = 0 \,, \qquad K_k'(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$



interpoly:

$$P_{2n+1}(x) = \sum_{k=0}^{n} \left[H_k(x) y_k + K_k(x) z_k \right]$$

is the interpolation Hermite polynomial