§ 12 Initial value problems / ODEs

$$\begin{cases} y'' + 2y' = 3y \\ f''(x) + 2f'(x) = 3f(x) \\ \dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 3y \end{cases}$$

identical, just different notation.

$y = y(x)$ is a function of x, ODEs are relations between functions and their derivatives. Solution is a function or a set of functions

One solution is $y(x) = e^{-3x}$ since:

$$y'(x) = -3e^{-3x}$$
$$y''(x) = 9e^{-3x}$$

$$y'' + 2y' = 9e^{-3x} + 2(-3e^{-3x}) = 3e^{-3x} = 3y$$

We will consider initial value problems

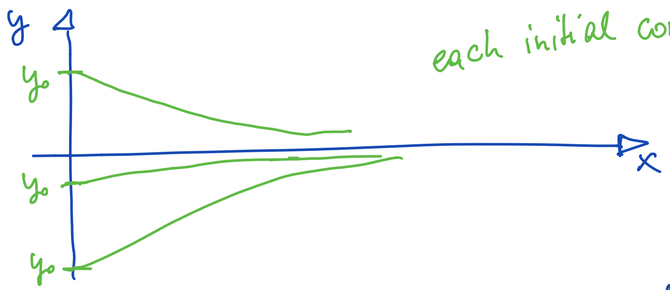
$$\text{(IVP)} \quad \begin{cases} y' = f(x,y) & \text{differential equation} \\ y(x_0) = y_0 & \text{initial value,} \end{cases}$$

Solution is a curve $y : [x_0, X_M] \to \mathbb{R}$ that starts at y_0.

Example:

$$y' = -2xy \quad \text{on } x \in [0,1]$$
$$y(0) = y_0 \in \mathbb{R} \quad \text{initial condition}$$

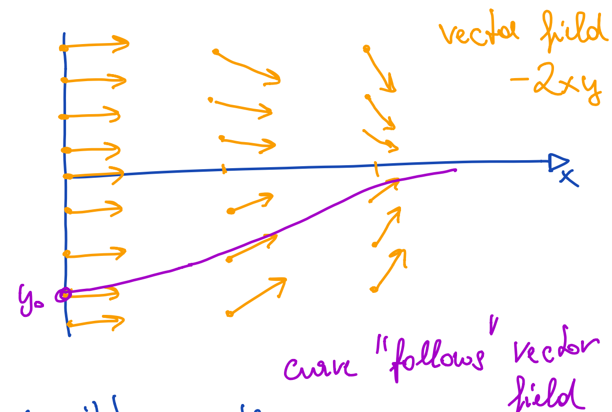
Solution is $y(x) = y_0 e^{-x^2}$



each initial condition gives me a different solution.

How about if we cannot find solution analytically or guess a solution? Then we have to rely on numerical approximate solutions

$$y' = f(x,y) = -2xy$$



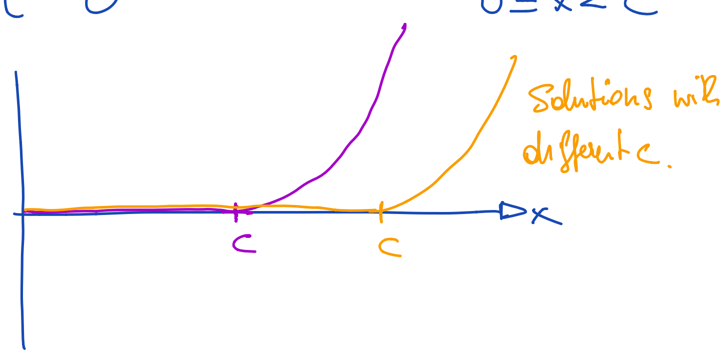
We have to ensure that a unique solution exists – otherwise it's pointless to find numerical approximations.

<u>Example</u>:

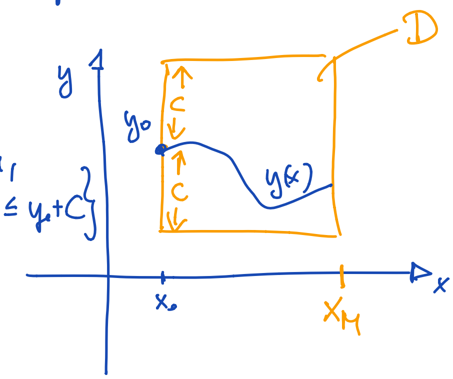
$$\begin{cases} y' = |y|^{\alpha} \\ y(0) = 0 \end{cases} \qquad \alpha \in (0,1)$$

does not have a unique solution

solutions: $$y_c(x) = \begin{cases} (1-\alpha)^{\frac{1}{1-\alpha}} (x-c)^{\frac{1}{1-\alpha}} & c \le x < \infty \\ 0 & 0 \le x < c \end{cases}$$



In general, we cannot hope for a unique solution – it will only exist and be unique if $f(\cdot,\cdot)$ satisfies certain conditions.



Theorem: $y' = f(x,y)$, $y(x_0) = y_0$

f: continuous in $\mathcal{D} = \{(x,y),\ x_0 \le x \le x_M,\ y_0 - C \le y \le y_0 + C\}$

$|f(x, y_0)| \le K \quad \forall x$

f Lipschitz' in 2nd variable, i.e.

$$|f(x,u) - f(x,v)| \le L\,|u-v|$$

$$C \ge \frac{K}{L}\left(e^{L(x_M - x_0)} - 1\right)$$

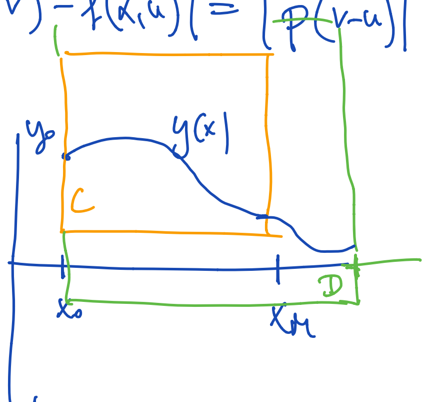
$\Longrightarrow$ There exists a unique solution $y \in C^1([x_0, x_M])$ in the box.

Example: $y' = py + q \qquad p, q \in \mathbb{R}$

$K = |p y_0| + |q|$ since $|f(x, y_0)| = |p y_0 + q| \le K$

$L = |p| \qquad |f(x,v) - f(x,u)| = |p(v-u)| \le \underbrace{|p|}_{L}\,|v-u|$

$\longrightarrow$ Solution exists for all $x \in [0, \infty)$



Example 2: $y' = y^2$, $y(0) = 1$

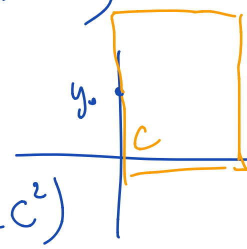
$|y_0^2| = 1 = K$

exact solution $y(x) = \frac{1}{1-x}$

$0 \le x < 1$

$$|v^2-u^2| = |v-u|\,|v+u| \le L\,|v-u|$$

$$L = 2(1+C)$$



$|u-1| \le C$

$|v-1| \le C$

$\Longrightarrow |u|+|v| \le 2(1+C)$

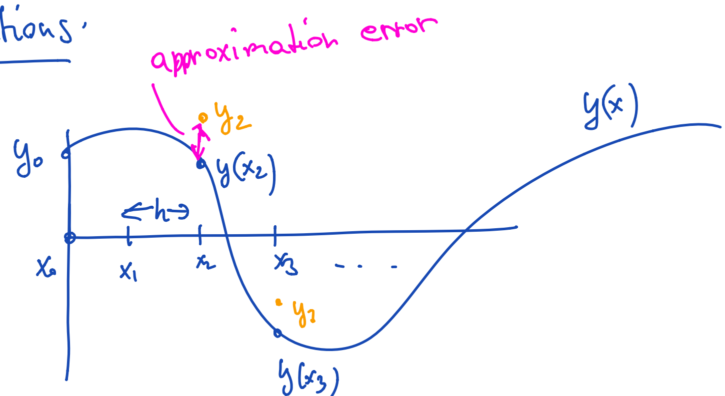
$$C \ge \frac{1}{2(1+C)}\left(e^{2(1+C)x_M}-1\right)$$

$$\Longrightarrow x_M \le \frac{1}{2(1+C)}\ln\left(1+2C+2C^2\right)$$

$$\Longrightarrow x_M \le 0.43.$$

$\longrightarrow$ theory only guarantees solution for $x \in [0, 0.43]$

Approximation of solutions.



We will use points $x_1, x_2, \dots x_N$ in $[x_0, x_M]$, $x_n = x_0 + nh$

$h = \dfrac{x_M - x_0}{N}$, $y_n \approx y(x_n)$ approximations, and we will compute $y_1, y_2, \dots$ in succession.