§ 1.4 Relaxation and Newton's method

Solve $f(x)=0$, f continuous

$$(**)\quad \boxed{\begin{array}{l} x_0 \text{ STARTING VALUE} \\ x_{k+1} = \underbrace{x_k - \lambda f(x_k)}_{g(x_k)} \qquad k=0,1,2,\ldots \\ \lambda \neq 0 \quad \text{relaxation parameter} \end{array}}$$

If CONVERGING TO $\xi \longrightarrow f(\xi)=0$

f diff'able $\Longrightarrow g'(x) = 1-\lambda f'(x)$ should be small in absolute value for fast convergence.

$\longrightarrow \lambda \approx \dfrac{1}{f'(\xi)}$ if $f'(\xi)\neq 0$

For such a choice of λ we can expect fast convergence.

__Theorem 1.7:__ $f(\xi)=0$, f cont, f' defined in a neighborhood of ξ, f' continuous, $f'(\xi)\neq 0$
Then $\exists\ \lambda, \delta>0$ such that x_k defined in $(**)$ converges for every x_0 in $[\xi-\delta, \xi+\delta]$.

"start close enough to the solution ξ, choose λ appropriately, then the relaxed iteration $(**)$ converges."

Proof (book).

Now allow λ to be a function of x:

$$x_{k+1} = \underbrace{x_k - \lambda(x_k) f(x_k)}_{g(x_k)} \qquad k=0,1,2,\ldots$$

Upon convergence, $f(\xi)=0$ if $\lambda(\xi)\neq 0$.

Asymptotic convergence depends on

$$g'(\xi) = 1 - \lambda(\xi) f'(\xi) - \cancel{\lambda'(\xi) f(\xi)} = 0 \text{ as } f(\xi)=0$$

$$\longrightarrow \underline{\lambda(x_k) = \frac{1}{f'(x_k)}}$$

leads to **Newton's method**:

x_0 initial point

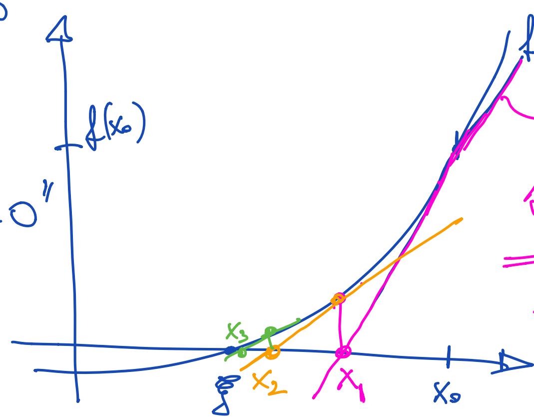
$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \qquad k=0,1,2,\dots}$$

assumes $f'(x_k) \neq 0$

linearization of f

$\longrightarrow$ solves "linearization $=0$"

$\longrightarrow$ iterates.



$f(x_0)+(x-x_0)f'(x_0)$

$f(x_0)+(x-x_0)f'(x_0)=0$

$\Longrightarrow x = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ is x_1 in Newton's method!

__Def:__ $x_k \longrightarrow \xi$ as $k \to \infty$,

If $\dfrac{|x_{k+1}-\xi|}{|x_k-\xi|^q} \xrightarrow{k\to\infty} \mu$, $q>1$, $\mu<\infty$

then x_k converges with order q.

In particular $q=2$, quadratic convergence.

How fast is that? Assume $|x_k-\xi| \sim 10^{-1}$

$\longrightarrow |x_{k+1}-\xi| \sim \mu(10^{-1})^2 = \mu\, 10^{-2}$

$|x_{k+2}-\xi| \sim \mu(\mu\, 10^{-2})^2 = \mu^3\, 10^{-4}$

$|x_{k+3}-\xi| \sim \mu^7\, 10^{-8}$

Theorem 1.8 (Convergence of Newton's method)
f continuous, f'' continuous on $I_\delta = [\xi-\delta, \xi+\delta]$, $\delta > 0$, $f(\xi) = 0$, $f''(\xi) \neq 0$, Suppose $\exists A > 0$

$$\left|\frac{f''(x)}{f'(y)}\right| \leq A \quad \text{for all } x, y \in I_\delta$$

Then: If $|\xi - x_0| \leq h$ $\quad h = \min(\delta, \frac{1}{A})$, then x_k defined by Newton's method converges quadratically to ξ.

Proof: Suppose $|\xi - x_k| \leq h$, Taylor:

$$0 = f(\xi) = f(x_k) + (\xi - x_k) f'(x_k) + \frac{(\xi - x_k)^2}{2} f''(\eta_k)$$

$$\eta_k \in (\xi, x_k)$$

Divide by $f'(x_k)$:

$$0 = \frac{f(x_k)}{f'(x_k)} + \xi - x_k + \frac{(\xi - x_k)^2}{2 f'(x_k)} f''(\eta_k)$$

$-x_{k+1}$

$|\cdot| \leq \frac{1}{A}$

$$\Longrightarrow \xi - x_{k+1} = -\frac{(\xi - x_k)^2}{2 f'(x_k)} f''(\eta_k) \quad (\#)$$

$|..| \leq A$

abs. values

$$\Longrightarrow |\xi - x_{k+1}| \leq \frac{|\xi - x_k|}{2}$$

$$\Longrightarrow |\xi - x_{k+1}| \leq 2^{-k-1} |\xi - x_0|$$

$$\Longrightarrow x_k \to \xi \quad \text{as } k \to \infty$$

$$\eta_k \to \xi \quad \text{as } k \to \infty \quad [\eta_k \in (\xi, x_k)]$$

$$(\#) \Longrightarrow \frac{|\xi - x_{k+1}|}{|\xi - x_k|^2} \longrightarrow \left|\frac{f''(\xi)}{2 f'(\xi)}\right| = \mu$$

$\Longrightarrow$ quadratic convergence $\blacksquare$