Orthogonal polynomials

Consider integrals of the form

$$I(f) = \int_a^b w(x) f(x) dx$$

$w(x) \geq 0$ weight function, s.t.
$0 < \int_a^b w(x)dx < \infty$, and $|\int_a^b x^j w(x) dx| < \infty$, $j = 1, 2, \ldots$.

Define the inner product

$$(f, g) = \int_a^b w(x) f(x) g(x) dx.$$

For specific choice of $w(x)$, orthogonal polynomials defined with respect to this inner product. Polynomials $p_j \in \Pi_j$, $j = 0, 1, 2, \ldots$,

$$(p_i, p_k) = 0 \quad \text{for } i \neq k,$$

with $p_0(x) \equiv 1$.

Examples

$w(x)$	*Interval*	*Polynomials*
1	$[-1, 1]$	Legendre polynomials (up to a factor)
$(1-x^2)^{-1/2}$	$[-1, 1]$	Chebyshev polynomials
e^{-x}	$[0, \infty)$	Laguerre polynomials
e^{-x^2}	$(-\infty, \infty)$	Hermite polynomials

Theorem Let $p_n \in \Pi_n$ be the nth orthogonal polynomial with respect to the inner product with weight function $w(x)$ on $[a, b]$. Then, all n roots of the orthogonal polynomial are simple and they all lie in the open interval (a, b).

Theorem

a) Let $x_1, \ldots, x_n$ be the n roots of the nth orthogonal polynomial $p_n(x)$ and let $w_1, \ldots, w_n$ be the solution of

$$\sum_{i=1}^{n} p_k(x_i) w_i = \begin{cases} (p_0, p_0) & \text{if } k = 0 \\ 0 & \text{if } k = 1, 2, \ldots, n-1 \end{cases} \tag{1}$$

Then $w_i > 0$, for $i = 1, \ldots, n$ and

$$\int_a^b w(x) p(x) dx = \sum_{i=1}^{n} w_i p(x_i), \tag{2}$$

hold for all polynomials $p \in \Pi_{2n-1}$.

Theorem, contd.

b) Conversely, if the numbers w_i, x_i, $i = 1, \ldots, n$, are such that Eq. (2) holds for all $p \in \Pi_{2n-1}$ then the x_i's are the roots of p_n and the weights w_i satisfy Eq. (1).

c) It is not possible to find numbers x_i, w_i $i = 1, \ldots, n$ such that Eq. (2) holds for polynomials $p \in \Pi_{2n}$.

Define quadrature rule with quadrature points x_i and quadrature weights w_i, $i = 1, \ldots, n$, to approximate $\int_a^b w(x) f(x) dx$.

Gauss quadrature

Most common weight function $w(x) \equiv 1$ on interval $[-1, 1]$ (approximation of integral $\int_{-1}^{1} f(x)dx$).

For this case, theorem above due to Gauss.

$$p_k(x) = \frac{k!}{(2k)!} \frac{d^k}{dx^k} (x^2 - 1)^k, \quad k = 0, 1, \ldots$$

(Legendre polynomials up to a factor).

n	w_i	x_i
2	$w_1 = w_2 = 1$	$x_2 = -x_1 = 0.5773502692\ldots$
3	$w_1 = w_3 = 5/9$	$x_3 = -x_1 = 0.7745966692\ldots$
	$w_2 = 8/9$	$x_2 = 0.$

Newton-Cotes formulas
$n+1$ equidistant points, (polynomial of degree n);
error of $O(h^{n+1})$.

Gauss quadrature
n points, integrating polynomial of degree $2n-1$ exactly;
error of $O(h^{2n})$.

(Optimal: Cannot do better! [Peano kernel theorem]).