EQUIVARIANT GEOMETRY OF LOW-DIMENSIONAL QUADRICS

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Abstract. We provide new stable linearizability constructions for regular actions of finite groups on homogeneous spaces and low-dimensional quadrics.

1. Introduction

Let G be a finite group, acting generically freely and regularly on a smooth projective variety X. Of particular interest are linear actions, i.e., generically free actions on $\mathbb{P}(V)$ arising from a linear representation V of G. Actions which are equivariantly birational to a linear action are called *linearizable* (or *linear*); in particular X is a rational variety. The classification of such actions, up to birationality, is an open problem even for linear actions on $\mathbb{P}^2$ (see [DI09], [TYZ24] and references therein).

A related problem is to understand *stable* linearizability, i.e., linearizability of G-actions on $X \times \mathbb{P}^m$, with trivial action on the second factor. Apart from its intrinsic interest, this property is relevant for the study of automorphisms of fields of invariants [Kol24]. Until recently, the only known instances of stable equivariant birationalities were those arising from faithful linear actions of G, and a particular nonlinearizable but stably linearizable action of the dihedral group $\mathfrak{D}_6$ (of order 12) on a quadric surface [LPR06]. New tools, such as the G-equivariant version of the universal torsor formalism of Colliot-Thélène-Sansuc [CTS87] allowed us to settle the stable linearizability problem for quadric surfaces [HT23, Section 7].

In this note, we focus on quadric hypersurfaces of dimension three and four. This is an interesting class of examples; indeed, it is already unknown whether or not every $\mathfrak{S}_3$-action on a smooth quadric threefold is linearizable. On the one hand, cohomological obstructions and the universal torsor formalism play a limited role for these varieties. On the other hand, this is a good testing ground for stable linearization: the Pfaffian constructions of [BvBT23], interpreted via coincidences among Lie groups; the reduction to 2-Sylow subgroups, in the spirit of of Springer's Theorem, studied in [DR15]; and techniques from the stable birational geometry of quadrics over nonclosed fields.

In Section 2 we recall basic notions from equivariant geometry and connect it to versality for group actions on varieties. In Section 3 we investigate the impact of isotropic subspaces on stable linearizability; see Theorem 3.5. As in the case of quadrics over nonclosed fields [Tot09], the anisotropic quadrics have the richest birational geometry. In Section 4 we turn to flag varieties for special linear groups, and establish stable linearizability of translation actions in this context, in Theorem 4.6. In Section 5, we provide Springer-type results, reducing stable

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linearization to 2-Sylow subgroups. We interpret arguments for special orthogonal groups in terms of the geometry of Pfaffians. In Section 6, we present key examples of stable linearizable actions in dimension 3, and reduce the stable linearization problem to a specific action of the dihedral group $\mathfrak{D}_4$; see Corollary 6.5.

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2. Groups actions, twists, and rationality

Throughout, we work over a base field k that is algebraically closed of characteristic zero.

2.1. **Notions of linearizability.** A G-variety is a smooth algebraic variety over k with a generically free action of G. Here varieties are assumed to be geometrically integral.

We use the following four birational properties of a G-action on X:

- *strictly linear* if there exists an equivariant birational map
$$V \overset{\sim}{\dashrightarrow} X,$$
where V is a linear representation of G;
- *linear* if there exists an equivariant birational map
$$\mathbb{P}(V) \overset{\sim}{\dashrightarrow} X,$$
where V is a linear representation of G;
- *stably linear* if $X \times \mathbb{P}^n$ is linear, with trivial action on the second factor;
- *projectively linear* if there exists a projective representation of G on $\mathbb{P}(V)$ and an equivariant birational map
$$\mathbb{P}(V) \overset{\sim}{\dashrightarrow} X.$$

In drawing analogies between equivariant geometry and geometry over nonclosed fields, one could view (strict, projective) linearizability as analogous to rationality, stable linearizability as analogous to stable rationality, etc.

Remark 2.1. The No-Name Lemma [CGR06, 4.3] implies that if $X \times W$ is linear for some linear G-representation W then X is stably linear. In particular, the notion of *strictly stably linear* is logically equivalent to the notion of *stably linear*.

2.2. **Projective bundles.** Our definition of linearizability requires generically free actions. Without this assumption, the notion behaves counter-intuitively:

Example 2.2. Consider the dihedral group
$$G = \mathfrak{D}_4 = \langle \sigma, \tau : \sigma^4 = \tau^2 = e, \tau\sigma = \sigma^3\tau \rangle$$
with representation V given by
$$\sigma = \begin{pmatrix} i & 0 \\ 0 & i^3 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This does *not* act generically freely on $\mathbb{P}(V)$ as σ^2 is trivial; the quotient has nonzero Amitsur invariant [HT23, Section 3.5]. Consider the product $\mathbb{P}(V) \times \mathbb{P}^1$ with a trivial action on the second factor; it also has nonzero Amitsur invariant, as this is a stable birational invariant. The product is not equivariantly birational to $\mathbb{P}(V')$ for any three-dimensional representation of G, which would have trivial Amitsur invariant.

Let G be a finite group and V a G-representation such that the induced action on $\mathbb{P}(V)$ is generically free. The quotient map $V \dashrightarrow \mathbb{P}(V)$ gives a line bundle

$$\mathrm{Bl}_0(V) \to \mathbb{P}(V),$$

so the No-Name Lemma implies V and $\mathbb{P}(V) \times \mathbb{A}^1$ are equivariantly birational. The inclusion $V \subset \mathbb{P}(V \oplus 1)$ yields that V and $\mathbb{P}(V) \times \mathbb{P}^1$ are linearizable. Induction implies that $\mathbb{P}(V) \times \mathbb{P}^n$ is as well. In each case, G acts trivially on the second factor.

We recall observations of [Kat92, §1], retaining the assumptions above:

- Suppose $\mathcal{E} \to \mathbb{P}(V)$ is a vector bundle of rank $n+1$ with a lifting of the G-action. Then the No-Name Lemma implies that $\mathcal{E}$ is birational to $\mathcal{O}^{n+1}_{\mathbb{P}(V)}$ over $\mathbb{P}(V)$, hence $\mathbb{P}(\mathcal{E})$ is birational to $\mathbb{P}(V) \times \mathbb{P}^n$.
- Suppose X is a smooth projective variety with generically-free G-action and $\mathcal{E} \to X$ a vector bundle with G-action. If X is linearizable then $\mathbb{P}(\mathcal{E})$ is also linearizable.

Here is an extension of these ideas:

Proposition 2.3. *Let G and H be finite groups acting generically freely on $\mathbb{P}(V)$ and $\mathbb{P}(W)$ respectively. Then the induced action of $G \times H$ on $\mathbb{P}(V) \times \mathbb{P}(W)$ is stably linearizable.*

Proof. Consider the dominant rational map

$$\mathbb{P}(V \oplus W) \dashrightarrow \mathbb{P}(V) \times \mathbb{P}(W)$$

which induces the morphism

$$\mathrm{Bl}_{\mathbb{P}(V) \sqcup \mathbb{P}(W)} \mathbb{P}(V \oplus W) \to \mathbb{P}(V) \times \mathbb{P}(W).$$

This is the projectivization of a rank-two vector bundle $\mathcal{E} \to \mathbb{P}(V) \times \mathbb{P}(W)$ that is equivariant for $G \times H$. The $G \times H$-action is generically free on $\mathbb{P}(V) \times \mathbb{P}(W)$ hence $\mathcal{E}$ is equivariantly birational to $\mathbb{P}(V) \times \mathbb{P}(W) \times \mathbb{A}^2$, with trivial action on the last factor. Thus $\mathbb{P}(V) \times \mathbb{P}(W)$ is stably linearizable. □

Example 2.4. This is the best possible; we cannot expect linearizability here. Fix odd integers k, ℓ and consider the actions of $\mathfrak{D}_k$ and C_ℓ on $\mathbb{P}(V)$ and $\mathbb{P}(U)$, where V is the standard two-dimensional representation of the dihedral group and U has weights ζ and ζ^{-1}. Here $C_\ell = \langle \zeta \rangle \simeq \mu_\ell$. Keep in mind that

$$\mathbb{P}(V) \times \mathbb{P}(U) \hookrightarrow \mathbb{P}(V \otimes U)$$

as a quadric surface. The classification of [DI09] – see the bottom of page 537 – shows that $\mathbb{P}(V) \times \mathbb{P}(U)$ is not birational to $\mathbb{P}(V')$ for any three-dimensional representation of $\mathfrak{D}_k \times C_\ell$.

2.3. **Notions of versality.** We recall the terminology of [DR15], restricting to finite groups:

- *weakly versal* **(WV)**: for every field K/k and every G-torsor $T \to \mathrm{Spec}(K)$ there is a G-equivariant k-morphism $T \to X$,
- *versal* **(V)**: every G-invariant open $U \subset X$ is weakly versal,
- *very versal* **(VV)**: there exists a linear representation $G \to \mathsf{GL}(V)$ and a dominant G-equivariant rational map $V \to X$,

- *stably linearizable* **(SL)**: $X \times W$ is equivariantly birational to a linear representation of G, where W is a linear space with trivial G-action.

Note that in the definitions above:

- T is viewed as a k-scheme;
- we could replace W with a non-trivial linear representation or the projectivization of a linear representation without changing the definition.

These notions are related:

$$(\mathbf{SL}) \Rightarrow (\mathbf{VV}) \Rightarrow (\mathbf{V}) \Rightarrow (\mathbf{WV}).$$

Remark 2.5. Not every projectively linear action is very versal. Indeed, suppose we are given a projective faithful representation

$$\rho : G \to \mathsf{PGL}(V)$$

and an associated central extension

$$1 \to \mu_n \to \widetilde{G} \to G \to 1 \tag{1}$$

admitting a linear representation

$$\widetilde{\rho} : \widetilde{G} \to \mathsf{GL}(V).$$

By [DR15, Proposition 9.1], the G-action on $\mathbb{P}(V)$ is very versal if and only if the exact sequence (1) splits.

We recall the notion of a *twisting pair*, a tuple (T, K) consisting of a field extension K/k and a G-torsor T over K; this gives rise to a twist ${}^TX$, the K-variety obtained by twisting via T.

The connection between versality and rationality over nonclosed fields is addressed in [DR15, Theorem 1.1]:

- **(WV)** $\Leftrightarrow$ ${}^TX(K) \neq \emptyset$, for all twisting pairs (T, K),
- **(V)** $\Leftrightarrow$ ${}^TX(K)$ are Zariski dense, for all (T, K),
- **(VV)** $\Leftrightarrow$ ${}^TX$ is unirational over K, for all (T, K),
- **(SL)** $\Leftrightarrow$ ${}^TX$ is stably rational over K, for all (T, K).

Remark 2.6. An analogous statement regarding rationality over K fails: there are examples of stably linearizable but not linearizable quadric surfaces [HT23].

3. Quadrics and isotropic subspaces

Here we follow [NP23, Section 2]. We continue to assume the base field is algebraically closed of characteristic zero.

Let (V, q) be a non-degenerate quadratic form invariant under the action of a finite group G.

Definition 3.1. An *isotropic subspace* $W \subseteq V$ is a G-invariant subspace such that $\mathsf{q}|W = 0$. We say (V, q) is *anisotropic* if it has no nonzero isotropic subspaces. A *hyperbolic subspace* of V is a pair of G-invariant isotropic subspaces $W, W^\vee \subset V$ such that

$$\mathsf{q}(w, f) = f(w), \qquad \text{for all } w \in W, f \in W^\vee.$$

In other words, the restriction of q to $H_W := W \oplus W^\vee$ has matrix

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Suppose an irreducible representation W of G admits a nonzero invariant quadratic form q. When q is non-degenerate, this induces a G-equivariant isomorphism

$$W \stackrel{\sim}{\to} W^\vee, \tag{2}$$

unique up to scalar.

Proposition 3.2. *Suppose (V, q) is non-degenerate and admits an irreducible isotropic subspace W. Then we obtain*

$$(V, \mathsf{q}) = (V', \mathsf{q}') \oplus_\perp H_W$$

where (V', q') is non-degenerate.

Proof. Since q is non-degenerate, there exists a copy of $W^\vee \subset V$ such that

$$\mathsf{q}|(W \oplus W^\vee) = \begin{pmatrix} 0 & I \\ I & R \end{pmatrix}$$

where R is a G-invariant quadratic form on $W^\vee$. If $R = 0$ then we have our hyperbolic form H_W. If $R \neq 0$ then, by (2), $R = \lambda I$ for some $0 \neq \lambda \in k$. Since k does not have characteristic two, after equivariant row and column operations $\mathsf{q}|W \oplus W^\vee$ becomes hyperbolic. □

Applying this inductively gives an equivariant version of Witt decomposition:

Corollary 3.3. *Every non-degenerate (V, q) is equivariantly equivalent to the orthogonal direct sum of a hyperbolic form and an anisotropic form.*

Proposition 3.4. *Let $X \subset \mathbb{P}(W \oplus W^\vee)$ be a quadric hypersurface associated with a hyperbolic form H_W, where W is a representation of G of dimension $d \geq 2$. If G acts generically freely on $\mathbb{P}(W)$ then X is linearizable. If G acts generically freely on X then X is stably linearizable.*

We already observed in Example 2.4 that we cannot expect X to be linearizable in general.

Proof. Consider the linear projection from $\mathbb{P}(W)$, which induces

$$\pi_{\mathbb{P}(W)} : \mathrm{Bl}_{\mathbb{P}(W)} X \to \mathbb{P}(W^\vee),$$

which is a $\mathbb{P}^{d-1}$ bundle – indeed, the projectivization of a vector bundle with G-action. Note that G acts generically freely on $\mathbb{P}(W)$ if and only if it acts generically freely on $\mathbb{P}(W^\vee)$. When G acts generically freely on $\mathbb{P}(W^\vee)$ then the observations of Section 2.2 give that X is linearizable. Suppose that G fails to acts generically freely on $\mathbb{P}(W)$. Since it *does* act generically freely on $X \subset \mathbb{P}(W \oplus W^\vee)$, we conclude there is a cyclic central subgroup

$$C_\ell = \langle \zeta \rangle \subset G, \quad \ell \text{ odd}, \, \zeta^\ell = 1,$$

acting via

$$\begin{aligned} C_\ell \times (W \oplus W^\vee) &\to W \oplus W^\vee \\ \zeta \cdot (w, f) &\mapsto (\zeta w, \zeta^{-1} f). \end{aligned}$$

Let U be the two-dimensional representation of C_ℓ with weights ζ and ζ^{-1}. Take the basechange of $\pi_{\mathbb{P}(W)}$

$$(\mathrm{Bl}_{\mathbb{P}(W)} X) \times \mathbb{P}(U) \to \mathbb{P}(W^\vee) \times \mathbb{P}(U)$$

which remains the projectivization of a G-equivariant vector bundle; now the base has a generically free action of G on the base. Since $\mathbb{P}(W^\vee) \times \mathbb{P}(U)$ is stably linearizable by Proposition 2.3, $X \times \mathbb{P}(U)$ is as well. An application of the No-Name Lemma, as in Section 2.2, implies that X is stably linearizable. □

The same projection argument yields the following:

Theorem 3.5. *Let (V, q) be a non-degenerate quadratic form invariant under the action of a finite group G. Let*

$$X = \{\mathsf{q} = 0\} \subset \mathbb{P}(V)$$

and assume that $\dim(X) > 0$. *Let $0 \neq W \subset V$ be an isotropic subspace for q and*

$$H_W \simeq W \oplus W^\vee \subseteq V$$

the hyperbolic subspace guaranteed by Proposition 3.2. If G acts generically freely on $\mathbb{P}(W^\vee \oplus H_W^\perp)$ then X is linearizable. If G acts generically freely on X then X is stably linearizable.

When $\dim(W) = 1$ the projection is birational, and we obtain the well-known

Corollary 3.6. *Retain the notation of Theorem 3.5. If X has a fixed point then X is linearizable.*

In light of Theorem 3.5 and Corollary 3.3, the natural question for future study is the stable birational classification of anisotropic quadrics, i.e.,

$$X = \{\mathsf{q} = 0\} \subset \mathbb{P}(V),$$

where

$$(V, \mathsf{q}) = (V_1, \mathsf{q}_1) \oplus_\perp \cdots \oplus_\perp (V_r, \mathsf{q}_r) \tag{3}$$

is an orthogonal direct sum of self-dual irreducible representations of G with their distinguished quadratic forms.

There are only finitely many possibilities to consider, thanks to our next result:

Proposition 3.7. *Suppose that q is direct sum of irreducible non-degenerate quadratic forms as in (3). If q is anisotropic then the factors (V_i, q_i) are not isomorphic.*

Proof. Suppose a summand, say (V_1, q_1), appears with multiplicity. After rescaling if necessary, we obtain

$$(V_1, \mathsf{q}_1) \oplus_\perp (V_1, \mathsf{q}_1) \subset (V, \mathsf{q}).$$

This contains an isotropic subspace – the image of V_1 under $v \mapsto (v, iv)$ – contradicting the assumption that (V, q) is anisotropic. □

4. Flag varieties and special groups

Here we consider natural actions of finite groups on homogeneous spaces for special classes of algebraic groups, with a view toward stable linearizability. The most fundamental construction is the No-Name Lemma mentioned in Remark 2.1. We will apply it freely as we present further applications below.

An algebraic group G over k is called *special* if

$$\mathrm{H}^1(K, \mathsf{G}) = 0, \quad \forall K/k.$$

Examples (listed in [CTS07, Section 4.2]) include

- split connected solvable groups,
- GL_n, SL_n, Sp_{2n}, split Spin_n, with $n \le 6$.

By [Ser58, p. 26], the only special semisimple groups over k are products of SL_n and Sp_n.

Proposition 4.1. *Let G be a special connected linear algebraic group and $G \subset \mathsf{G}$ a finite subgroup. Then the translation action of G on G is stably linearizable.*

This generalizes [BPT10, Proposition 3.7].

Proof. We are following the strategy of [CTS07, Prop. 4.9]. Choose a representation $G \hookrightarrow \mathsf{GL}_n$ for some n. Note that G is rational over k. Consider the diagonal embedding

$$G \hookrightarrow \mathsf{GL}_n \times \mathsf{G}$$

and the projection onto G. We claim that this is G-birational to $\mathsf{GL}_n \times \mathsf{G}$, with trivial action on the first factor. Indeed, a finite group action on GL_n is tautologically linearizable over any field. Since G acts generically freely on G, the No-Name Lemma implies the desired birational map.

On the other hand, consider the projection

$$\pi_1 : \mathsf{GL}_n \times \mathsf{G} \to \mathsf{GL}_n.$$

This is a torsor for G in the sense that it admits a section s after basechange

$$G \times \mathsf{GL}_n \to \mathsf{GL}_n,$$

namely,

$$s(\gamma, g) \mapsto (\gamma \cdot 1_{\mathsf{G}}, \gamma \cdot g).$$

However, the speciality assumption implies that there is a section even over the function field of GL_n. Thus $\mathsf{G} \times \mathsf{GL}_n$ with the diagonal action of G is equivariantly birational to $\mathsf{G} \times \mathsf{GL}_n$, with trivial action on the first factor.

Putting this together, we find that G and GL_n are stably birational as G-varieties. It follows that G is stably linearizable. □

Corollary 4.2. *Let G be special and $\mathsf{U} \subset \mathsf{G}$ a unipotent subgroup. Fix a finite subgroup $G \subset \mathsf{G}$. Then the induced left action on G/U is stably linearizable.*

Proof. Since U is unipotent and the characteristic is zero, $G \cap \mathsf{U} = \{1\}$ and G acts generically freely on G/U. Consider the projection

$$\mathsf{G} \to \mathsf{G}/\mathsf{U},$$

which is a vector bundle as the group U is special [CTS07, Section 4.2]. The No-Name Lemma implies G is G-birational to the product

$$\mathsf{U} \times (\mathsf{G}/\mathsf{U})$$

with G acting trivially on the first factor. Since the G-action on G is stably linearizable the same holds for the quotient. □

Remark 4.3. Even when G is not special, one can sometimes establish the linearizability of the translation action. For example, $\mathsf{G} = \mathsf{PGL}_2$ admits an equivariant compactification to $\mathbb{P}^3$, and the translation action extends as a projectively linear action on $\mathbb{P}^3$. The obstruction to linearizability of this action is captured by the Amitsur invariant (see, e.g. [HT23, Section 3.5]).

Proposition 4.4. *Let G be special and $\mathsf{B} \subset \mathsf{G}$ a Borel subgroup. Let $G \subset \mathsf{G}$ be a finite subgroup such that the induced left action on the quotient G/B is generically free. Then this action is stably linearizable.*

Proof. Express $\mathsf{B} = \mathsf{U} \rtimes \mathsf{T}$ where T is a maximal torus which gives

$$\varpi : \mathsf{G}/\mathsf{U} \to \mathsf{G}/\mathsf{B} =: \mathrm{Fl}$$

with fibers isomorphic to T Indeed, if $L_1, \ldots, L_r$ are line bundles forming a basis for $\mathrm{Pic}(\mathrm{Fl})$ then we may interpret

$$\mathsf{G}/\mathsf{U} = L_1^* \times_{\mathrm{Fl}} \cdots \times_{\mathrm{Fl}} L_r^*, \quad L_i^* = L_i \setminus 0;$$

in particular, the action of our finite group linearizes to each of the L_i. Our generic-freeness assumption means that G/U is equivariantly birational to $\mathbb{A}^r \times \mathrm{Fl}$ with trivial action on the first factor. Thus G/B is stably linearizable. □

We record an application of the Serre-Grothendieck classification of special groups [Ser58, Section 4]:

Lemma 4.5. *Let G denote a special semisimple linear algebraic group and $\mathsf{P} \subset \mathsf{G}$ a (split) parabolic subgroup. Then the Levi factors of P are also special.*

Theorem 4.6. *Let G denote a special semisimple group, $\mathsf{P} \subset \mathsf{G}$ a split parabolic subgroup, and $G \hookrightarrow \mathsf{G}$ a finite group. If G acts generically freely on G/P then this action is stably linearizable.*

Proof. We proceed by induction on the dimension of G.

Choose a maximal sequence of parabolic subgroups

$$\mathsf{P} =: \mathsf{P}_1 \subsetneq \mathsf{P}_2 \subsetneq \cdots \mathsf{P}_{m-1} \subsetneq \mathsf{P}_m = \mathsf{B}$$

and consider the tower

$$\mathsf{G}/\mathsf{B} \to \mathsf{G}/\mathsf{P}_{n-1} \to \cdots \to \mathsf{G}/\mathsf{P}.$$

Each step of this tower is fibered in projective homogeneous spaces for L, a Levi subgroup of some parabolic subgroup of G. These groups are special by Lemma 4.5, and their homogeneous spaces are stably linearizable by induction. As before, iterating the No-Name Lemma yields

$$\mathsf{G}/\mathsf{B} \text{ stab. lin.} \Rightarrow \mathsf{G}/\mathsf{P}_{n-1} \text{ stab. lin.} \Rightarrow \cdots \Rightarrow \mathsf{G}/\mathsf{P} \text{ stab. lin.}$$

□

5. Quadric hypersurfaces

5.1. **Arbitrary dimensions.** Before considering specific actions of finite groups on quadrics, we record general considerations regarding the presentation of such actions. Let

$$X \subset \mathbb{P}^n$$

be a smooth quadric hypersurface. Let q be the associated quadratic form, which is unique, up to scalars. We consider finite subgroups $G \subset \mathsf{PGL}_{n+1}$ preserving X.

The Amitsur invariant [HT23, Section 3.5] yields:

Proposition 5.1. *If the action of G on X is stably linearizable then we can lift $G \subset \mathsf{GL}_{n+1}$.*

When the number of variables is odd, we can always work with the special orthogonal group:

Proposition 5.2. *If $n = 2m$ is even then we may assume $G \subset \mathsf{SL}_{2m+1}$.*

Proof. Given

$$\varrho : G \hookrightarrow \mathsf{O}_{2m+1}$$

there is a modified representation

$$\rho = \det(\varrho) \cdot \varrho : G \to \mathsf{SO}_{2m+1}$$

that is projectively equivalent to ϱ. Note that ρ is injective if and only if the image of ϱ does not contain $-I$, i.e. ϱ acts generically freely on $\mathbb{P}^{2m}$. □

5.2. **Pfaffian constructions.** We recall the Pfaffian construction, for quadric hypersurfaces; see [BvBT23, Section 7] for further details.

Let M denote an antisymmetric $2r \times 2r$ matrix. The *Pfaffian form* $\mathrm{Pf}(M)$ is a homogeneous form of degree r such that

$$\mathrm{Pf}(M)^2 = \det(M).$$

When $r = 2$

$$M = \begin{pmatrix} 0 & m_{12} & m_{13} & m_{14} \\ -m_{12} & 0 & m_{23} & m_{24} \\ -m_{13} & -m_{23} & 0 & m_{34} \\ -m_{14} & -m_{24} & -m_{34} & 0 \end{pmatrix}$$

we have

$$\mathrm{Pf}(M) = m_{12}m_{34} - m_{13}m_{23} + m_{14}m_{23}$$

the Plücker relation for the Grassmannian

$$\mathrm{Gr}(2, W) \subset \mathbb{P}(\wedge^2 W), \quad \dim(W) = 4.$$

We recall a few properties:

- Let (V, q) be a non-degenerate quadratic form where V is a six-dimensional representation of G. This is Pfaffian if and only if

$$V \simeq \wedge^2 W, \quad \dim(W) = 4$$

  and q coincides with the (symmetric!) wedge pairing

$$\wedge^2 W \times \wedge^2 W \to \wedge^4 W.$$

- Let V be a non-degenerate quadratic form with $\dim(V) = 5$. Then V is Pfaffian if and only if there exists a symplectic representation

$$(W, \omega), \quad \dim(W) = 4$$

  of G, with

$$\wedge^2 W = \langle \omega \rangle \oplus V.$$

- In either case, assuming G acts generically freely on

$$X = \{\mathsf{q} = 0\} \subset \mathbb{P}(V),$$

  then X is stably birational to W.

The last statement arises from the inclusion and projection morphisms

$$\mathcal{S}|X \hookrightarrow W \times X \to W,$$

where $\mathcal{S} \to \mathrm{Gr}(2, W)$ is the universal subbundle.

5.3. **Springer-type results.** The following is a corollary of [DR15, Theorem 10.2]:

Proposition 5.3. *Let $X \subset \mathbb{P}^n, n \geq 2$, denote a smooth quadric hypersurface, with a generically-free action by a finite group G. Let $G_2 \subseteq G$ denote a 2-Sylow subgroup. Then the G-action on X is stably linearizable if and only if the G_2-action on X is stably linearizable.*

Recall the equivalences stated in Section 2.3: In particular, a group action on X is weakly versal if each twist by the group admits rational points and stably linearizable if each twist is stably rational. However, Springer's Theorem and stereographic projection guarantee that, for a smooth positive-dimensional quadric hypersurface Y over a field L, the following are equivalent:

- Y is rational over L;
- Y is stably rational over L;
- Y has a rational point over L;
- Y has a rational point over an odd-degree extension of L.

Now the last condition corresponds to passing from G to a 2-Sylow subgroup G_2, thus Proposition 5.3 follows.

Remark 5.4. This argument is a bit vexing, as we are *not* showing that linearizability can be checked on passage to a 2-Sylow subgroup! The dictionary of Section 2.3 leaves out birational equivalence to a G-representation or its projectivization.

Example 5.5. Consider the action of $G = \mathfrak{S}_3 \times C_2$ on

$$X = \{x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0\} \subset \mathbb{P}^3$$

where C_2 acts via $x_4 \mapsto -x_4$ and $\mathfrak{S}_3$ permutes x_1, x_2, and x_3. By [Isk08] this action is not linearizable; a proof using Burnside invariants may be found in [HKT21, Section 7.6]. On the other hand, the 2-Sylow subgroup G_2 fixing x_3 has fixed points $\{x_4 = x_1 - x_2 = 0\} \cap X$. The stable linearizability of this action has been shown, using different techniques, in [LPR06] and [HT23, Section 6].

5.4. **Quadric threefolds.** Theorem 4.6 – combined with the symplectic interpretation of Spin_5 and the fact that symplectic groups are special – yields the following:

Theorem 5.6. *Let $X \subset \mathbb{P}^4$ be a smooth three-dimensional quadric. Suppose that $G \subset \mathsf{SO}_5$ acts on X generically freely. The action of G is stably linearizable if the induced extension*

$$\begin{tikzcd}
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d, equal] & \widetilde{G} \arrow[r] \arrow[d, hook] & G \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mu_2 \arrow[r] & \mathsf{Sp}_4 \arrow[r] & \mathsf{SO}_5 \arrow[r] & 1
\end{tikzcd}$$

is trivial, i.e., the restriction homomorphism

$$\mathrm{H}^2(\mathsf{SO}_5, \mu_2) \to \mathrm{H}^2(G, \mu_2)$$

vanishes on the distinguished extension.

Here is a geometric interpretation of this theorem: An action of $G \subset \mathsf{SO}_5$ on $X \subset \mathbb{P}^4$ lifts to the spin group if and only if there is a G-equivariant imbedding

$$X \hookrightarrow \mathrm{Gr}(2, 4)$$

arising from a representation $G \to \mathsf{SL}_4$ leaving a non-degenerate 2-form invariant. This is the Pfaffian construction from Section 5.2.

Example 5.7. The converse of Theorem 5.6 is not true: There are linearizable quadric threefolds $X \subset \mathbb{P}^4$ such that $G \subset \mathsf{SO}_5$ does not lift to Sp_4. In geometric terms, the variety of lines $F_1(X)$ – a four-dimensional projective representation of G – may have non-vanishing invariant.

Here is a construction: Let C denote a conic with non-trivial Amitsur invariant, corresponding to a projective representation

$$\phi : G \to \mathsf{PGL}_2$$

not lifting to a linear representation. Let V denote the linear representation associated with the symmetric square of ϕ; there is an embedding

$$C \hookrightarrow \mathbb{P}(V) \subset \mathbb{P}(\mathbf{1} \oplus V).$$

The blowup of this projective space along C admits a morphism

$$\varpi : \mathrm{Bl}_C(\mathbb{P}(\mathbf{1} \oplus V)) \xrightarrow{\sim} X \subset \mathbb{P}^4$$

given by the linear system of quadrics vanishing along C. Write

$$x = \varpi(\mathbb{P}(V)) \in X$$

for the image of the proper transform of the plane spanned by C. Consider the lines in $\mathbb{P}(\mathbf{1} \oplus V)$ meeting C at a point, a projective plane bundle $W \to C$. (Secants to C are counted twice!) The No-Name Lemma implies that

$$W \overset{\sim}{\dashrightarrow} \mathbb{P}^2 \times C,$$

where the first factor has trivial G-action. The morphism ϖ induces

$$\pi : W \xrightarrow{\sim} F_1(X)$$

that blows up the lines

$$\{\ell : x \in \ell \subset X\} \simeq C.$$

We conclude that $F_1(X)$ – birational to $C \times \mathbb{P}^2$ – also has non-trivial Amitsur invariant.

5.5. **Quadric fourfolds.** Similarly, we observe:

Theorem 5.8. *Let $X \subset \mathbb{P}^5$ be a smooth four-dimensional quadric. Suppose that $G \subset \mathsf{SO}_6$ acts on X generically freely. The action of G is stably linearizable if the induced extension*

$$\begin{tikzcd}
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d, equal] & \widetilde{G} \arrow[r] \arrow[d, hook] & G \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mu_2 \arrow[r] & \mathsf{SL}_4 \arrow[r] & \mathsf{SO}_6 \arrow[r] & 1
\end{tikzcd}$$

is trivial, i.e., the restriction homomorphism

$$\mathrm{H}^2(\mathsf{SO}_6, \mu_2) \to \mathrm{H}^2(G, \mu_2)$$

vanishes on the distinguished extension.

Compare this with the Pfaffian interpretation in Section 5.2.

Example 5.9. Let V be a 4-dimensional representation of $\mathfrak{S}_5$. Its exterior square is the 6-dimensional representation; and there is a unique invariant quadric $X \subset \mathbb{P}^5$. The $\mathfrak{S}_5$-action on X is not known to be linearizable. Theorem 5.8, combined with the Pfaffian construction, yields stable linearizability for the action on X.

6. Applications to threefolds

In this section, we present examples of stable linearizability constructions, focusing on cases where linearizability is not known.

We let X be a smooth quadric threefold with a generically free regular action of a finite group G. We recall a "nonstandard" linearizability construction, see, e.g., [ACC+23, Sect. 5.8]: The infinite dihedral group $\mathbb{G}_m \rtimes \mu_2$ acts on the quintic del Pezzo threefold $V_5 \subset \mathbb{P}^6$ – which has automorphism group PGL_2. The action lifts to a linear representation in GL_7; and it stabilizes

- a twisted cubic curve $R_3 \subset V_5$;
- a conic $R_2 \subset V_5$;
- a line $R_1 \subset V_5$,

see [ACC+23, Cor. 5.39]: Projection from R_2 gives an equivariant birational map

$$V_5 \overset{\sim}{\dashrightarrow} \mathbb{P}^3.$$

Projection from R_1 gives an equivariant birational map

$$\pi_{C_1} : V_5 \overset{\sim}{\dashrightarrow} X$$

onto a smooth quadric threefold.

The action on

$$X = \{x_1x_2 = x_3x_4 + x_5^2\} \subset \mathbb{P}^4_{x_1,\ldots,x_5}$$

is given by

$$\begin{array}{lcl} \tau : (x_1,x_2,x_3,x_4,x_5) & \mapsto & (x_2,x_1,x_4,x_3,x_5) \\ \sigma : (x_1,x_2,x_3,x_4,x_5) & \mapsto & (\lambda^{-3}x_1,\lambda^3x_2,\lambda^{-1}x_3,\lambda x_4,x_5), \end{array}$$

where λ is a character of $\mathbb{G}_m$.

For example, setting $\lambda = e^{2\pi i/3}$ gives a linearization of the $\mathfrak{S}_3$-action

$$X \subset \mathbb{P}(V), \quad V = V_\pm \oplus V_2 \oplus \mathbf{1},$$

where $V_\pm$ is the permutation representation on x_1, x_2, and V_2 is the unique faithful 2-dimensional representation of $\mathfrak{S}_3$. To our knowledge, linearizability of other actions of $\mathfrak{S}_3$ is unknown. For $\mathfrak{D}_8$ ($\lambda = e^{\pi i/4}$) we obtain linearizability when

$$X \subset \mathbb{P}(V) \quad V = V_2 \oplus V_2' \oplus \mathbf{1},$$

where V_2 and V_2' are *different* faithful 2-dimensional representations of $\mathfrak{D}_8$.

Proposition 6.1. *Let*

$$X := \{x_1x_2 = x_3x_4 + x_5^2\}$$

and $G = \mathfrak{D}_{4n}$ with n odd, acting via faithful 2-dimensional representations on $\{x_1,x_2\}$ and $\{x_3,x_4\}$, and trivially on x_5. Then the G-action on X is stably linearizable.

Proof. Proposition 5.3 reduces us to the 2-Sylow subgroup $\mathfrak{D}_4 \subset \mathfrak{D}_{2n}$, which acts via

$$\begin{array}{rcl} \tau : (x_1, x_2, x_3, x_4, x_5) & \mapsto & (x_2, x_1, x_4, x_3, x_5) \\ \sigma : (x_1, x_2, x_3, x_4, x_5) & \mapsto & (ix_1, i^3x_2, \iota^3 x_3, \iota x_4, x_5), \end{array}$$

where i and ι are primitive fourth roots of unity. This has an isotropic subspace by Proposition 3.7 and hence is linearizable (for $\mathfrak{D}_4$) by Theorem 3.5. □

We now turn to the diagonal quadric

$$X = \{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0\} \subset \mathbb{P}^4, \tag{4}$$

with an action of a subgroup $G \subset W(\mathsf{D}_5)$, the Weyl group of D_5, via signed permutations. In [TYZ24, Section 9], there is a classification of G such that

- all abelian subgroups $H \subset G$ have fixed points,
- G does not have a fixed point.

Recall that existence of fixed points for generically free actions of abelian groups is a birational invariant of smooth projective varieties and that linear actions of abelian groups have fixed points. Furthermore, a fixed point yields a linearization for the action.

The maximal groups on the list in [TYZ24] are:

$$\mathfrak{S}_5, \quad \mathfrak{S}_4, \quad C_4 \wr C_2, \quad \mathsf{GL}_2(\mathbb{F}_3), \quad \mathfrak{S}_3 \times \mathfrak{D}_4, \quad \mathfrak{D}_8,$$

and the maximal 2-Sylow subgroups without fixed points are

$$\mathfrak{D}_4 \subset \mathfrak{S}_4, \quad C_4 \wr C_2, \quad SD16 \subset \mathsf{GL}_2(\mathbb{F}_3), \quad \mathfrak{D}_8,$$

with specific 5-dimensional representations V, giving rise to $X \subset \mathbb{P}(V)$.

Proposition 6.2. *The G-action on X in* (4) *is stably linearizable if:*

1. *$G = \mathfrak{S}_5$, with the standard permutation action,*
2. *$G = \mathfrak{S}_3 \times C_2^2 \subset \mathfrak{S}_3 \times \mathfrak{D}_4$, with the standard permutation action of $\mathfrak{S}_3$ on the first variables and sign changes on x_4, x_5.*

Proof. This is another corollary of Proposition 5.3: the 2-Sylow subgroup $G_2 \subset G$ has fixed points. □

Remark 6.3. The G-action in Case (1) is known to admit only two Mori fiber space models, and in particular is not linearizable, by [CSZ23, Theorem 3.1]; in Case (2), the Burnside obstructions of [KT22] prevent linearizability, see [TYZ24, Example 9.2].

Theorem 3.5 yields the stronger result:

Proposition 6.4. *The following G-actions on the quadric*

$$X := \{x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0\} \subset \mathbb{P}^4,$$

are linearizable:

- *the unique (up to conjugation) $G = C_4 \wr C_2 \subset W(\mathsf{D}_5)$,*
- *the unique semidihedral group $SD16 \subset W(\mathsf{D}_5)$,*
- *the unique dihedral group $\mathfrak{D}_8 \subset W(\mathsf{D}_5)$.*

Proof. The restriction of the $W(\mathsf{D}_5)$-action to the (unique) $C_4 \wr C_2$ has character $(5,-3,1,1,1,1,1,-3,1,1,-3,1,-1,-1)$. Concretely, the action is given by

$$\sigma_1 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\sigma_4 := \mathrm{diag}(-1,-1,1,1), \quad \sigma_5 := \mathrm{diag}(1,1,-1,-1),$$

with σ_i acting on the variables $x_1, \dots, x_4$ as indicated, with $\sigma_1 : x_5 \mapsto -x_5$, and all other generators acting trivially on x_5. In particular,

$$X \subset \mathbb{P}^4 = \mathbb{P}(V_2 \oplus V_2' \oplus \chi),$$

where the characters are given by

$$\begin{array}{rcl} \mathrm{char}(V_2) & = & (2,-2,0,0,2i,-2i,1-i,0,1+i,-1-i,-1+i,0,0,0), \\ \mathrm{char}(V_2') & = & (2,-2,0,0,-2i,2i,1+i,0,1-i,-1+i,-1-i,0,0,0). \end{array}$$

Theorem 3.5 applies.

We turn to $SD16$: the restriction of the $W(\mathsf{D}_5)$-action to $G = SD16$ yields a representation with character $(5,-3,-1,1,1,-1,-1)$. The action is given by

$$\sigma_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\sigma_4 := \mathrm{diag}(-1,-1,-1,-1),$$

with σ_i acting on $x_1, \dots, x_4$ as indicated and σ_1, σ_2 acting via $x_5 \mapsto -x_5$, and σ_3, σ_4 acting trivially on x_5. Thus

$$X \subset \mathbb{P}^4 = \mathbb{P}(V_2 \oplus V_2' \oplus \chi)$$

as an invariant quadric with generically free G-action, where V_2, V_2' are 2-dimensional faithful, complex conjugate, representations of G, and χ is a character. Since $V_2' = V_2^\vee$, Theorem 3.5 applies.

We repeat the analysis for $G = \mathfrak{D}_8$. The restriction of the $W(\mathsf{D}_5)$-action to G yields a representation V with character $(5,-3,-1,1,1,-1,-1)$, which decomposes as

$$V = V_2 \oplus V_2' \oplus \chi,$$

where the characters are given by

$$\begin{array}{rcl} \mathrm{char}(V_2) & = & (2,-2,0,0,0,\sqrt{2},-\sqrt{2}) \\ \mathrm{char}(V_2') & = & (2,-2,0,0,0,-\sqrt{2},\sqrt{2}) \\ \chi & = & (1,1,-1,1,1,-1,-1), \end{array}$$

and the standard linearizability via a twisted cubic applies. $\square$

Corollary 6.5. *Let $G \subset W(\mathsf{D}_5)$ be such that the induced action on the diagonal quadric*

$$X \subset \mathbb{P}^4 = \mathbb{P}(V),$$

via the standard irreducible 5-dimensional representation V of $W(\mathsf{D}_5)$ satisfies the following properties:

- *for every abelian $H \subseteq G$ one has $X^H \neq \emptyset$,*

- *G does not contain a subgroup $H \simeq \mathfrak{D}_4$ such that the restriction of the representation V to H decomposes as*
$$V|_H = V_2 \oplus \chi \oplus \chi' \oplus \chi'',$$
where χ, χ', χ'' are pairwise distinct characters of H.

Then the G-action on X is stably linearizable.

Remark 6.6. The methods above give no information about (stable) linearizability of the following G-actions on smooth quadric threefolds $X \subset \mathbb{P}(V)$:

- $G = \mathfrak{D}_4$ and $V = V_2 \oplus \chi \oplus \chi' \oplus \chi''$, where V_2 is the unique irreducible 2-dimensional representation and χ, χ', χ'' are pairwise distinct characters.
- $G = \mathfrak{D}_8$ and $V = V_2 \oplus V_2' \oplus \mathbf{1}$, where V_2 is the non-faithful 2-dimensional irreducible representation and V_2' is a faithful representation of G.

References

[ACC+23] C. Araujo, A.-M. Castravet, I. Cheltsov, K. Fujita, A.-S. Kaloghiros, J. Martinez-Garcia, C. Shramov, H. Süß, and N. Viswanathan. *The Calabi problem for Fano threefolds*, volume 485 of *Lond. Math. Soc. Lect. Note Ser.* Cambridge: Cambridge University Press, 2023.

[BPT10] F. Bogomolov, T. Petrov, and Yu. Tschinkel. Unramified cohomology of finite groups of Lie type. In *Cohomological and geometric approaches to rationality problems*, volume 282 of *Progr. Math.*, pages 55–73. Birkhäuser Boston, Boston, MA, 2010.

[BvBT23] Chr. Böhning, H.-Chr. Graf von Bothmer, and Yu. Tschinkel. Equivariant birational geometry of cubic fourfolds and derived categories, 2023. `arxiv:2303.17678`.

[CGR06] V. Chernousov, P. Gille, and Z. Reichstein. Resolving G-torsors by abelian base extensions. *J. Algebra*, 296(2):561–581, 2006.

[CSZ23] I. Cheltsov, A. Sarikyan, and Z. Zhuang. Birational rigidity and alpha invariants of Fano varieties, 2023. `arxiv:2304.11333`.

[CTS87] J.-L. Colliot-Thélène and J.-J. Sansuc. La descente sur les variétés rationnelles. II. *Duke Math. J.*, 54(2):375–492, 1987.

[CTS07] J.-L. Colliot-Thélène and J.-J. Sansuc. The rationality problem for fields of invariants under linear algebraic groups (with special regards to the Brauer group). In *Algebraic groups and homogeneous spaces*, volume 19 of *Tata Inst. Fund. Res. Stud. Math.*, pages 113–186. Tata Inst. Fund. Res., Mumbai, 2007.

[DI09] I. V. Dolgachev and V. A. Iskovskikh. Finite subgroups of the plane Cremona group. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I*, volume 269 of *Progr. Math.*, pages 443–548. Birkhäuser Boston, Boston, MA, 2009.

[DR15] A. Duncan and Z. Reichstein. Versality of algebraic group actions and rational points on twisted varieties. *J. Algebr. Geom.*, 24(3):499–530, 2015.

[HKT21] B. Hassett, A. Kresch, and Yu. Tschinkel. Symbols and equivariant birational geometry in small dimensions. In *Rationality of Varieties*, volume 342 of *Progr. Math.*, pages 201–236. Birkhäuser, Cham, 2021.

[HT23] B. Hassett and Yu. Tschinkel. Torsors and stable equivariant birational geometry. *Nagoya Math. J.*, 250:275–297, 2023.

[Isk08] V. A. Iskovskikh. Two non-conjugate embeddings of $S_3 \times Z_2$ into the Cremona group. II. In *Algebraic geometry in East Asia—Hanoi 2005*, volume 50 of *Adv. Stud. Pure Math.*, pages 251–267. Math. Soc. Japan, Tokyo, 2008.

[Kat92] P. Katsylo. On the birational classification of linear representations, 1992. MPI preprint.

[Kol24] J. Kollár. Automorphisms and twisted forms of rings of invariants. *Milan J. Math.*, 2024.

[KT22] A. Kresch and Yu. Tschinkel. Equivariant birational types and Burnside volume. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 23(2):1013–1052, 2022.

[LPR06] N. Lemire, V. L. Popov, and Z. Reichstein. Cayley groups. *J. Amer. Math. Soc.*, 19(4):921–967, 2006.

[NP23] G. Nebe and R. Parker. Equivariant quadratic forms in characteristic 2. *Bull. Lond. Math. Soc.*, 55(2):668–679, 2023.

[Ser58] J.-P. Serre. Espaces fibrés algébriques. *Séminaire Claude Chevalley*, 3:1–37, 1958.

[Tot09] B. Totaro. Birational geometry of quadrics. *Bull. Soc. Math. France*, 137(2):253–276, 2009.

[TYZ24] Yu. Tschinkel, K. Yang, and Zh. Zhang. Equivariant birational geometry of linear actions. *EMS Surv. Math. Sci.*, 11(2):235–276, 2024.