STABLE EQUIVARIANT BIRATIONALITIES OF CUBIC AND DEGREE 14 FANO THREEFOLDS

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Abstract. We develop an equivariant version of the Pfaffian-Grassmannian correspondence and apply it to produce examples of nontrivial twisted equivariant stable birationalities between cubic threefolds and degree 14 Fano threefolds.

1. Introduction

Over algebraically closed fields, birationality of smooth cubic threefolds

$$Y \subset \mathbb{P}^4$$

and associated Fano threefolds X of degree 14 and Picard rank 1,

$$X = \mathbb{P}^9 \cap \mathrm{Gr}(2,6) \subset \mathbb{P}^{14},$$

is well-known classically. Recently, it has been reconsidered from the perspective of vector bundles and derived categories [Kuz04]. However, birationality in presence of group actions is widely open.

In this paper, we work over an algebraically closed field k of characteristic zero and focus on equivariant birationalities. Our starting point is the following beautiful example: There is a distinguished smooth cubic threefold Y, the *Klein cubic*, with $G := \mathrm{Aut}(Y) = \mathsf{PSL}_2(\mathbb{F}_{11})$; it is given by

$$y_1^2y_2 + y_2^2y_3 + y_3^2y_4 + y_4^2y_5 + y_5^2y_1 = 0.$$

There is also a unique smooth Fano threefold X of degree 14 and Picard number 1, admitting a regular, generically free, action of G. The threefolds X and Y are birational. However, these G-actions are birationally rigid, and thus not equivariantly birational to each other [BCDP23, Theorem 4.3]. We complement this result by showing:

Theorem 1.1 (Proposition 4.1)**.** *Let Y be the Klein cubic threefold and X the associated Fano threefold of degree 14, with a generically*

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free regular action of $G = \mathsf{PSL}_2(\mathbb{F}_{11})$. Then there is a G-equivariant birationality

$$Y \times \mathbb{P}^2 \times \mathbb{P}(V) \sim_G X \times \mathbb{P}^2 \times \mathbb{P}(V),$$

with trivial G-action on $\mathbb{P}^2$, and projectively linear G-action on the projectivization of an irreducible 6-dimensional representation V of the central extension $\tilde{G} = \mathsf{SL}_2(\mathbb{F}_{11})$.

Birational rigidity techniques work well when the group under consideration is *large*, with large orbits. We are not aware of results such as [BCDP23, Theorem 4.3] for actions of any other finite groups on cubic threefolds Y and the associated X.

In this note, we apply the recently developed formalism of equivariant Burnside groups [KT22] to exhibit actions failing equivariant birationality for such pairs. We introduce the notion of *twisted* equivariant stable birationality (in Section 2) and produce examples of equivariantly nonbirational but twisted equivariantly stably birational varieties, for actions of finite groups; this relies on vector bundle techniques, and in particular, the no-name lemma, which is ubiquitous in equivariant birational geometry. For example, we prove in Section 5:

Theorem 1.2. *There exist smooth cubic threefolds Y, with associated birational Fano threefolds X of degree 14, such that*

- *Y and X carry a generically free regular action of $G = \mathfrak{K}_4$, the Klein four-group,*
- *the G-actions are not equivariantly birational,*
- *the G-actions are twisted equivariantly stably birational.*

The proofs proceed via an equivariant version of the classically known Pfaffian-Grassmannian construction, recalled in Section 3. We classify automorphisms of smooth cubic threefolds admitting an equivariant Pfaffian realization, in Proposition 3.7. In Section 4, we explain our approach in the case of $G = \mathsf{PSL}_2(\mathbb{F}_{11})$; we show that the recently introduced Burnside invariants *do not* distinguish the actions on Y and X. In Section 5, we turn to $G = C_3 \rtimes \mathfrak{D}_4$ and its subgroup $\mathfrak{K}_4$. In this case, birational rigidity techniques fail, but the Burnside invariants show nonbirationality of the actions. In Sections 6 and 7, we consider actions of dihedral groups and of the symmetric group $\mathfrak{S}_5$, on *singular* Y and X.

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2. Equivariant birational geometry

Birationality and stable birationality. We consider G-varieties, i.e., projective varieties X with a regular, generically free action of a linear algebraic group G; in most of our applications, G is a finite group. Equivariant birationality of G-varieties Y, X is denoted by

$$Y \sim_G X.$$

Equivariant stable birationality means that

$$Y \times \mathbb{P}^m \sim_G X \times \mathbb{P}^m,$$

for some m, with *trivial* action on the second factor. Examples of equivariantly nonbirational but stably birational actions are sought after, see [Kol22, Section 4], and produced in [HT23], [BGvBT23a], [BGvBT23b].

These notions should be viewed as analogous to birationality and stable birationality of varieties over nonclosed fields.

Twisted equivariant stable birationality. One important distinction between equivariant geometry and geometry over nonclosed fields is that there is only *one* projective space of a given dimension, but possibly several, nonbirational $\mathbb{P}(V)$ for a (projectively) linear action of a finite group, see [TYZ23]. This leads us to the notion of *twisted equivariant stable birationality*, when

$$Y \times \prod_j \mathbb{P}(V_j) \sim_G X \times \prod_j \mathbb{P}(V_j),$$

where V_j are linear representations of extensions of G such that G acts (projectively) linearly on $\mathbb{P}(V_j)$, for all j. This does not preclude equivariant stable birationality of Y and X, a priori. See, e.g., [HT22, Section 2] for a discussion of (projectively) linear actions, linearizability, and stable linearizability.

Over nonclosed fields, this notion is analogous to birationality after multiplication with Brauer-Severi varieties.

No-name lemma. Let $\mathcal{E} \to X$ be a G-vector bundle of rank m, with a generically free G-action on X and $\mathcal{E}$. Then

$$\mathcal{E} \sim_G X \times \mathbb{P}^m,$$

with trivial action on $\mathbb{P}^m$, see, e.g., [CGR06, Section 4.3]. In particular, for any G-representation V of dimension m, one has

$$X \times V \sim_G X \times \mathbb{P}^m,$$

with trivial action on $\mathbb{P}^m$. This also implies that all faithful *linear* actions of a finite group G on projective spaces are stably birational. For *projectively linear* actions of G arising from faithful representations V, W of a central extension $\tilde{G}$ of G, with the same central character, the no-name lemma applied to the extension

$$1 \to \mathbb{G}_m \to G' \to G \to 1,$$

(where G' contains $\tilde{G}$) shows that

$$\mathbb{P}(W) \times \mathbb{P}^m \sim_G \mathbb{P}(W \oplus V), \quad m = \dim(V), \tag{2.1}$$

with trivial action on $\mathbb{P}^m$. The no-name lemma substantially simplifies the birational geometry of equivariant vector bundles; we will apply it in Section 3.

Equivariant Burnside groups. This formalism produces a homomorphism from the free abelian group on *equivariant birational types*, i.e., equivariant birationality classes of n-dimensional varieties, to

$$\mathrm{Burn}_n(G),$$

a group defined by generators

$$\mathfrak{s} = (H, Z \circlearrowright k(F), \beta),$$

subject to conjugacy and blowup relations [KT22]. The class of a G-action on a variety is computed on a *standard model*, see [HKT21, Section 7.2]. On such a model X, all stabilizers are abelian, and the class

$$[X \circlearrowleft G] := \sum_{H,F} (H, Z \circlearrowright k(F), \beta) \in \mathrm{Burn}_n(G)$$

is a sum over all (conjugacy classes of) abelian subgroups $H \subseteq G$, and strata $F \subseteq X$, of dimension $d \leq n$, with generic stabilizer H and residual action of $Z \subseteq Z_G(H)/H$; here $\beta = (b_1, \ldots, b_{n-d})$ is the collection of weights of H in the normal bundle to F.

One of the relations in $\mathrm{Burn}_n(G)$ states that the symbol $\mathfrak{s}$ vanishes if there exists a nonempty $I \subseteq [1, \ldots, n-d]$ such that $\sum_{i \in I} b_i = 0$.

There is a subgroup

$$\mathrm{Burn}_n^{\mathrm{inc}}(G) \subset \mathrm{Burn}_n(G)$$

freely generated by *incompressible* symbols, see, e.g., [TYZ23, Section 3.6]. In many situations, we can distinguish G-actions already via projection of $[X \circlearrowleft G]$ to this subgroup. We record:

Example 2.1. Let $Y \subset \mathbb{P}^4$ be a cubic threefold with the action of $G = \mathfrak{K}_4$ via (3.7). Note that one of the generating involutions fixes a cubic surface $S \subset Y$, and the fixed locus of the residual involution on S is a cubic curve $C \subset S$.

Assume that $C = Y^G$ is *smooth*. Then the symbol

$$(C_2, C_2 \circlearrowright k(S), (1))$$

is incompressible, and the G-action is not linearizable, see [CTZ24, Proposition 2.6].

Such a symbol also arises from a $\mathfrak{K}_4$-action on a 4-nodal cubic threefold [CTZ24, Example 5.2]; or from a C_4-action in [CTZ24, Example 2.7].

Example 2.2. Let $Y \subset \mathbb{P}^4$ be a singular cubic threefold given by

$$y_1y_2y_3 + f_3(y_3, y_4, y_5) = 0,$$

where f_3 is a cubic form. It carries the action of the dihedral group $G := \mathfrak{D}_{2n}$ of order $4n \geq 8$, via

$$y_1 \leftrightarrow y_2, \quad (y_1, y_2, y_3, y_4, y_5) \mapsto (\zeta y_1, \zeta^{-1} y_2, y_3, y_4, y_5),$$

where ζ is a primitive root of unity of order $2n$.

Then G fixes a plane cubic curve $C \subset \mathbb{P}^2_{y_3,y_4,y_5}$ given by $f_3 = 0$. Assume that C is *smooth*. To reach a standard model, one has to blow up C. Computing the class $[Y \circlearrowleft \mathfrak{D}_{2n}]$ on such a model, we find the *incompressible* symbol

$$(C_2, \mathfrak{D}_n \circlearrowright k(C)(t), (1)), \quad n \geq 2, \tag{2.2}$$

where C is a genus 1 curve, see [CMTZ24, Proposition 5.17]. Choosing roots of unity ζ, ζ' such that $\zeta \neq \pm\zeta'$, we obtain equivariantly nonbirational, nonlinearizable G-actions on the rational cubic threefold Y.

3. Dualities

Pfaffian-Grassmannian correspondence. We follow the presentation in [Kuz04, Section 2], which builds on classical constructions, see, e.g., [Put82], [IM00]. Let A and V be vector spaces over k of dimension 5, respectively, 6. Let

$$f : A \to \wedge^2(V^\vee)$$

be an injective linear map, called *A-net of skew forms on V*. It is called *regular*, if $\mathrm{rk}(f(a)) \geq 4$, for all nonzero $a \in A$.

We consider the following varieties, which are smooth for generic f:

$$Y_f = \mathbb{P}(f(A)) \cap \mathrm{Gr}(2, V)^\vee, \text{a cubic threefold,}$$
$$X_f = \mathbb{P}(f(A)^\perp) \cap \mathrm{Gr}(2, V), \text{a degree 14 Fano threefold,}$$

where $\mathrm{Gr}(2, V)^\vee \subset \wedge^2(V^\vee)$ is the projective dual of $\mathrm{Gr}(2, V)$, given by the vanishing of the Pfaffian cubic form $\mathrm{Pf} \in \mathrm{Sym}^3(\wedge^2(V^\vee))$. Note that $\mathrm{Gr}(2, V)^\vee$ parametrizes skew forms on V of rank at most four. Let $\mathcal{U}_f$ be the restriction of the tautological rank two bundle on $\mathrm{Gr}(2, V)$ to X_f. For a *regular* f, there is a natural rank two theta-bundle $\mathcal{E}_f$ over Y_f, see [Kuz04, Section 2] and [IM00, Theorem 2.2]; the dual bundle $\mathcal{E}_f^\vee$ is the rank two subbundle of the rank six trivial bundle $Y_f \times V$ given by

$$\mathcal{E}_f^\vee = \{(y, v) \in Y_f \times V \mid v \in \ker(y)\}.$$

The injective classifying morphism

$$\kappa : Y_f \to \mathrm{Gr}(2, V), \quad y \mapsto \ker(y)$$

induces an embedding of Y_f in $\mathrm{Gr}(2, V)$. Under this embedding, $\mathcal{E}_f^\vee$ is naturally identified with the restriction of the tautological bundle from $\mathrm{Gr}(2, V)$ to Y_f. Furthermore,

$$\mathrm{Sing}(X_f) = \mathrm{Sing}(Y_f) = X_f \cap Y_f \subset \mathrm{Gr}(2, V). \tag{3.1}$$

By [Kuz04, Theorem 2.18], for a *regular* f, we have a diagram

$$\begin{array}{ccccc} \mathcal{E}_f^\vee & & \overset{\theta}{\dashrightarrow} & & \mathcal{U}_f \\ \downarrow & \overset{\psi}{\searrow} & & \overset{\phi}{\swarrow} & \downarrow \\ Y_f & & V & & X_f \end{array} \tag{3.2}$$

where the morphisms ψ, ϕ are induced by the natural projection from the tautological bundle over $\mathrm{Gr}(2, V)$ to V. The images of ψ and ϕ can be described as follows

$$\psi(\mathcal{E}_f^\vee) = \{v \in V \mid v \in \ker(f(a)) \text{ for some nonzero } a \in A\},$$
$$\phi(\mathcal{U}_f) = \{v \in V \mid v \in \ell \text{ for some } \ell \in X_f \subset \mathrm{Gr}(2, V)\}.$$

Linear algebra shows that $\psi(\mathcal{E}_f^\vee) = \phi(\mathcal{U}_f)$, see e.g., [Kuz04, Proposition 2.11, 2.15], [Put82, Theorem B]. In fact, the common image of ψ and ϕ is a quartic hypersurface

$$Q_f \subset V,$$

singular along the affine cone $\tilde{C}_f$ over a curve $C_f \subset \mathbb{P}(Q_f) \subset \mathbb{P}(V)$. Both ψ and ϕ are isomorphisms on the complement $Q_f \setminus \tilde{C}_f$. The composition $\theta := \phi^{-1} \circ \psi$ is then a birational map between vector bundles. After projectivization, θ induces a birational map between $\mathbb{P}^1$-bundles

$$\mathbb{P}(\mathcal{E}_f^\vee) \dashrightarrow \mathbb{P}(\mathcal{U}_f),$$

which is a flop in a ruled surface [Kuz04, Theorem 2.17].

Remark 3.1. In the literature, the existence of the diagram (3.2) and the birationality of θ are proved for the projectivizations $\mathbb{P}(\mathcal{E}_f^\vee), \mathbb{P}(\mathcal{U}_f)$ and $\mathbb{P}(V)$. However, the underlying linear algebra proof applies to the vector bundles verbatim. To study the equivariant geometry of this construction, we use the diagram of vector bundles (3.2).

As explained in [Kuz04, Remark 2.19], fixing a hyperplane $\Pi \subset \mathbb{P}(V)$, there are induced birational maps

$$\varrho_\Pi : Y_f \overset{\sim}{\dashrightarrow} \Pi \cap \mathbb{P}(Q_f) \overset{\sim}{\dashleftarrow} X_f. \tag{3.3}$$

Equivariant Pfaffian-Grassmannians. To arrive at an action of a finite group G on Y and X, we start with a faithful 6-dimensional $\tilde{G}$-representation V of a central extension

$$1 \to \mathbb{G}_m \to \tilde{G} \to G \to 1$$

which induces a generically free action of G on $\mathbb{P}(V)$, i.e., the central $\mathbb{G}_m$ is acting trivially on $\mathbb{P}(V)$. Let A be a 5-dimensional vector space and

$$f : A \to \wedge^2(V^\vee) \tag{3.4}$$

a regular A-net such that

$$f(A) \subset \wedge^2(V^\vee)$$

is a 5-dimensional $\tilde{G}$-invariant subspace. Assume that the induced G-actions on $\mathbb{P}(f(A))$ and on $\mathbb{P}(f(A)^\perp)$ are also generically free. Then

$$Y_f := \mathrm{Gr}(2, V)^\vee \cap \mathbb{P}(f(A)), \quad X_f := \mathrm{Gr}(2, V) \cap \mathbb{P}(f(A)^\perp), \tag{3.5}$$

carry G-actions, which are again generically free. The $\tilde{G}$-actions naturally lift to the vector bundles $\mathcal{E}_f^\vee$ and $\mathcal{U}_f$.

We refer to this construction as *equivariant Pfaffian-Grassmannian correspondence*. A generically free G-action on a smooth cubic threefold Y is called *equivariantly Pfaffian* if it arises from an equivariant Pfaffian-Grassmannian correspondence.

Remark 3.2. Recall that every smooth cubic threefold over $\mathbb{C}$ admits a Pfaffian representation [AR96], [MT01]; in fact, this holds also for singular cubics [Com20]. By [Bea00, Theorem 8.2], a smooth cubic threefold Y over a nonclosed field k is Pfaffian if and only if there is an arithmetically Cohen-Macaulay curve $C \subset Y$, not contained in a hyperplane, with $K_C = \mathcal{O}_C$, i.e., an elliptic normal quintic, defined over k.

In the equivariant context, this criterion fails: the presence of a G-stable elliptic quintic C on Y implies equivariant birationality with the corresponding X, see, e.g., [IM00, Theorem 1.1]. However, in Section 5, we produce examples of equivariantly Pfaffian actions on smooth Y and X, which are not equivariantly birational.

Twisted equivariant stable birationality. Given an equivariant Pfaffian-Grassmannian correspondence, the diagram (3.2) constructed from a regular A-net (3.4) is $\tilde{G}$-equivariant. In particular, the corresponding birational maps ψ, ϕ and θ in (3.2) are $\tilde{G}$-equivariant since their constructions are canonical. We have an equivariant birationality

$$\mathcal{E}_f^\vee \sim_{\tilde{G}} \mathcal{U}_f.$$

However, the no-name lemma *does not* apply directly to this situation, since the $\tilde{G}$-action has a nontrivial generic stabilizer on the bases Y_f and X_f. Therefore, we use a variant: let W be a faithful representation of $\tilde{G}$, inducing a generically free G-action on $\mathbb{P}(W)$, with the central $\mathbb{G}_m$ acting via scalars. E.g., we could put $W = V$. Consider the diagram

$$\begin{array}{ccc} \mathcal{E}_f^\vee \times W & \overset{\sim}{\dashrightarrow} & \mathcal{U}_f \times W \\ \downarrow & & \downarrow \\ Y_f \times W & & X_f \times W \end{array}$$

The horizontal map is a $\tilde{G}$-equivariant birational isomorphism. The vertical maps are $\tilde{G}$-equivariant rank two vector bundles, and the action of $\tilde{G}$ on the respective bases is generically free. By the no-name lemma, we have

$$Y_f \times \mathbb{A}^2 \times W \sim_{\tilde{G}} X_f \times \mathbb{A}^2 \times W,$$

with trivial $\tilde{G}$-action on the $\mathbb{A}^2$-factors. The projections to the respective bases are equivariant under the action of $\mathbb{G}_m = \ker(\tilde{G} \to G)$. This implies G-equivariant birationality of the quotients

$$(Y_f \times \mathbb{A}^2 \times W)/\mathbb{G}_m \sim_G (X_f \times \mathbb{A}^2 \times W)/\mathbb{G}_m,$$

and therefore

$$Y_f \times \mathbb{P}^2 \times \mathbb{P}(W) \sim_G X_f \times \mathbb{P}^2 \times \mathbb{P}(W),$$

with trivial G-action on $\mathbb{P}^2$ and projectively linear G-action on $\mathbb{P}(W)$. This yields:

Proposition 3.3. *Given an equivariant Pfaffian-Grassmannian correspondence, we have a twisted equivariant stable birationality between the corresponding cubic threefold Y_f and the associated degree 14 Fano threefold X_f.*

Remark 3.4. We do not know whether or not Y_f and X_f are equivariantly stably birational, unless we can apply the no-name lemma directly to $\mathcal{E}_f^\vee$ and $\mathcal{U}_f$, i.e., when V can be chosen to be a G-representation.

Remark 3.5. If there is a G-stable hyperplane $\Pi \subset \mathbb{P}(V)$, with a genericaly free action of G, then ϱ_Π from (3.3) is G-equivariant, and

$$Y \sim_G X.$$

This happens, e.g., when G is cyclic.

Equivariantly Pfaffian actions. Finite groups which can act regularly and generically freely on smooth cubic threefolds have been classified in [WY20]. We recall:

- There are 6 maximal groups of automorphisms of smooth cubic threefolds, by [WY20, Theorem 1.1]:

$$C_3^4 \rtimes \mathfrak{S}_5,\ ((C_3^2 \rtimes C_3) \rtimes C_4) \times \mathfrak{S}_3,\ C_{24},\ C_{16},\ \mathsf{PSL}_2(\mathbb{F}_{11}),\ C_3 \times \mathfrak{S}_5. \tag{3.6}$$

- There are 2 types of actions of the Klein four-group $\mathfrak{K}_4$, given (in suitable coordinates) in [WY20, Table 2]:

$$\mathrm{diag}(-1,1,1,1,1), \quad \mathrm{diag}(1,1,-1,1,1), \tag{3.7}$$

$$\mathrm{diag}(-1,-1,1,1,1), \quad \mathrm{diag}(1,-1,-1,1,1). \tag{3.8}$$

Remark 3.6. The unique (up to conjugation) $\mathfrak{K}_4 \subset \mathsf{PSL}_2(\mathbb{F}_{11})$ acts via (3.8) on the Klein cubic threefold. On the other hand, the $\mathfrak{S}_5$ with the permutation action on $\mathbb{P}^4$ contains a $\mathfrak{K}_4$ acting via (3.7) on the Fermat cubic threefold. By Example 2.1, this second $\mathfrak{K}_4$ contributes an incompressible symbol to the class of the action in the Burnside group.

Here we investigate which groups admit equivariantly Pfaffian actions on smooth cubic threefolds. This is an algorithmic task. With `Magma`, we implement the following steps:

(1) Let $\tilde{G}$ be one of the Schur covers of G. List faithful 6-dimensional linear $\tilde{G}$-representations V which induce generically free G-actions on $\mathbb{P}(V)$.
(2) For each such V, and each isomorphism class of 5-dimensional subrepresentations of $\wedge^2(V)$, choose a generic such $A \subset \wedge^2(V)$; check whether or not the induced G-action on $\mathbb{P}(A)$ is generically free and
$$\mathbb{P}(A) \cap \mathrm{Gr}(2, V)^\vee$$
is a smooth cubic threefold.

By construction, any equivariantly Pfaffian action on a smooth cubic threefold can be obtained in this way. Going through the steps (1) and (2), we find

Proposition 3.7. *Let G be a finite group. Then G admits an equivariantly Pfaffian action on a smooth cubic threefold if and only if G is a subgroup of one of the following groups:*
$$\mathsf{PSL}_2(\mathbb{F}_{11}), \quad C_3 \rtimes \mathfrak{D}_4, \quad \mathfrak{F}_5, \quad C_8.$$

Proof. We start with cyclic G, listed in [WY20, Table 2]; in these cases, the Schur cover of G is itself. Applying steps (1) and (2) with $\tilde{G} = G$, we find that the following do not arise from this construction:
$$G = C_3, \quad \text{with weights } (1, 1, 1, \zeta_3, \zeta_3^a), \quad a = 0, 1, 2,$$
$$G = C_4, \quad \text{with weights } (1, 1, 1, -1, \zeta_4).$$
Excluding all cases in [WY20, Table 2] containing one of the C_3 or C_4 actions above, we find that the actions of the following abelian groups
$$C_4 \times C_2, \quad C_3^2, \quad C_9, \quad C_{15}, \quad C_{16}, \quad C_{18}, \quad C_2 \times C_3^2, \quad C_{24},$$
$$C_4 \times C_6, \quad C_3^3, \quad C_3 \times C_9, \quad C_4 \times C_3^2, \quad C_2^2 \times C_3^2, \quad C_2 \times C_3^3, \quad C_3^4$$
on smooth Y and X are not equivariantly Pfaffian.

Then excluding subgroups of the 6 maximal groups in [WY20, Theorem 1.1] which contain a subgroup isomorphic to one of these abelian groups, we are left with
$$\mathsf{PSL}_2(\mathbb{F}_{11}), \quad C_3 \rtimes \mathfrak{D}_4, \quad \mathfrak{S}_5, \quad C_8,$$
and their subgroups.

Applying steps (1) and (2) above to the Schur cover of $\mathfrak{S}_4$, we find that $\mathfrak{S}_4$ and thus also $\mathfrak{S}_5$ does not admit equivariantly Pfaffian actions on *smooth* cubic threefolds.

Excluding $\mathfrak{S}_4$ and $\mathfrak{S}_5$, we are left with 4 maximal groups

$$\mathsf{PSL}_2(\mathbb{F}_{11}), \quad C_3 \rtimes \mathfrak{D}_4, \quad \mathfrak{F}_5, \quad C_8.$$

These admit equivariantly Pfaffian actions on smooth cubic threefolds, see explicit constructions in Sections 4, 5, and [TZ24]. □

Since every faithful 6-dimensional representation of $\mathfrak{F}_5$ and C_8 admits a 1-dimensional subrepresentation, the corresponding smooth cubic Y and degree 14 Fano threefolds X are equivariantly birational, by Remark 3.5. Given our interest in *nontrivial* (twisted) stable birationalities, we present in Sections 4 and 5 explicit Pfaffian constructions for

$$\mathsf{PSL}_2(\mathbb{F}_{11}), \quad C_3 \rtimes \mathfrak{D}_4.$$

Remark 3.8. The proof of Proposition 3.7 confirms computations by Böhning and von Bothmer indicating that regular C_3-actions on smooth cubic threefold with weights $(1,1,1,1,\zeta_3)$ are not equivariantly Pfaffian. On the other hand, we will see in Section 5 that C_3-actions with weights $(1,1,\zeta_3,\zeta_3^2,\zeta_3^2)$ are equivariantly Pfaffian. Proposition 3.7 also shows that $\mathfrak{S}_5$ does not admit equivariantly Pfaffian actions on smooth cubic threefolds. However, an $\mathfrak{S}_5$-action on *singular* cubics may arise from the Pfaffian construction, see Section 7.

Remark 3.9. The papers [CTZ23], [CTZ24], [CMTZ24] classify actions on cubic threefolds with isolated singularities, under the assumption that the action does not fix any of the singular points, and apply this classification to linearizability questions. It would be interesting to explore the equivariant Pfaffian-Grassmannian correspondence for singular cubic threefolds.

4. Klein cubic threefold and $\mathsf{PSL}_2(\mathbb{F}_{11})$-actions

Here we apply the equivariant Pfaffian-Grassmannian correspondence of Section 3 to the Klein cubic threefold and the associated degree 14 Fano threefold, equipped with the action of $G = \mathsf{PSL}_2(\mathbb{F}_{11})$.

Writing the representation. Let V be one of the two irreducible 6-dimensional representations of

$$\tilde{G} := \mathsf{SL}_2(\mathbb{F}_{11}).$$

Assume V has character

$$\mathrm{char}(V) = (6,-6,0,0,1,1,0,-1,-1,-\lambda,\lambda+1,0,0,-\lambda-1,\lambda),$$

$$\lambda := \zeta_{11}^9 + \zeta_{11}^5 + \zeta_{11}^4 + \zeta_{11}^3 + \zeta_{11}.$$

Then

$$\wedge^2(V^\vee) = A \oplus V_{10},$$

where A and the dual $V_{10}^\vee = A^\perp \subset \wedge^2(V)$ are faithful irreducible representations of G of dimension 5, respectively, 10. The Pfaffian cubic

$$Y := \mathbb{P}(A) \cap \mathrm{Gr}(2, V)^\vee$$

is a smooth cubic threefold with a generically free action of G. Such a cubic (the Klein cubic threefold) is unique; up to a change of variables, it is given by the equation

$$\{y_1^2 y_2 + y_2^2 y_3 + y_3^2 y_4 + y_4^2 y_5 + y_5^2 y_1 = 0\} \subset \mathbb{P}^4_{y_1,\ldots,y_5}.$$

The dual Fano threefold

$$X := \mathrm{Gr}(2, V) \cap \mathbb{P}(A^\perp) \subset \mathbb{P}^9$$

of degree 14 is also smooth. The equations can be found at [TZ24].

Stabilizer stratification. We compute the fixed loci stratification of the G-action on Y and X with `Magma`, recording the data for (orbit representatives of) loci F with nontrivial generic stabilizeres:

- stabilizer of F,
- the residual action on F,
- dimension of F,
- degree of F,
- characters of the induced action on the normal bundle to F.

The fixed loci stratification for the G-action on Y is given by

	Stabilizer	Residue	dim	deg	Characters
1	C_6	triv	0	1	$(4,1,5)$
2–3	C_2^2	triv	0	1	$((1,0),(0,1),(1,1))$
4	C_{11}	triv	0	1	$(2,3,4)$
5	C_5	triv	0	1	$(1,3,4)$
6	C_5	triv	0	1	$(3,2,1)$
7	C_3	triv	0	1	$(2,1,2)$
8	C_2	$\mathfrak{S}_3$	1	3	$(1,1)$
9	C_2	$\mathfrak{S}_3$	1	1	$(1,1)$

On the smooth degree 14 Fano threefold X, it is:

	Stabilizer	Residue	dim	deg	Characters
1	$\mathfrak{A}_4$	triv	0	1	N/A
2	$\mathfrak{S}_3$	triv	0	1	N/A
3	C_6	triv	0	1	$(3,1,5)$
4	$\mathfrak{S}_3$	triv	0	1	N/A
5	C_2^2	triv	0	1	$((1,1),(0,1),(1,0))$
6	C_{11}	triv	0	1	$(3,10,5)$
7–8	C_5	triv	0	1	$(3,1,2)$
9	C_3	triv	0	1	$(2,2,1)$
10	C_3	C_2^2	1	2	$(2,1)$
11	C_2	$\mathfrak{S}_3$	1	6	$(1,1)$

Burnside invariants. The vanishing relation in $\mathrm{Burn}_3(G)$ mentioned in Section 2 implies that the *only* nontrivial contribution to $[Y \circlearrowleft G]$ comes from points with stabilizer C_{11}; and similarly, for X. (This is justified even though both Y and X are not standard models.) A direct computation shows that

$$\begin{aligned}[][Y \circlearrowleft G] &= (C_{11}, 1 \circlearrowright k, (2,3,4)) \\ &= (C_{11}, 1 \circlearrowright k, (3,10,5)) = [X \circlearrowleft G] \in \mathrm{Burn}_3(G).\end{aligned}$$

This is not surprising, given Remark 3.5: Y and X are C_{11}-equivariantly birational. In particular, the Burnside formalism does not allow to distinguish these actions.

Twisted equivariant stable birationality. Birational rigidity techniques yield (see [BCDP23, Theorem 4.3])

$$Y \not\sim_G X.$$

On the other hand, applying Proposition 3.3, we have a twisted G-equivariant stable birationality:

Proposition 4.1. *Let $G = \mathsf{PSL}_2(\mathbb{F}_{11})$, acting on the Klein cubic threefold Y and the associated degree 14 Fano threefold X. Then*

$$Y \times \mathbb{P}^2 \times \mathbb{P}(V) \sim_G X \times \mathbb{P}^2 \times \mathbb{P}(V),$$

with trivial G-action on $\mathbb{P}^2$ and projectively linear G-action on $\mathbb{P}(V)$, arising from a 6-dimensional irreducible representation V of $\mathsf{SL}_2(\mathbb{F}_{11})$.

5. $C_3 \rtimes \mathfrak{D}_4$-ACTIONS

We provide additional examples of nonbirational, but twisted equivariantly stably birational, actions on smooth Y and X. Here, the nonbirationality of the actions is established via Burnside invariants.

Writing the representation. Let $G = C_3 \rtimes \mathfrak{D}_4$ and V be a faithful 6-dimensional representation of $\tilde{G} \simeq C_3 \rtimes \mathfrak{D}_8$, decomposing as

$$V = V_2 \oplus V_4,$$

where V_2 and V_4 are irreducible $\tilde{G}$-representations with character

$$\begin{aligned} &\mathrm{char}(V_2) = (2, -2, 0, 0, 2, 0, -2, 0, 0, \lambda, -\lambda, 0), \quad \lambda = \zeta_8^3 - \zeta_8, \\ &\mathrm{char}(V_4) = (4, -4, 0, 0, -2, 0, 2, 0, 0, 0, 0, 0). \end{aligned} \tag{5.1}$$

Then

$$\wedge^2(V^\vee) = U_1^{\oplus 2} \oplus U_2 \oplus W_1^{\oplus 2} \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5, \tag{5.2}$$

where U_1 and U_2 are distinct 1-dimensional and $W_i, i = 1, \dots, 5$ distinct 2-dimensional representations of G.

Let $A \subset \wedge^2(V^\vee)$ be a generic 5-dimensional subspace isomorphic, as a G-representation to

$$U_1 \oplus W_1 \oplus W_2.$$

One can check that the Pfaffian cubic

$$Y := \mathbb{P}(A) \cap \mathrm{Gr}(2, V)^\vee$$

and the associated degree 14 Fano threefold

$$X := \mathbb{P}(A^\perp) \cap \mathrm{Gr}(2, V)$$

are smooth, and carry a generically free action of G; see [TZ24] for an example and equations.

Stabilizer stratification. Note that G contains *two* conjugacy classes of the Klein four-group $\mathfrak{K}_4$, corresponding to the two $\mathfrak{K}_4$-actions (3.7) and (3.8), with respective characters in A given by

$$(5, -3, 1, -3), \quad (5, -1, 1, -1).$$

The fixed loci stratification for the first $\mathfrak{K}_4$-action is given by

	Stabilizer	Residue	dim	deg	Characters
1	$\mathfrak{K}_4$	triv	1	3	$((1,1),(1,0))$
2	$\mathfrak{K}_4$	triv	0	1	$((1,0),(0,1),(1,0))$
3	$\mathfrak{K}_4$	triv	0	1	$((1,1),(0,1),(1,1))$
4	C_2	C_2	2	3	(1)
5	C_2	C_2	2	3	(1)
6	C_2	C_2	1	1	$(1,1)$

The fourth and fifth strata are smooth cubic surfaces, and the first stratum is a smooth cubic curve contained in both cubic surfaces.

On the smooth degree 14 Fano threefold X, we have:

	Stabilizer	Residue	dim	deg	Characters
1–7	C_2	triv	0	1	$(1,1,1)$
8	C_2	triv	1	1	$(1,1)$
9	C_2	triv	1	1	$(1,1)$
10	C_2	C_2	1	6	$(1,1)$

The last stratum is a degree 6 smooth curve of genus 1.

Burnside invariants. Using the stabilizer stratification above, we obtain:

Proposition 5.1. *For all $G' \subseteq G$ containing a conjugate of $\mathfrak{K}_4 \subset G$ which acts on A with character*

$$(5,-3,1,-3),$$

we have

$$Y \not\sim_{G'} X.$$

Proof. To establish nonbirationality, consider the specified action of $\mathfrak{K}_4 \subset G$ on Y. In the stabilizer stratification, we find incompressible symbols

$$(C_2, C_2 \circlearrowright k(S), (1)),$$

where S is a cubic surface and the residual C_2-action fixes a smooth cubic curve on S. This contributes to the class

$$[Y \circlearrowleft \mathfrak{K}_4] \in \mathrm{Burn}_3(\mathfrak{K}_4),$$

and is an instance of Example 2.1. On the other hand, we see from the stabilizer stratification for X that no such symbols arise in $[X \circlearrowleft \mathfrak{K}_4]$. This implies that

$$Y \not\sim_{\mathfrak{K}_4} X.$$

□

Twisted equivariant stable birationality. Applying Proposition 3.3 and the construction there with $W = V_4$, we obtain:

Proposition 5.2. *We have*

$$Y \times \mathbb{P}^2 \times \mathbb{P}(V_4) \sim_G X \times \mathbb{P}^2 \times \mathbb{P}(V_4),$$

with trivial G-action on $\mathbb{P}^2$ and projectively linear G-action on $\mathbb{P}(V_4)$, arising from the faithful $\tilde{G}$-representation V_4 given by (5.1).

6. Singular examples

Here, we consider actions of dihedral groups $\mathfrak{D}_{2n}$ of order $4n$ on cubic threefolds with $2\mathsf{D}_4$-singularities.

Writing the representation. We start with a faithful irreducible 2-dimensional representation $V_2 = V_2(\chi)$ of $\mathfrak{D}_{4n}$, determined by a primitive character $\chi = \chi_{4n}$ of C_{4n}. In detail, choose generators

$$\mathfrak{D}_{4n} = \langle s, t \mid s^{4n} = t^2 = tsts = 1 \rangle,$$

and consider the representation V_2 given by

$$s \mapsto \begin{pmatrix} \zeta_{4n}^{-r} & 0 \\ 0 & \zeta_{4n}^{r} \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where r is determined by χ_{4n}, $\gcd(r, 4n) = 1$. Put

$$V = V_2^{\oplus 3},$$

this is a faithful 6-dimensional representation of $\mathfrak{D}_{4n}$, with a generically free action of $\mathfrak{D}_{2n}$ on its projectivization $\mathbb{P}(\wedge^2 V)$.

Decomposing the representations. We have:

$$\wedge^2(V) = I^{\oplus 3} \oplus W_1^{\oplus 6} \oplus W_2^{\oplus 3}, \tag{6.1}$$

where

- I is the trivial representation of $\mathfrak{D}_{2n}$,
- W_2 is the irreducible 2-dimensional representation of $\mathfrak{D}_{2n}$ determined by the character χ_{4n}^2, and
- W_1 is the 1-dimensional representation of $\mathfrak{D}_{2n}$ given by

$$s \mapsto (1), \quad t \mapsto (-1).$$

Let $z_1, \dots, z_6$ be coordinates of V, and fix the following basis of $\wedge^2(V)$:

$$u_1 = z_1 \wedge z_2, \quad u_2 = z_1 \wedge z_3, \quad \dots, \quad u_5 = z_1 \wedge z_6, \quad u_6 = z_2 \wedge z_3,$$
$$u_7 = z_2 \wedge z_4, \quad \dots, \quad u_{15} = z_5 \wedge z_6.$$

We can rewrite the decomposition in an appropriate basis

$$\begin{aligned} I^{\oplus 3} &= \langle u_3 + u_6 \rangle \oplus \langle u_5 + u_8 \rangle \oplus \langle u_{12} + u_{13} \rangle, \\ W_1^{\oplus 6} &= \langle u_1 \rangle \oplus \langle u_{10} \rangle \oplus \langle u_{15} \rangle \oplus \langle u_3 - u_6 \rangle \oplus \langle u_5 - u_8 \rangle \oplus \langle u_{12} - u_{13} \rangle, \\ W_2^{\oplus 3} &= \langle u_2, u_7 \rangle \oplus \langle u_4, u_9 \rangle \oplus \langle u_{11}, u_{14} \rangle. \end{aligned}$$

The invariant Pfaffian cubic. Choose a basis $v_1, \dots, v_{15}$ of $\wedge^2(V^\vee)$ corresponding to $u_1, \dots, u_{15}$, we have a similar decomposition of $\wedge^2(V^\vee)$ as in (6.1). Fix the 5-dimensional subspace $A \subset \wedge^2(V^\vee)$, with basis

$$\begin{aligned} y_1 &= v_2, \\ y_2 &= v_7, \\ y_3 &= v_5 - v_8 - v_{10} + v_{12} - v_{13} - v_{15}, \\ y_4 &= -12v_1 + v_5 - v_8 + 2v_{10} + v_{12} - v_{13}, \\ y_5 &= -3v_1 + v_5 - v_8 - v_{10}. \end{aligned} \tag{6.2}$$

Then $\mathfrak{D}_{2n}$ acts generically freely on $\mathbb{P}(A) = \mathbb{P}^4_{y_1,\dots,y_5}$, via

$$\begin{aligned} s &: (\mathbf{y}) \mapsto (\zeta_{2n}^{-r} y_1, \zeta_{2n}^{r} y_2, y_3, y_4, y_5), \\ t &: (\mathbf{y}) \mapsto (y_2, y_1, -y_3, -y_4, -y_5). \end{aligned}$$

This induces a $\mathfrak{D}_{2n}$-action on

$$Y := \mathbb{P}(A) \cap \mathrm{Gr}(2, V)^\vee,$$

a cubic threefold with $2\mathsf{D}_4$-singularities given by

$$Y = \{y_1 y_2 y_3 + y_3^3 + 3y_3^2 y_5 + 45 y_3 y_4^2 + 10 y_4^3 + y_5^3 = 0\} \subset \mathbb{P}^4_{y_1,\dots,y_5}. \tag{6.3}$$

The invariant Fano threefold. The annihilator $A^\perp \subset \wedge^2(V)$ of A is a 10-dimensional subspace, with basis

$$\begin{aligned} x_1 &= u_1 - 3u_{10} + 9u_{12} - 9u_{13} + 21u_{15}, & x_2 &= u_3, \quad x_3 = u_4, \\ x_4 &= u_5 - u_8 + 2u_{10} - 3u_{12} + 3u_{13} - 6u_{15}, & x_5 &= u_6, \\ x_6 &= u_5 + u_8, & x_7 &= u_9, \quad x_8 = u_{11}, \\ x_9 &= u_{12} + u_{13}, & x_{10} &= u_{14}. \end{aligned}$$

The degree 14 Fano threefold

$$X = \mathrm{Gr}(2, V) \cap \mathbb{P}(A^\perp) \subset \mathbb{P}^9_{x_1,\dots,x_{10}}$$

is given by the vanishing of the following polynomials:

$$
\begin{aligned}
& -144x_1^2+114x_1x_4-21x_4^2-x_8x_{10}+x_9^2,\\
& 3x_1x_4-9x_1x_5-3x_1x_6-2x_4^2+3x_4x_5+2x_4x_6+x_5x_9,\\
& -9x_1x_7+3x_4x_7+x_4x_{10}-x_6x_{10}+x_7x_9,\\
& 21x_1x_2-9x_1x_4-9x_1x_6-6x_2x_4-x_3x_{10}+3x_4^2+3x_4x_6+x_4x_9+x_6x_9,\\
& -9x_1x_3+3x_3x_4-x_3x_9+x_4x_8+x_6x_8\\
& x_1x_8+x_3x_5,\\
& -9x_1^2+3x_1x_4+x_1x_9+x_2x_4-x_2x_6,\\
& -3x_1^2+2x_1x_4+x_2x_5,\\
& x_1x_{10}-x_2x_7,\\
& -9x_1x_2-3x_1x_4-3x_1x_6+3x_2x_4-x_2x_9+2x_4^2+2x_4x_6,\\
& -3x_1x_7+2x_4x_7+x_5x_{10},\\
& 9x_1^2-3x_1x_4+x_1x_9+x_4x_5+x_5x_6,\\
& 21x_1^2-6x_1x_4-x_3x_7-x_4^2+x_6^2,\\
& -3x_1x_3-x_2x_8+2x_3x_4,\\
& 9x_1x_4+21x_1x_5-9x_1x_6-3x_4^2-6x_4x_5+3x_4x_6+x_4x_9-x_6x_9+x_7x_8.
\end{aligned}
$$

The group $\mathfrak{D}_{2n}$ acts generically freely on $\mathbb{P}^9$ and X, via

$$
\begin{aligned}
s &: (\mathbf{x}) \mapsto (x_1, x_2, \zeta^{-r}x_3, x_4, x_5, x_6, \zeta^r x_7, \zeta^{-r}x_8, x_9, \zeta^r x_{10}), \quad \zeta = \zeta_{2n},\\
t &: (\mathbf{x}) \mapsto (-x_1, x_5, x_7, -x_4, x_2, x_6, x_3, x_{10}, x_9, x_8).
\end{aligned}
$$

The Fano threefold X is *singular* along two disjoint *lines*

$$\{x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_8 = x_9 = 0\}$$

and

$$\{x_1 = x_2 = x_4 = x_5 = x_6 = x_7 = x_9 = x_{10} = 0\}.$$

Remark 6.1. Note that here $\mathrm{Sing}(X) \not\simeq \mathrm{Sing}(Y)$, contrary to [Kuz04, Proposition A.4]. This is explained by the fact that the embedding $f : A \hookrightarrow \wedge^2(V^\vee)$ is *not* regular, which requires that *all* forms in A have rank ≥ 4. In our case, for general subrepresentations in $\wedge^2(V^\vee)$ isomorphic to A, from (6.2), the locus $\mathbb{P}(A) \cap \mathrm{Gr}(2, V^\vee)$ consists of two *points*, representing skew-symmetric forms of rank 2 in $\wedge^2(V^\vee)$. These two points are the singular points of the cubic hypersurface Y.

Stabilizer stratification. We record the stabilizer stratification for the G-action on the singular cubic threefold Y:

	Stabilizer	Residue	dim	deg	Character
1	$\mathfrak{D}_{2n}$	triv	1	3	N/A
2	C_n	triv	0	1	Singular point
3	C_2^2	triv	0	1	$((0,1),(1,0),(0,1))$
4	C_2^2	triv	0	1	$((0,1),(1,0),(0,1))$
5	C_2	C_2	2	3	(1)
6	C_2	C_2	2	3	(1)
7	C_2	C_n	1	1	$(1,1)$

As in Section 5, the strata 5 and 6 are cubic surfaces, with residual action fixing the same smooth cubic curve E_Y in the stratum 1.

The stratification on the degree 14 Fano threefold X is:

	Stabilizer	Residue	dim	deg	Character
1	C_n	triv	1	1	Singular line
2	C_n	C_2	1	6	$(2r,-2r)$
3–8	C_2	triv	0	1	$(1,1,1)$
9	C_2	triv	1	1	$(1,1)$
10	C_2	triv	1	1	$(1,1)$

Burnside invariants. The first stratum on Y has generic stabilizer $\mathfrak{D}_{2n}$, it is a smooth cubic curve

$$E_Y := \{y_1 = y_2 = 0\} \cap Y.$$

We are in the situation of Example 2.2: the model Y is not in standard form; after blowing up E_Y, we obtain the incompressible symbol

$$(C_2, \mathfrak{D}_n \circlearrowright k(E_Y)(t), (1)). \tag{6.4}$$

Note that the action on the generic fiber of the projectivization of the normal bundle to E_Y is nonabelian.

On the other side, the second stratum in the stabilizer stratification of the degree 14 Fano threefold X is a degree 6, smooth genus 1 curve

$$E_X := \{x_3 = x_7 = x_8 = x_{10} = 0\} \cap X,$$

with stabilizer $\langle s\rangle \simeq C_n$. We checked, via `Magma`, that E_Y and E_X have the same j-invariant and thus are isomorphic.

We know that the G-action on X is not in standard form. However, even after blowups, the resulting class in $\mathrm{Burn}_3(G)$ differs from the symbol (6.4) – we cannot get a nonabelian action on the fibers of a $\mathbb{P}^1$-bundle over E_X, from an abelian stabilizer.

Therefore, the $\mathfrak{D}_{2n}$-actions on X and Y are not equivariantly birational.

7. $\mathfrak{S}_5$-actions

Here we investigate the Pfaffian construction for $G = \mathfrak{S}_5$. Consider the central extension

$$\tilde{G} = C_2 \rtimes \mathsf{SL}_2(\mathbb{F}_5)$$

with GAP ID (240,90). Note that $\tilde{G}$ is one of the Schur covers of $\mathfrak{S}_5$ (the other central extensions yield a picture similar to the one described below).

The group $\tilde{G}$ has two faithful irreducible 6-dimensional linear representations, differing by the sign character. Let V be the one with character

$$(6, -6, 0, 0, 0, 1, 0, 0, 0, -\zeta^3 - \zeta, \zeta^3 + \zeta, -1), \quad \zeta = \zeta_8.$$

Then $\wedge^2(V)$ is a faithful G-representation decomposing as

$$\wedge^2(V^\vee) = V_1 \oplus V_4 \oplus V_5 \oplus V_5',$$

where the character of each summand is given by

$$\begin{aligned} V_1 &: (1, -1, 1, 1, -1, 1, -1), \\ V_4 &: (4, -2, 0, 1, 0, -1, 1), \\ V_5 &: (5, -1, 1, -1, 1, 0, -1), \\ V_5' &: (5, 1, 1, -1, -1, 0, 1). \end{aligned}$$

There are three distinct 5-dimensional subspaces $A \subset \wedge^2(V^\vee)$, namely V_5, V_5' and $V_1 \oplus V_4$. The resulting Pfaffian cubic threefolds

$$Y = \mathbb{P}(A) \ \cap \ \mathrm{Gr}(2, V)^\vee$$

have different singularity types.

$A = V_5$. The cubic threefold Y has 6 A_1-singularities. The $\mathfrak{S}_5$-action on Y is unique, see [CTZ24, Proposition 7.3].

$A = V_5'$. The cubic Y is the Segre cubic threefold, the unique cubic with 10 A_1-singularities. The $\mathfrak{S}_5$-action with the prescribed character is unique. It is the *nonstandard* $\mathfrak{S}_5$ in $\mathrm{Aut}(Y) = \mathfrak{S}_6$, and is linearizable, see [CTZ24, Section 6].

$A = V_1 \oplus V_4$. The cubic threefold Y has 5 A_1-singularities, with a transitive action of $\mathfrak{S}_5$. Such an $\mathfrak{S}_5$-action is unique [CTZ24, Section 6]. By [CSZ23, Theorem 3.1], the only G-Mori fiber spaces which are G-birational to Y are a smooth quadric threefold and Y itself; under the standard Cremona involution, Y is $\mathfrak{S}_5$-equivariantly birationally transformed to the smooth quadric threefold given by

$$y_1y_2+y_1y_3+y_2y_3+y_1y_4+y_2y_4+y_3y_4+y_1y_5+y_2y_5+y_3y_5+y_4y_5=0,$$

with the same $\mathfrak{S}_5$ permutation action on the coordinates. The singular locus of the dual Fano threefold X consists of 10 points, i.e., for each singularity on Y there *two* singular points on X – the corresponding A-net is not regular.

References

[AR96] A. Adler and S. Ramanan. *Moduli of abelian varieties*, volume 1644 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996.

[BCDP23] J. Blanc, I. Cheltsov, A. Duncan, and Yu. Prokhorov. Finite quasisimple groups acting on rationally connected threefolds. *Math. Proc. Cambridge Philos. Soc.*, 174(3):531–568, 2023.

[Bea00] A. Beauville. Determinantal hypersurfaces. *Michigan Math. J.*, 48:39–64, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[BGvBT23a] Chr. Böhning, H.-Chr. Graf von Bothmer, and Yu. Tschinkel. Equivariant birational geometry of cubic fourfolds and derived categories, 2023. `arXiv:2303.17678`.

[BGvBT23b] Chr. Böhning, H.-Chr. Graf von Bothmer, and Yu. Tschinkel. Equivariant birational types and derived categories, 2023. `arXiv:2311.15881`.

[CGR06] V. Chernousov, P. Gille, and Z. Reichstein. Resolving G-torsors by abelian base extensions. *J. Algebra*, 296(2):561–581, 2006.

[CMTZ24] I. Cheltsov, L. Marquand, Yu. Tschinkel, and Zh. Zhang. Equivariant geometry of singular cubic threefolds, II, 2024. `arXiv:2405.02744`.

[Com20] G. Comaschi. Pfaffian representations of cubic threefolds, 2020. `arXiv:2005.06593`.

[CSZ23] I. Cheltsov, A. Sarikyan, and Z. Zhuang. Birational rigidity and alpha invariants of Fano varieties, 2023. `arXiv:2304.11333`.

[CTZ23] I. Cheltsov, Yu. Tschinkel, and Zh. Zhang. Equivariant geometry of the Segre cubic and the Burkhardt quartic, 2023. `arXiv:2308.15271`.

[CTZ24] I. Cheltsov, Yu. Tschinkel, and Zh. Zhang. Equivariant geometry of singular cubic threefolds, 2024. `arXiv:2401.10974`.

[HKT21] B. Hassett, A. Kresch, and Yu. Tschinkel. Symbols and equivariant birational geometry in small dimensions. In *Rationality of varieties*, volume 342 of *Progr. Math.*, pages 201–236. Birkhäuser/Springer, Cham, [2021] ©2021.

[HT22] B. Hassett and Yu. Tschinkel. Equivariant geometry of odd-dimensional complete intersections of two quadrics. *Pure Appl. Math. Q.*, 18(4):1555–1597, 2022.

[HT23] B. Hassett and Yu. Tschinkel. Torsors and stable equivariant birational geometry. *Nagoya Math. J.*, 250:275–297, 2023.

[IM00] A. Iliev and D. Markushevich. The Abel-Jacobi map for a cubic threefold and periods of Fano threefolds of degree 14. *Doc. Math.*, 5:23–47, 2000.

[Kol22] J. Kollár. Automorphisms and twisted forms of rings of invariants, 2022. `arXiv:2212.03772`.

[KT22] A. Kresch and Yu. Tschinkel. Equivariant birational types and Burnside volume. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 23(2):1013–1052, 2022.

[Kuz04] A. G. Kuznetsov. Derived category of a cubic threefold and the variety V_{14}. *Tr. Mat. Inst. Steklova*, 246(Algebr. Geom. Metody, Svyazi i Prilozh.):183–207, 2004. Translation in *Proc. Steklov Inst. Math.* 2004, no. 3(246), 171–194.

[MT01] D. Markushevich and A. S. Tikhomirov. The Abel-Jacobi map of a moduli component of vector bundles on the cubic threefold. *J. Algebraic Geom.*, 10(1):37–62, 2001.

[Put82] P. J. Puts. On some Fano-threefolds that are sections of Grassmannians. *Nederl. Akad. Wetensch. Indag. Math.*, 44(1):77–90, 1982.

[TYZ23] Yu. Tschinkel, K. Yang, and Zh. Zhang. Equivariant birational geometry of linear actions, 2023. `arXiv:2302.02296`.

[TZ24] Yu. Tschinkel and Zh. Zhang. Supporting materials, 2024. `https://zhijiazhangz.github.io/pfaffian` .

[WY20] L. Wei and X. Yu. Automorphism groups of smooth cubic threefolds. *J. Math. Soc. Japan*, 72(4):1327–1343, 2020.

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