

RATIONALITY OF SINGULAR CUBIC THREEFOLDS OVER \mathbb{R}

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ABSTRACT. We study rationality properties of real singular cubic threefolds.

1. INTRODUCTION

In this note we focus on rationality properties of singular cubic threefolds over the real numbers. We are guided by results in equivariant birational geometry in [CTZ24], [CMTZ24], which elucidated birationality constructions as well as obstructions to birationality, in the equivariant context.

One of the main problems in birational geometry is the (stable) rationality problem; see, e.g., [MT86], [CT95], [Che05], [Pro18], [Pir18], [Voi19], [Sch21] for reports on milestones in this area. Classically, one was interested in algebraically closed ground fields, of characteristic zero, mostly the complex numbers \mathbb{C} . However, many (stable) rationality constructions rely crucially on results over nonclosed fields, e.g., function fields of smaller-dimensional varieties: in [BCTSSD85] this is used to produce nonrational but stably rational threefolds over \mathbb{C} , via a degree 4 Del Pezzo fibration over \mathbb{P}^1 , interpreted as a degree 4 Del Pezzo surface over the nonclosed field $k = \mathbb{C}(\mathbb{P}^1)$. New examples of rational cubic fourfolds, via a degree 6 Del Pezzo fibration over \mathbb{P}^2 , can be found in [AHTVA19]. The survey [CT19] contains many results concerning rationality over arbitrary fields.

Of particular interest are varieties with simple presentations, such as hypersurfaces in (weighted) projective spaces, or linear sections of Grassmannians. Among the simplest such varieties are quadric and cubic hypersurfaces. Rationality of quadrics is completely settled: a necessary and sufficient condition is the existence of a rational point. For cubics, we do not currently have such criteria. Smooth complex

cubic threefolds $X_{\mathbb{C}} \subset \mathbb{P}^4$ are irrational [CG72], while the singular ones are rational unless they are cones over smooth cubic curves.

In recent years, there have been several results concerning rationality of geometrically rational varieties over \mathbb{R} . For example, it is now known that rationality of smooth intersections of two quadrics $X_{2,2} \subset \mathbb{P}^n$, for $n = 5$, is governed by the existence of rational points and lines [HT21]. For $n = 6$, a necessary and sufficient condition is that $X(\mathbb{R})$ is connected [HKT22]. The papers [BW20], [JJ23], [CTP24] explore rationality of conic and quadric bundles over \mathbb{R} .

Here, we focus on rationality of singular cubic threefolds

$$X = X_{\mathbb{R}} \subset \mathbb{P}^4$$

over \mathbb{R} . Every such X has real points. We recall standard rationality constructions, in presence of distinguished loci, over \mathbb{R} :

- Projection from a singular point gives a birational map to \mathbb{P}^3 ,
- Projection from a line and a disjoint plane gives a birational map $X \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^1$.

In particular, if the cubic is not a cone and there exists a real singular point, then X is rational over \mathbb{R} . Hence, throughout the paper, we assume that X is not a cone and that

$$X^{\text{sing}}(\mathbb{R}) = \emptyset.$$

This implies that either the singularities of $X_{\mathbb{C}}$ are isolated and their number $s = s(X_{\mathbb{C}})$ is even, i.e., $s = 2, 4, 6, 8, 10$, or that the singular locus has positive dimension. By [Avi20], a real form of a cubic with 10 isolated singularities is rational over \mathbb{R} – geometrically, there is only one such cubic, the *Segre* cubic.

Let $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2$ be the group generated by complex conjugation. In addition to an action of Γ on special loci of X , such as the singular locus X^{sing} , we have an action on the geometric Picard group $\text{Pic}(\tilde{X}_{\mathbb{C}})$ of the minimal resolution of singularities \tilde{X} of X . We investigate (stable) rationality over \mathbb{R} , using the following tools:

- *classification*: we compute all possibilities for the action of Γ , configurations of singularities, and normal forms; we freely use the terminology and techniques of [CTZ24] and [CMTZ24];
- *topology*: if $X(\mathbb{R})$ is disconnected then X is not stably rational;
- *cohomology*: if

$$(1.1) \quad H^1(\Gamma, \text{Pic}(\tilde{X}_{\mathbb{C}})) \neq 0,$$

then X is not stably rational; we refer to this as the **(H1)**-obstruction.

We summarize the results:

- $s = 2$:
 - $2A_1, 2A_2$: not rational over \mathbb{R} .
 - $2A_3$ with no plane: criteria for disconnectedness of $X(\mathbb{R})$, and examples with $X(\mathbb{R})$ connected or disconnected.
 - $2A_3$ with a plane: rational if and only if there is a real line disjoint from the plane.
 - $2A_4$: criteria for disconnectedness, and examples with $X(\mathbb{R})$ connected or disconnected.
 - $2A_5$: all not stably rational.
 - $2D_4$: criteria for disconnectedness, examples with rational X , and examples with $X(\mathbb{R})$ disconnected.
- $s = 4$:
 - $4A_1$ with no plane, $4A_2, 2A_2 + 2A_1$ are birational over \mathbb{R} ; criteria for disconnectedness, and examples of connected and disconnected $X(\mathbb{R})$.
 - $4A_1$ with a plane: rational if and only if there is a real line disjoint from the plane.
 - $2A_3 + 2A_1$ with three planes: criteria for disconnectedness, examples with rational X , and with $X(\mathbb{R})$ disconnected.
 - $2A_3 + 2A_1$ with one plane: always rational.
 - $2D_4 + 2A_1$: rational if and only if it contains three planes.
- $s = 6$:
 - $6A_1$ with no plane: rational if and only if it contains a real cubic scroll.
 - $6A_1$ with one plane: always rational.
 - $6A_1$ with three planes: criteria for disconnectedness, examples with rational X , and with $X(\mathbb{R})$ disconnected.
 - $2A_2 + 4A_1, 2A_3 + 4A_1$: always rational.
- $s = 8$: rational if and only if it contains three planes.

Geometrically, cubic threefolds with nonisolated singularities are of four types [Yok02], [All03]: with X^{sing} consisting of a plane, a line, a conic, or a twisted quartic. We show that all real forms of such cubics are rational, unless X^{sing} is a conic. In this last case, we have examples with connected or disconnected $X(\mathbb{R})$. The rationality of such cubics is open, see, e.g., [CTP24].

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2. COHOMOLOGY

Here, we investigate the **(H1)**-obstruction for cubics with isolated singularities.

Proposition 2.1. *Let X be a real cubic threefold with isolated singularities over \mathbb{C} . We have*

$$H^1(\Gamma, \text{Pic}(\tilde{X}_{\mathbb{C}})) \neq 0$$

if and only if one of the following holds:

- (1) $X_{\mathbb{C}}$ has $2A_5$ -singularities,
- (2) $X_{\mathbb{C}}$ has $2D_4 + 2A_1$ -singularities, and only one real plane,
- (3) $X_{\mathbb{C}}$ has $6A_1$ -singularities in linearly general position, and contains no normal cubic scrolls over \mathbb{R} ,
- (4) $X_{\mathbb{C}}$ has $8A_1$ -singularities, and only one real plane.

Proof. When $X_{\mathbb{C}}$ has the indicated configurations of singularities, the proof follows from Propositions 3.5, 4.5, 5.3, and 6.1. In all other cases, the **(H1)**-obstruction is trivial, as shown in [CTZ24], [CMTZ24]. \square

We now list representative examples with nontrivial cohomology for each of the cases in Proposition 2.1.

Example 2.2. $2A_5$: Over \mathbb{C} , every such cubic is given by

$$x_1x_2x_3 + x_1x_4^2 + x_2x_5^2 + x_3^3 + bx_3x_4x_5 = 0, \quad b^2 \neq -4.$$

Over \mathbb{R} , every such cubic is given by

$$(2.1) \quad X = \{(x_1^2 + x_2^2)x_3 + x_3^3 - 2x_1(x_4^2 - x_5^2) - 4x_2x_4x_5 + bx_3(x_4^2 + x_5^2) = 0\},$$

for some $b \in \mathbb{R}$, with singular points

$$[1 : i : 0 : 0 : 0], \quad [1 : -i : 0 : 0 : 0].$$

Consider the projection from the real line through the singularities

$$\pi : X \dashrightarrow \mathbb{P}^2, \quad (x_1, x_2, x_3, x_4, x_5) \mapsto (x_3, x_4, x_5).$$

The image $\pi(X(\mathbb{R}))$, given by

$$\left\{ x_3^2 + \frac{b - \sqrt{b^2 + 4}}{2}x_4^2 + \frac{b - \sqrt{b^2 + 4}}{2}x_5^2 \leq 0 \right\} \subset \mathbb{P}^2,$$

is connected; it is easy to see that $X(\mathbb{R})$ is also connected.

The Galois action on $\text{Pic}(\tilde{X}_{\mathbb{C}})$ is isomorphic to the C_2 -action considered in [CMTZ24, Proposition 5.12], which implies that

$$(2.2) \quad H^1(\Gamma, \text{Pic}(\tilde{X}_{\mathbb{C}})) = \mathbb{Z}/2\mathbb{Z},$$

hence X is not stably rational over \mathbb{R} . See Proposition 3.5 for details.

Example 2.3. 6A₁: In the equivariant context, assuming that the singularities are in linearly general positions, we obtain nontrivial cohomology from any $C_2 \subseteq \text{Aut}(X)$, which does not fix a node [CTZ24, Proposition 7.5]. Over \mathbb{R} , we realize this by

$$(x_1^2 + x_2^2)(x_3 + 2x_4 + 39x_5) + 2x_1(x_3^2 + x_4^2 + 4x_5^2 + x_3x_5) + 2x_2(x_3x_4 - x_3^2 + x_4^2 + 4x_5^2) + x_4^2x_3 + 4x_5^2x_3 - 2x_3^2x_4 + 39x_3^2x_5 = 0,$$

with singular points

$$\begin{aligned} [0 : 0 : 0 : -2i : 1], \quad [0 : 0 : 0 : 2i : 1], \quad [0 : -1 : -i : 1 : 0], \\ [0 : -1 : i : 1 : 0], \quad [-i : 1 : 0 : 0 : 0], \quad [i : 1 : 0 : 0 : 0]. \end{aligned}$$

The equation

$$x_3^2 + x_4^2 + 20x_3x_5 + 4x_5^2 = 0$$

cuts out two complex conjugate irreducible divisors on $X_{\mathbb{C}}$, so that complex conjugation acts nontrivially on the class group $\text{Cl}(X_{\mathbb{C}})$. Computation in [CTZ24, Proposition 7.5] implies (2.2); it follows that X is not stably rational over \mathbb{R} .

The image in $\mathbb{P}_{x_3, x_4, x_5}^2$ of the projection from the line passing through the last two singular points is connected, and given by

$$\{(x_3^2 + x_4^2 + 20x_3x_5 + 4x_5^2)(x_3^2 + x_4^2 - \frac{77}{2}x_3x_5 + 4x_5^2) \geq 0\} \subset \mathbb{P}_{x_3, x_4, x_5}^2,$$

which shows that $X(\mathbb{R})$ is also connected.

Example 2.4. 8A₁: In the equivariant context, $X_{\mathbb{C}}$ is given by

$$(ax_1 + x_2 + bx_3)x_4x_5 + x_4(x_3^2 - x_1^2) + x_5(x_3^2 - x_2^2) = 0, \quad a, b \in \mathbb{C},$$

and cohomology from C_2 generated by

$$(\mathbf{x}) \mapsto (ax_5 - x_1, x_4 - x_2, x_3, x_4, x_5).$$

Over \mathbb{R} , we can realize the corresponding Γ -action on cohomology on X given by

$$a_1(x_1^2 + x_3^2)x_4 + x_4x_5^2 + a_2(x_2^2x_5 + x_3^2x_5 + x_4^2x_5) + a_3x_3x_4x_5 = 0,$$

with $a_1, a_2, a_3 \in \mathbb{R}$ and singularities at

$$[0 : -i : 0 : 1 : 0], [0 : i : 0 : 1 : 0], [-i : 0 : 0 : 0 : 1], [i : 0 : 0 : 0 : 1],$$

$$[-i : -i : 1 : 0 : 0], [i : -i : 1 : 0 : 0], [-i : i : 1 : 0 : 0], [i : i : 1 : 0 : 0].$$

A direct computation of $\text{Cl}(X_{\mathbb{C}})$ as in Proposition 6.1 shows that the Γ -action gives nontrivial cohomology (2.2).

Example 2.5. $2D_4 + 2A_1$: In the equivariant context, $X_{\mathbb{C}}$ is given by

$$x_1x_2x_3 + x_1x_2x_4 + x_5^2(a_1x_3 + a_2x_4) + a_3x_3x_4x_5 = 0, \quad a_1, a_2, a_3 \in \mathbb{C}^{\times},$$

with nontrivial cohomology from the involution

$$(\mathbf{x}) \mapsto (x_2, x_1, x_4 + \frac{a_1 - a_2}{a_3}x_5, x_3 - \frac{a_1 - a_2}{a_3}x_5, x_5).$$

Over \mathbb{R} , we can realize this by

$$(x_1^2 + x_2^2)x_3 + x_5(x_3^2 + x_4^2) + (b_1x_3 + b_2x_4)x_5^2 + ax_5^3 = 0$$

for $a, b_1, b_2 \in \mathbb{R}$ with $b_2^2 < 4a$. The D_4 -singularities are

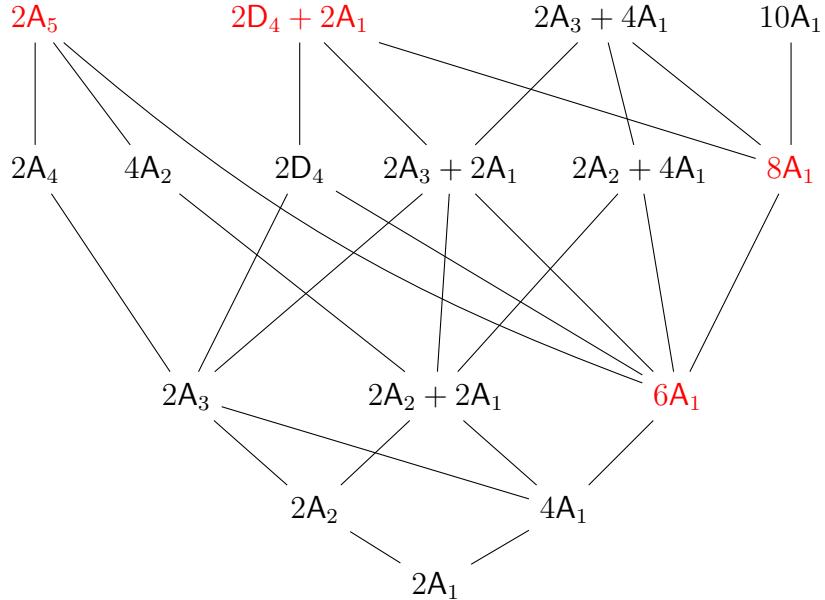
$$[1 : i : 0 : 0 : 0], \quad [1 : -i : 0 : 0 : 0],$$

and the A_1 -singularities are

$$[0 : 0 : 1 : i : 0], \quad [0 : 0 : 1 : -i : 0].$$

In these cases, the Γ -action gives nontrivial cohomology (1.1), see Proposition 4.5 for more details.

The following diagram shows specialization patterns, obtained in investigations of equivariant birationalities in [CTZ24], [CMTZ24]. The red-labeled cases (i.e., cubic threefolds with indicated singularity types) admit forms over \mathbb{R} with nontrivial cohomology $H^1(\Gamma, \text{Pic}(\tilde{X}_{\mathbb{C}}))$ from a specific action of Γ ; and general cubic threefolds, indicated in black and specializing to these, have trivial cohomology.



Remark 2.6. In contrast to the equivariant case treated in [CTZ24], specialization arguments do not work over \mathbb{R} , and we do not know whether or not a very general cubic threefold with trivial cohomology specializing to one with nontrivial cohomology is rational over \mathbb{R} . They can be applied to show the existence of cubics failing stable rationality over fields such as $\mathbb{R}(t)$.

3. TWO SINGULAR POINTS

Assume that $X_{\mathbb{C}}$ has two complex-conjugated singular points, e.g.,

$$p_1 = [1 : i : 0 : 0 : 0], \quad \text{and} \quad p_2 = [1 : -i : 0 : 0 : 0],$$

so that $X \subset \mathbb{P}_{x_1, \dots, x_5}^4$ is given by

$$(3.1) \quad (x_1^2 + x_2^2)x_3 + x_1q_1 + x_2q_2 + f_3 = 0,$$

where $q_1, q_2 \in \mathbb{R}[x_4, x_5]$ are quadratic forms and $f_3 \in \mathbb{R}[x_3, x_4, x_5]$ is a cubic form. One checks that q_1 and q_2 cannot both vanish if $X_{\mathbb{C}}$ has A_n -singularities. If $q_1 \not\equiv 0$ and $q_2 \equiv 0$, then up to a change of variables, we have $q_1 = q_2$. Thus, we may assume that none of q_1 or q_2 is 0.

Each of q_1 and q_2 defines two points in $\mathbb{P}_{x_4, x_5}^1(\mathbb{C})$, with one of the following possibilities:

- one real point with multiplicity 2, or
- two real points, or

- two conjugate points.

Combining the configuration patterns of four points defined by q_1 and q_2 , we find all possibilities of the pair (q_1, q_2) , up to isomorphism:

q_1	q_2				
x_4^2	λx_4^2	x_5^2	$x_4 x_5$	$\lambda(x_4^2 - x_5^2)$	$\lambda(x_4^2 + x_5^2)$
$x_4 x_5$	$\lambda x_4 x_5$	$\lambda x_4(x_4 - x_5)$	$\lambda(x_4^2 - x_5^2)$	$\lambda(x_4^2 + x_5^2)$	
$x_4^2 + x_5^2$	$x_4^2 + \lambda x_5^2$			$\lambda > 0$ everywhere	

Introducing new coordinates

$$y_1 = x_1 x_3 + \frac{q_1}{2}, \quad \text{and} \quad y_2 = x_2 x_3 + \frac{q_2}{2},$$

and multiplying (3.1) by x_3 , we birationally transform X over \mathbb{R} to

$$(3.2) \quad Y = \{y_1^2 + y_2^2 + f_3 x_3 - \frac{q_1^2 + q_2^2}{4} = 0\} \subset \mathbb{P}_{y_1, y_2, x_1, x_2, x_3}(2, 2, 1, 1, 1).$$

Note that Y is a conic bundle over \mathbb{P}^2 , with

$$\pi : Y \rightarrow \mathbb{P}_{x_1, x_2, x_3}^2, \quad (y_1, y_2, x_1, x_2, x_3) \mapsto (x_1, x_2, x_3).$$

The image $\pi(Y(\mathbb{R}))$ is given by

$$(3.3) \quad \frac{q_1^2 + q_2^2}{4} - f_3 x_3 \geq 0.$$

In particular, we write

$$(3.4) \quad f_3 := t_1 x_3^3 + x_3^2(t_2 x_4 + t_3 x_5) + x_3(t_4 x_4^2 + t_5 x_5^2 + t_6 x_4 x_5) + t_7 x_4^2 x_5 + t_8 x_5^2 x_4 + t_9 x_4^3 + t_{10} x_5^3,$$

for $t_1, \dots, t_{10} \in \mathbb{R}$.

2A₁ and 2A₂-singularities.

Proposition 3.1. *Assume that X has no real singular points and that $X_{\mathbb{C}}$ has 2A₁ or 2A₂-singularities. Then X is not rational over \mathbb{R} .*

Proof. Recall that X is birational to the conic bundle (3.2). When $X_{\mathbb{C}}$ has 2A₁ or 2A₂-singularities, the conic bundle has a smooth quartic discriminant, with trivial double cover. By [BW20] or [KP24, Theorem 6.10], X is not rational over \mathbb{R} . \square

2A₃-singularities with no plane. When p_1 and p_2 are A₃-points and $X_{\mathbb{C}}$ contains no plane, one can check (cf. [CMTZ24, Section 5]) that the only possibility for q_1 and q_2 from the table above is

$$q_1 = x_4 x_5 \quad \text{and} \quad q_2 = \frac{1}{2}(x_4^2 - x_5^2).$$

The A₃-singularities impose the following conditions on (3.4)

$$t_7 = t_{10}, \quad t_8 = t_9.$$

Remark 3.2. Consider the conic bundle $\pi : Y \rightarrow \mathbb{P}^2$ birational to X given by (3.2). Note that the real locus of the discriminant curve in \mathbb{P}^2 is smooth. It follows that $Y(\mathbb{R})$ (and thus $X(\mathbb{R})$) is connected if and only if the image $\pi(Y(\mathbb{R}))$ in $\mathbb{P}^2(\mathbb{R})$ given by (3.2) is connected. Using this, we present examples where $X(\mathbb{R})$ is disconnected or connected:

- Disconnected: $t_1 = -1, t_2 = -10, t_3 = -2, t_4 = t_5 = t_6 = t_7 = 1, t_8 = 5,$
- Connected: $t_1 = t_2 = t_3 = t_4 = t_5 = t_6 = t_7 = t_8 = 1.$

2A₃-singularities with one plane. Here, $X_{\mathbb{C}}$ contains a unique plane Π , which is thus defined over \mathbb{R} . This implies that X is birational to a smooth intersection of two quadrics $X_{2,2} \subset \mathbb{P}^5$, via unprojection from the plane, see [CTZ24, Proposition 5]. Rationality of $X_{2,2}$ over \mathbb{R} is determined by the existence of a real line, see [HT21]; on X this translates into the existence of a real line disjoint from Π .

Example 3.3. The cubic X given by

$$(x_1^2 + x_2^2)x_3 + (x_1 + x_2)x_4^2 + (x_3 + x_4 + x_5)(x_3^2 + x_4x_5) - 2x_3x_5^2 = 0$$

is rational. Indeed, the line

$$\{x_1 + x_2 = x_1 - x_5 = x_3 + x_4 + x_5 = 0\}$$

in X is disjoint from the unique plane $\{x_3 = x_4 = 0\} \subset X$.

Topological types of $X_{2,2}$ with real points but without real lines are listed in [HT21, Section 11.4].

2A₄-singularities. Cubic threefolds $X_{\mathbb{C}}$ with 2A₄-singularities do not contain a plane. As in the case of 2A₃-singularities with no plane, we find that

$$q_1 = x_4 x_5, \quad q_2 = \frac{1}{2}(x_4^2 - x_5^2), \quad t_7 = t_{10}, \quad t_8 = t_9,$$

together with the following conditions ensuring 2A₄-singularities

$$t_6 = -8t_7 t_8, \quad t_4 = t_5 + 4t_7^2 - 4t_8^2.$$

Example 3.4. Similarly as before, $X(\mathbb{R})$ can be connected or disconnected depending on the parameters. We give examples:

- Disconnected: $t_1 = 2, t_2 = -10, t_3 = t_5 = t_6 = t_7 = 0, t_4 = -16, t_8 = -2$.
- Connected: $t_1 = t_2 = t_3 = t_4 = t_5 = t_7 = t_8 = 1, t_6 = -8$.

Again, this can be seen via checking the connectedness of (3.3).

$2A_5$ -singularities. Such cubics X are given by

$$(3.5) \quad (x_1^2 + x_2^2)x_3 + x_3^3 + \frac{1}{2}x_2(x_4^2 - x_5^2) + x_1x_4x_5 + bx_3(x_4^2 + x_5^2) = 0, \quad b \in \mathbb{R}.$$

Note that $X(\mathbb{R})$ is connected, see Example 2.2. However X is not stably rational:

Proposition 3.5. *Assume that X has no real singular points and that $X_{\mathbb{C}}$ has $2A_5$ -singularities. Then X is not stably rational.*

Proof. We may assume that X is given by (3.5). The equation

$$q(x_1, \dots, x_5) = x_3^2 + \frac{2b - \sqrt{4b^2 + 2}}{4}x_4^2 + \frac{2b - \sqrt{4b^2 + 2}}{4}x_5^2$$

cuts out complex conjugated divisors on $X_{\mathbb{C}}$, we have

$$\{q = 0\} \cap X = S \cup \bar{S},$$

where

$$\begin{aligned} S = \{q = x_3(x_1 + ix_2) + \frac{i}{4}(x_4 - ix_5)^2 = x_1x_4 + ix_2x_4 - x_2x_5 + \\ + ix_1x_5 + (\sqrt{4b^2 + 2} + 2b)ix_3x_4 + (\sqrt{4b^2 + 2} + 2b)x_3x_5 = 0\}. \end{aligned}$$

By [CMTZ24, Section 5], the class group $\text{Cl}(X_{\mathbb{C}})$ is generated by the classes of S, \bar{S} and the class of a hyperplane section F subject to the relation $S + \bar{S} = 2F$. As in [CMTZ24, Proposition 5.12],

$$H^1(\Gamma, \text{Pic}(\tilde{X}_{\mathbb{C}})) = \mathbb{Z}/2,$$

hence X is not stably rational over \mathbb{R} . \square

Here we also record an explicit specialization pattern. See Remark 2.6 for a discussion of specializations over \mathbb{R} .

Example 3.6. Consider the family of cubics $\mathcal{X} \rightarrow \mathbb{A}_{t_1, \dots, t_8}^8$ given by

$$\begin{aligned} (x_1^2 + x_2^2)x_3 + x_1x_4x_5 + \frac{1}{2}x_2(x_4^2 - x_5^2) + t_1x_3^3 + x_3^2(t_2x_4 + t_3x_5) + \\ + x_3(t_4x_4^2 + t_5x_5^2 + t_6x_4x_5) + (x_4^2 + x_5^2)(t_7x_5 + t_8x_4) = 0. \end{aligned}$$

Let \mathcal{X}_1 be the subfamily of \mathcal{X} consisting of fibers above the locus

$$\{t_6 + 8t_7t_8 = t_5 - t_4 + 4t_7^2 - 4t_8^2 = 0\} \subset \mathbb{A}^8,$$

and \mathcal{X}_2 be the subfamily of \mathcal{X} consisting of fibers above the line

$$\{t_1 - 1 = t_2 = t_3 = t_5 - t_4 = t_6 = t_7 = t_8 = 0\} \subset \mathbb{A}^8.$$

We have a natural inclusion $\mathcal{X} \supset \mathcal{X}_1 \supset \mathcal{X}_2$. Then a very general fiber of \mathcal{X} is a real cubic with $2\mathbb{A}_3$ -singularities, that of \mathcal{X}_1 has $2\mathbb{A}_4$ -singularities, and that of \mathcal{X}_2 has $2\mathbb{A}_5$ -singularities, over \mathbb{C} .

2D₄-singularities. As in the proof of [CMTZ24, Proposition 5.15], we find that such X are given by

$$(3.6) \quad (x_1^2 + x_2^2)x_3 + t_1x_3^3 + x_3^2(t_2x_4 + t_3x_5) + x_3(t_4x_4^2 + t_5x_5^2 + t_6x_4x_5) + x_4q(x_4, x_5) = 0,$$

for general parameters $t_1, \dots, t_6 \in \mathbb{R}$, and

$$q(x_4, x_5) = x_4^2 - x_5^2 \quad \text{or} \quad x_4^2 + x_5^2.$$

In the first case, X contains three real planes

$$\Pi_1 = \{x_3 = x_4 = 0\}, \Pi_2 = \{x_3 = x_4 + x_5 = 0\}, \Pi_3 = \{x_3 = x_4 - x_5 = 0\}.$$

In the second case, X contains only one real plane Π_1 . In both cases, projection from Π_1 induces a quadric surface bundle

$$\pi : X \dashrightarrow \mathbb{P}_{x_3, x_4}^1.$$

Note that the discriminant curve of the conic bundle structure (3.2) in this case has real singularities. It is therefore more appropriate to use the quadric surface bundle structure to study the topology of $X(\mathbb{R})$. We have

$$\# \text{ components of } X(\mathbb{R}) = \# \text{ components of } \pi(X(\mathbb{R})).$$

First, we consider the case $q = x_4^2 - x_5^2$. There are three possibilities:

- (1) X contains a line disjoint from a plane, then X is rational,
- (2) $X(\mathbb{R})$ has two components, and X is not stably rational,
- (3) $X(\mathbb{R})$ is connected but X does not contain any line disjoint from one of the planes Π_1, Π_2, Π_3 .

In the third case, we do not know whether or not X is rational. Each possibility is realized, e.g., by

- (1) $(x_1^2 + x_2^2)x_3 - x_3^3 + x_4(x_4^2 - x_5^2)$,
- (2) $(x_1^2 + x_2^2)x_3 + 300x_3^3 + x_3^2(35x_4 + 24\sqrt{21}x_5) + 10x_3x_5^2 + x_4(x_4^2 - x_5^2)$,
- (3) $(x_1^2 + x_2^2)x_3 + x_3^3 + x_4(x_4^2 - x_5^2)$.

Let us explain when these possibilities occur, using π . Alternatively, we can use projection from Π_2 or Π_3 , but this is essentially the same, up to a change of coordinates. Put

$$\begin{aligned}\Delta(x_3, x_4) := & -x_4^4 + (t_5 - t_4)x_3x_4^3 + (t_4t_5 - t_2 - \frac{1}{4}t_6^2)x_3^2x_4^2 + \\ & + (t_2t_5 - t_1 - \frac{1}{2}t_3t_6)x_3^3x_4 + (t_1t_5 - \frac{1}{4}t_3^2)x_3^4.\end{aligned}$$

Then $\Delta(x_3, x_4) \cdot x_3^2$ is the discriminant of π . Set $D_1(x) = \Delta(1, x)$. Note that $D_1(x)$ can only have simple real roots since X has only three planes and no real singular points. Every line in X intersects exactly one plane among Π_1, Π_2, Π_3 , since $\Pi_1 + \Pi_2 + \Pi_3$ is cut out on X by the hyperplane $x_3 = 0$.

Lemma 3.7. *The cubic X contains a line disjoint from Π_2 and Π_3 if and only if $t_3 = -t_5t_6$ and $t_1 + t_2t_5 + t_4t_5^2 + t_5^3 < 0$, or D_1 has real roots and t_5 is smaller than its largest root.*

Proof. Observe that X contains a line disjoint from Π_2 and Π_3 if and only if this line is contained in a fiber of π over $(x_3, x_4) \neq (0, 1) \in \mathbb{P}^1$. Thus, we may set $x_3 = 1$. We have to understand when the real locus of the quadric $\pi^{-1}(x_4)$ contains a real line. If $x_4 = t_5$, the real locus of $\pi^{-1}(t_5)$ is isomorphic to the quadric in $\mathbb{P}_{u_1, u_2, u_3, u_4}^3$ given by

$$u_1^2 + u_2^2 + (t_1 + t_2t_5 + t_4t_5^2 + t_5^3)u_3^2 - \frac{(t_3 + t_5t_6)^2}{4(t_1 + t_2t_5 + t_4t_5^2 + t_5^3)}u_4^2 = 0.$$

If $t_3 = -t_5t_6$ and $t_1 + t_2t_5 + t_4t_5^2 + t_5^3 < 0$, the real locus of this quadric is a cone, so the fiber $\pi^{-1}(t_5)$ contains a line that is disjoint from Π_2 and Π_3 . Similarly, if $t_3 = -t_5t_6$ and $t_1 + t_2t_5 + t_4t_5^2 + t_5^3 > 0$, the real locus of this quadric is just a point. In all other cases, the real locus is a sphere, so it does not contain lines either.

When $x_4 \neq t_5$, the fiber $\pi^{-1}(t_5)$ is isomorphic to the quadric surface

$$(3.7) \quad S = \{u_1^2 + u_2^2 + (t_5 - x_4)u_3^2 + \frac{D_1(x_4)}{t_5 - x_4}u_4^2 = 0\} \subset \mathbb{P}_{u_1, u_2, u_3, u_4}^3,$$

and we may have the following possibilities:

- $D_1(x_4) < 0$, $S(\mathbb{R})$ a sphere;
- $D_1(x_4) > 0$ and $x_4 > t_5$, $S(\mathbb{R})$ is a hyperboloid, so the fiber contains a line disjoint from Π_2 and Π_3 ;
- $D_1(x_4) > 0$ and $x_4 < t_5$, $S(\mathbb{R})$ is empty;
- $D_1(x_4) = 0$ and $x_4 < t_5$, $S(\mathbb{R})$ is a point;

- $D_1(x_4) = 0$ and $x_4 > t_5$, $S(\mathbb{R})$ is a quadric cone, so the fiber contains a line disjoint from Π_2 and Π_3 .

This implies the required assertion. \square

Similarly, we obtain:

Lemma 3.8. *The locus $X(\mathbb{R})$ has two connected components if and only if D_1 has 4 real roots and t_5 is greater than these roots.*

Now, we consider the case $q = x_4^2 + x_5^2$. In this case X contains only one real plane, X does not contain real lines disjoint this plane, and the fiber of π over $x_3 = 0$ is empty. As before, we have two possibilities:

- (1) $X(\mathbb{R})$ is disconnected, so X is not stably rational over \mathbb{R} ,
- (2) $X(\mathbb{R})$ is connected.

In the second case, we do not know whether or not X is rational. Both possibilities are realized:

- (1) $X = \{(x_1^2 + x_2^2)x_3 + x_3^3 + x_4(x_4^2 + x_5^2) = 0\}$,
- (2) $X = \{(x_1^2 + x_2^2)x_3 + x_3^3 - x_3^2x_4 + 2x_3(x_5^2 - x_4^2) + x_4(x_4^2 + x_5^2) = 0\}$.

To explain these cases in terms of the defining equation (3.6), set

$$\begin{aligned} D_2(x) := & x^4 + (t_4 + t_5)x^3 + (t_2 + t_4t_5 - \frac{1}{4}t_6^2)x^2 + \\ & + (t_1 + t_2t_5 - \frac{1}{2}t_3t_6)x + t_1t_5 - \frac{1}{4}t_3^2. \end{aligned}$$

We have:

Lemma 3.9. *The locus $X(\mathbb{R})$ is connected if and only if one of the following holds:*

- D_2 has no real root,
- D_2 has 2 real roots $\alpha_1 < \alpha_2$ and $\alpha_1 \leq -t_5$,
- D_2 has 4 real roots $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ and $\alpha_3 \leq -t_5$.

Proof. Recall that $X(\mathbb{R})$ is connected if and only if $\pi(X(\mathbb{R}))$ is. Set $x_3 = 1$. When $x_4 = -t_5$, the real locus of $\pi^{-1}(x_4)$ is isomorphic to the quadric in $\mathbb{P}_{u_1, u_2, u_3, u_4}^3$ given by

$$u_1^2 + u_2^2 + (t_1 - t_2t_5 + t_4t_5^2 - t_5^3)u_3^2 - \frac{(t_3 - t_5t_6)^2}{4(t_1 - t_2t_5 + t_4t_5^2 - t_5^3)}u_4^2 = 0.$$

One can check that the real locus of this quadric is always nonempty. When $x_4 \neq -t_5$, a general fiber of π is isomorphic to the quadric surface

in $\mathbb{P}_{u_1, u_2, u_3, u_4}^3$

$$S = \{u_1^2 + u_2^2 + (x_4 + t_5)u_3^2 + \frac{D_2(x_4)}{x_4 + t_5}u_4^2 = 0\}$$

and we may have the following possibilities:

- when $D_2(x_4) < 0$, $S(\mathbb{R})$ is a sphere,
- when $D_2(x_4) > 0$ and $x_4 < -t_5$, $S(\mathbb{R})$ is a hyperboloid,
- when $D_2(x_4) > 0$ and $x_4 > -t_5$, $S(\mathbb{R})$ is empty,
- when $D_2(x_4) = 0$ and $x_4 < -t_5$, $S(\mathbb{R})$ is a quadric cone,
- when $D_2(x_4) = 0$ and $x_4 > -t_5$, $S(\mathbb{R})$ is a point.

From this, one obtains the cases when $X(\mathbb{R})$ is connected, based on roots of D_2 , as is stated. \square

4. FOUR SINGULAR POINTS

When four singular points are not in general position, by [CTZ24], all of them are $4\mathbb{A}_1$ -singularities, contained in a plane $\Pi \subset X$.

Proposition 4.1. *Assume that X has no real singular points and that $X_{\mathbb{C}}$ has $4\mathbb{A}_1$ -singularities in a plane $\Pi \subset X$. Then X is rational if and only if X contains a real line disjoint from Π .*

Proof. Unprojecting from Π , X is birational to a smooth intersection of two quadrics $X_{2,2}$ in \mathbb{P}^5 , which is rational if and only if it contains a real line, by [HT21]. The existence of a real line in $X_{2,2}$ is equivalent to the existence of a real line in X disjoint from Π . \square

Example 4.2. The cubic X given by

$$(x_1^2 + x_2^2 + x_3^2)x_4 + x_2(x_3x_5 + x_4^2) - x_5(x_4^2 + 3x_4x_5 + x_5^2) = 0$$

is rational. Indeed, the line

$$\{x_1 - x_2 = x_1 - x_3 = x_1 - x_5 = 0\}$$

in X is disjoint from the unique plane $\{x_4 = x_5 = 0\} \subset X$.

For the rest of the section, we assume that the singular points of $X_{\mathbb{C}}$ are in general position and have coordinates

$$p_1 = [1 : i : 0 : 0 : 0], \quad p_2 = [1 : -i : 0 : 0 : 0],$$

$$p_3 = [0 : 0 : 1 : i : 0], \quad p_4 = [0 : 0 : 1 : -i : 0].$$

Then the defining equation of X is

$$(4.1) \quad (r_1x_1 + r_2x_2)(x_3^2 + x_4^2) + (r_3x_3 + r_4x_4)(x_1^2 + x_2^2) + \\ + ax_5^3 + x_5^2(b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4) + \\ + x_5(t_1x_1x_3 + t_2x_1x_4 + t_3x_2x_3 + t_4x_2x_4 + t_5(x_1^2 + x_2^2) + t_6(x_3^2 + x_4^2)),$$

with parameters $r_1, \dots, r_4, b_1, \dots, b_4, t_1, \dots, t_6, a \in \mathbb{R}$.

4A₁, 2A₂ + 2A₁, 4A₂-singularities. In these cases, at least one of the r_1 and r_2 , and one of the r_3 and r_4 is nonzero. After changing variables,

$$r_1 = r_2 = r_3 = r_4 = 1, \quad t_1 = t_4, \quad t_2 = t_3, \quad t_5 = t_6 = 0.$$

It follows that X is given by

$$(4.2) \quad (x_1 + x_2)(x_3^2 + x_4^2) + (x_3 + x_4)(x_1^2 + x_2^2) + ax_5^3 + \\ + x_5^2(b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4) + \\ + x_5(t_1(x_1x_3 + x_2x_4) + t_2(x_1x_4 + x_2x_3)) = 0,$$

with parameters $a, b_1, b_2, b_3, b_4, t_1, t_2 \in \mathbb{R}$. The singularity types of X are determined by the parameters as follows

- (1) 4A₁: general $a, b_1, b_2, b_3, b_4, t_1, t_2 \in \mathbb{R}$,
- (2) 2A₂ + 2A₁: $b_1 = b_2 = -\frac{(t_1-t_2)^2}{8}$,
- (3) 4A₂: $b_1 = b_2 = b_3 = b_4 = -\frac{(t_1-t_2)^2}{8}$.

Cubics in the second and third cases are in fact birational to cubics with 4A₁-singularities. Common birational models for all three types are forms of smooth divisors in $(\mathbb{P}^1)^4$ of degree (1, 1, 1, 1), see [CTZ24, Section 5]. We describe these models over \mathbb{R} .

Let $\pi: \tilde{X} \rightarrow X$ be the blowup of the singularities of X . Then there exists a commutative diagram:

$$\begin{array}{ccc} \tilde{X} & \overset{\rho}{\dashrightarrow} & \hat{X} \\ \pi \downarrow & & \downarrow \phi \\ X & \overset{\chi}{\dashrightarrow} & Y \end{array}$$

where

- ρ is a composition of flops in the strict transform of the lines passing through pairs of singular points,
- ϕ is a contraction of the strict transform of the hyperplane section containing p_1, \dots, p_4 to a smooth point q of the threefold Y , and

- Y is a form of a smooth divisor in $(\mathbb{P}^1)^4$ of degree $(1, 1, 1, 1)$ with Picard rank 2. Such divisors with Picard rank 1 are conjecturally irrational, see [KP24, Conjecture 1.3].

Over \mathbb{R} , Y can be embedded in $\mathbb{P}_{y_1, \dots, y_4}^3 \times \mathbb{P}_{z_1, \dots, z_4}^3$ with equations

$$\begin{cases} y_1^2 = y_2^2 + y_3^2 + y_4^2, \\ z_1^2 = z_2^2 + z_3^2 + z_4^2, \\ \mathbf{y} \cdot M \cdot \mathbf{z}^t = 0, \end{cases}$$

where $\mathbf{y} = (y_1, \dots, y_4)$, $\mathbf{z} = (z_1, \dots, z_4)$, and M is a 4×4 matrix with real entries. Finding simple normal forms of Y is related to the singular value decomposition in Minkowski space [Ren96].

The map χ^{-1} is given by the linear system $|\phi^*(-K_Y) - 3E|$, where E is the exceptional divisor of ϕ above q . Different choices of q on Y lead to birational maps to different cubic threefolds with 4 singular points. This allows to find birational maps between cubics with $4\mathbf{A}_1$, $4\mathbf{A}_2$ and $4\mathbf{A}_2 + 2\mathbf{A}_1$ -singularities. Explicitly, let X be a cubic given by (4.2) with $2\mathbf{A}_2 + 2\mathbf{A}_1$ or $4\mathbf{A}_2$ -singularities. Let p_5 be a general point on X . Then the restriction to X of the linear subsystem in $|\mathcal{O}_{\mathbb{P}^4}(4)|$ consisting of quartics having singularities of multiplicity 3 at the points p_1, \dots, p_5 gives a birational map $X \dashrightarrow X'$, where X' is a cubic threefold with $4\mathbf{A}_1$ -singularities in general position.

We study the topology of $X(\mathbb{R})$. As in (3.2), X is birational to the conic bundle

$$Y = \{y_1^2 + y_2^2 + (x_3 + x_4)f_3 - \frac{q_1^2 + q_2^2}{4}\} \subset \mathbb{P}_{y_1, y_2, x_1, x_2, x_3}(2, 2, 1, 1, 1),$$

with

$$(4.3) \quad \pi : Y \rightarrow \mathbb{P}_{x_3, x_4, x_5}^2, \quad (y_1, y_2, x_3, x_4, x_5) \mapsto (x_3, x_4, x_5),$$

where

$$\begin{aligned} q_1 &= x_3^2 + x_4^2 + b_1 x_5^2 + t_1 x_3 x_5 + t_2 x_4 x_5, \\ q_2 &= x_3^2 + x_4^2 + b_2 x_5^2 + t_1 x_4 x_5 + t_2 x_3 x_5, \\ f_3 &= a x_5^3 + b_3 x_3 x_5^2 + b_4 x_4 x_5^2. \end{aligned}$$

The real locus of the discriminant curve in \mathbb{P}^2 is smooth for all three singularity types of X . It follows that $Y(\mathbb{R})$ and $X(\mathbb{R})$ are connected if and only if $\pi(Y(\mathbb{R}))$ is.

Example 4.3. We provide examples with $X(\mathbb{R})$

- disconnected: $a = -3, b_1 = b_2 = b_3 = b_4 = 0, t_1 = 3, t_2 = 4$,

- connected: $a = 0, b_1 = b_2 = b_3 = b_4 = t_1 = t_2 = 1$.

$2A_3 + 2A_1$ -singularities. Assume that p_1, p_2 are A_3 -singularities and p_3, p_4 are A_1 -singularities. From [CMTZ24, Section 7], we know that the defect of $X_{\mathbb{C}}$ is 1 or 2. When the defect is 1, by [CMTZ24, Lemma 7.3], X is rational over \mathbb{R} . When the defect is 2, $X_{\mathbb{C}}$ contains three planes. Two of them are spanned by

$$\Pi_1 = \langle p_2, p_3, p_4 \rangle, \quad \Pi_2 = \langle p_1, p_3, p_4 \rangle,$$

and thus are complex conjugate. This implies that $r_1 = r_2 = 0$ in (4.1). The normal form of such X is

$$\begin{aligned} x_4(x_1^2 + x_2^2) + ax_5^3 + x_5^2(b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4) + \\ + x_5(t_2(x_1 + x_2)x_4 + t_6(x_3^2 + x_4^2)), \end{aligned}$$

with parameters $b_1, b_2, b_3, b_4, t_2, t_6, a \in \mathbb{R}$. The third plane is defined over \mathbb{R} and is given by

$$\Pi_3 = \{x_4 = x_5 = 0\}.$$

As before, projecting from Π_3 , the projection map

$$\pi : X \rightarrow \mathbb{P}^1, \quad (\mathbf{x}) \mapsto (x_4, x_5),$$

endows X with the structure of a quadric surface bundle.

Proposition 4.4. *Let $\Delta \in \mathbb{R}[x_5]$ be given by*

$$\Delta(x_5) := -\frac{b_1^2 + b_2^2}{4}x_5^3 + \frac{-2t_2t_6(b_1 + b_2) - b_3^2 + 4at_6}{4t_6}x_5^2 + (b_4 - \frac{1}{2}t_2^2)x_5 + t_6.$$

The locus $X(\mathbb{R})$ is disconnected if and only if $\Delta(x_5)$ has three real roots $\alpha_1 < \alpha_2 < \alpha_3$ and one of the following holds

- $t_6 > 0$ and $0 < \alpha_1$, or
- $t_6 < 0$ and $\alpha_3 < 0$.

Proof. Set $x_4 = 1$. The fiber $\pi^{-1}(x_5)$ is isomorphic to the quadric surface

$$\{u_1^2 + u_2^2 + t_6x_5u_3^2 + x_5\Delta(x_5)u_4^2 = 0\} \subset \mathbb{P}_{u_1, \dots, u_4}^3$$

Its real locus is empty if and only if t_6, x_5 and $\Delta(x_5)$ have the same sign. Note that $t_6 \neq 0$ due to the singularities of $X_{\mathbb{C}}$. \square

2D₄ + 2A₁-singularities. Assume p_1 and p_2 are D₄-singularities and p_3 and p_4 are A₁-singularities of $X_{\mathbb{C}}$. We recall some basic geometry from [CMTZ24, Section 7]. The class group $\text{Cl}(X_{\mathbb{C}})$ is generated by the classes of five planes Π_1, \dots, Π_5 in $X_{\mathbb{C}}$ and the class of a general hyperplane section F , subject to relations

$$\Pi_1 + \Pi_2 + \Pi_4 = \Pi_3 + \Pi_4 + \Pi_5 = F \in \text{Cl}(X_{\mathbb{C}}).$$

One observes that Π_4 is defined over \mathbb{R} . We have

$$(4.4) \quad \Pi_1 \supset \{p_2, p_3, p_4\}, \quad \Pi_2 \supset \{p_1, p_3, p_4\}, \quad \Pi_3, \Pi_4, \Pi_5 \supset \{p_1, p_2\}.$$

The line passing through p_3 and p_4 is disjoint from Π_3 and Π_5 .

Now consider the equation (4.1). The inclusions in (4.2) imply that $r_1 = r_2 = 0$. Up to isomorphism, we may assume

$$r_4 = t_1 = t_4 = t_5 = 0, \quad r_3 = 1.$$

The D₄-singularities impose the conditions

$$t_2 + it_3 = -it_3t_5 - b_1 + ib_2 = 0.$$

Since all parameters are real numbers, we have

$$t_2 = t_3 = b_1 = b_2 = 0.$$

After a change of variables, the normal form of X is given by

$$(4.5) \quad (x_1^2 + x_2^2)x_3 + x_5(x_3^2 + x_4^2) + (b_3x_3 + b_4x_4)x_5^2 + ax_5^3 = 0,$$

for $a \in \mathbb{R}^{\times}$ and $b_3, b_4 \in \mathbb{R}$. Note that a further change of variables can set $a = \pm 1$. Over \mathbb{C} , the 5 planes are given by

$$\Pi_1 = \{x_1 + ix_2 = x_5 = 0\}, \quad \Pi_2 = \{x_1 - ix_2 = x_5 = 0\},$$

$$\Pi_3 = \{x_3 = x_5 = 0\}, \quad \Pi_4 = \{x_3 = 2x_4 + \left(b_4 + \sqrt{b_4^2 - 4a}\right)x_5 = 0\},$$

$$\Pi_5 = \{x_3 = 2x_4 + \left(b_4 - \sqrt{b_4^2 - 4a}\right)x_5 = 0\}.$$

Rationality of X is determined by the **(H1)**-obstruction. The following is analogous to [CMTZ24, Corollary 7.8].

Proposition 4.5. *Assume that X has no real singular points and that $X_{\mathbb{C}}$ has 2D₄ + 2A₁ singularities. Then X is not (stably) rational over \mathbb{R} if and only if X contains only one real plane.*

Proof. From the equations we see that either X contains only one plane Π_4 or three planes Π_3, Π_4, Π_5 . Assume X contains only Π_4 . Then the Galois group Γ switches $\Pi_1 \leftrightarrow \Pi_2$ and $\Pi_3 \leftrightarrow \Pi_5$. One computes, cf. [CMTZ24, Proposition 7.7], that

$$(4.6) \quad H^1(\Gamma, \text{Pic}(\tilde{X}_{\mathbb{C}})) = \mathbb{Z}/2,$$

and thus X is not stably rational over \mathbb{R} .

When X contains three planes, the real line passing through p_3 and p_4 is disjoint from Π_3 , thus X is rational over \mathbb{R} . \square

Corollary 4.6. *Let X be given by (4.5). Then X is (stably) rational if and only if $b_4^2 > 4a$.*

Remark 4.7. The quadric surface bundle structure $\pi : X \rightarrow \mathbb{P}^1$ obtained via projection from Π_4 has been used, in [CTP24, Section 11.2], to establish the nontriviality of cohomology in (4.6), via a computation of the Brauer group and the identity

$$H^1(\Gamma, \text{Pic}(\tilde{X}_{\mathbb{C}})) = \text{Br}(\tilde{X}_{\mathbb{R}})/\text{Br}(\mathbb{R}).$$

5. SIX SINGULAR POINTS

6A₁-singularities with no plane.

Proposition 5.1. *Let X be a cubic without real singular points and such that $X_{\mathbb{C}}$ has 6A₁-singularities in linearly general position. Then X is (stably) rational over \mathbb{R} if and only if X contains a normal rational cubic scroll defined over \mathbb{R} .*

Proof. Recall from [CTZ24, Section 7] that the class group of $X_{\mathbb{C}}$ is generated by the hyperplane section H and two classes of normal rational cubic scrolls S_1 and S_2 , subject to the relation $2H = S_1 + S_2$.

First we show that X contains a real normal rational cubic scroll if and only if the classes S_1 and S_2 are Γ -invariant. Assume that S_1 and S_2 are Γ -invariant, then a general section of the linear system $|S_1|$ is such a cubic scroll. The other direction is easy to see.

When X does not contain a real cubic scroll, Γ switches S_1 and S_2 , and one can compute, cf. [CTZ24, Proposition 7.5], that

$$H^1(\Gamma, \text{Pic}(\tilde{X}_{\mathbb{C}})) = \mathbb{Z}/2\mathbb{Z}.$$

It follows that X is not stably rational over \mathbb{R} .

When X contains a real normal rational cubic scroll. The linear system $|\mathcal{O}_X(2) - S|$ gives a rational map $X \dashrightarrow \mathbb{P}^2$, whose resolution

gives X the structure of a \mathbb{P}^1 -bundle over \mathbb{P}^2 , see [CTZ24, Section 7]. Such a \mathbb{P}^1 -bundle over \mathbb{R} admits a section and is therefore rational. \square

Remark 5.2. In [CTZ24, Proposition 7.1], it is shown that any involution $\iota \in \text{Aut}(X_{\mathbb{C}})$ not fixing any singular points acts nontrivially on $\text{Cl}(X_{\mathbb{C}})$, see also Example 2.3.

However, this does not hold for the Galois action. For example, let X be the cubic given by

$$x_1(x_3^2 - x_4^2 - x_5^2) + 5x_2x_3x_4 - x_5(x_1^2 + 4x_2^2 - x_3^2) = 0.$$

Then $X_{\mathbb{C}}$ has three pairs of conjugate \mathbb{A}_1 -singular points in linearly general position. But Γ acts trivially on $\text{Cl}(X_{\mathbb{C}})$. In particular, X contains a real cubic scroll given by

$$\begin{aligned} \{-4x_2x_3 - x_3^2 + x_1x_4 + x_1x_5 = x_1x_3 + x_2x_4 - x_3x_4 + \\ + x_2x_5 + x_3x_5 = x_1^2 + 4x_2^2 - 3x_2x_3 - x_3^2 + 2x_1x_5 = 0\} \subset X. \end{aligned}$$

Therefore X is rational. In this case, the Galois action does not fix any singular points but acts trivially on $\text{Cl}(X_{\mathbb{C}})$ and satisfies **(H1)**. This does not happen in the equivariant setting.

Recall that there are also examples where the Galois action fails **(H1)** and the corresponding cubic is not stably rational, see Example 2.3.

6 \mathbb{A}_1 -singularities with one plane. All such cubics are rational:

Proposition 5.3. *Let X be a cubic such that $X_{\mathbb{C}}$ has 6 \mathbb{A}_1 -singularities and one plane. Then X is rational over \mathbb{R} .*

Proof. The unique plane is defined over \mathbb{R} and contains four of the \mathbb{A}_1 -points, see [CTZ24, Section 7]. The line through the two other \mathbb{A}_1 -points is also defined over \mathbb{R} and is disjoint from the plane. This implies rationality of X over \mathbb{R} . \square

6 \mathbb{A}_1 -singularities with three planes.

Proposition 5.4. *Let X be a cubic without real singular points and such that $X_{\mathbb{C}}$ has 6 \mathbb{A}_1 -singularities and three planes. Then up to isomorphism, one of the following holds:*

- X contains three real planes, and is given by

(5.1)

$$x_2x_3x_4 + ax_1^3 + x_1^2(a_1x_2 + a_2x_3 + a_3x_4) + x_1(x_2^2 + x_3^2 + x_4^2 + x_5^2) = 0$$

with $a, a_1, a_2, a_3 \in \mathbb{R}$.

- X contains only one real plane, and is given by

$$(5.2) \quad x_2(x_3^2 + x_4^2) + ax_1^3 + x_1^2(a_1x_2 + a_2x_3 + a_3x_4) + x_1(x_2^2 - x_4x_5 + x_5^2) = 0,$$

with $a, a_1, a_2, a_3 \in \mathbb{R}$.

Proof. One may assume that the three planes of $X_{\mathbb{C}}$ are contained in the hyperplane $\{x_5 = 0\}$. If the planes are defined over \mathbb{R} then we may assume that they are given by

$$\Pi_1 = \{x_1 = x_2 = 0\}, \quad \Pi_2 = \{x_1 = x_3 = 0\}, \quad \Pi_3 = \{x_1 = x_4 = 0\}$$

and that the singular points of X are

$$\begin{aligned} p_1 &= [0 : 0 : 0 : 1 : i], & p_2 &= [0 : 0 : 0 : 1 : -i], & p_3 &= [0 : 1 : 0 : 0 : i], \\ p_4 &= [0 : 1 : 0 : 0 : -i], & p_5 &= [0 : 0 : 1 : 0 : i], & p_6 &= [0 : 0 : 1 : 0 : -i]. \end{aligned}$$

All cubic threefolds singular at p_1, \dots, p_6 are given by

$$\begin{aligned} &x_2x_3x_4 + ax_1^3 + x_1^2(a_1x_2 + a_2x_3 + a_3x_4 + a_4x_5) + \\ &+ bx_1(x_2^2 + x_3^2 + x_4^2 + x_5^2) + x_1(c_1x_2x_3 + c_2x_2x_4 + c_3x_3x_4) = 0 \end{aligned}$$

with $a, b, a_1, a_2, a_3, a_4, c_1, c_2, c_3 \in \mathbb{R}$. Up to a change of variables,

$$a_4 = c_1 = c_2 = c_3 = 0 \quad \text{and} \quad b = 1.$$

We obtain (5.1).

If only one plane is defined over \mathbb{R} , the three planes on $X_{\mathbb{C}}$ can be given by

$$\Pi_1 = \{x_1 = x_2 = 0\},$$

$$\Pi'_2 = \{x_1 = x_3 + ix_4 = 0\}, \quad \Pi'_3 = \{x_1 = x_3 - ix_4 = 0\},$$

and the singular points of $X_{\mathbb{C}}$ are

$$\begin{aligned} p_1 &= [0 : 0 : 1 : i : 0], & p_3 &= [0 : 1 : 0 : 0 : i], & p_5 &= [0 : 0 : 1 : -i : 0], \\ p_2 &= [0 : 0 : 1 : i : i], & p_4 &= [0 : 1 : 0 : 0 : -i], & p_6 &= [0 : 0 : 1 : -i : -i]. \end{aligned}$$

Similarly as above, one can check that X is given by (5.2). \square

Projecting from Π_1 , the projection map

$$(5.3) \quad \pi : X \rightarrow \mathbb{P}^1, \quad (\mathbf{x}) \mapsto (x_1, x_2),$$

endows X with the structure of a quadric surface bundle.

Proposition 5.5. *Let X be a cubic given by (5.1). Put*

$$\Delta_1(t_1, t_2, t_3, x) := x^4 + t_1x^3 + (a - 4)x^2 - (4t_1 + t_2t_3)x + (-4a + t_2^2 + t_3^2).$$

Then one of the following holds:

(1) *When at least one of the following polynomials (in x_2)*

$$\Delta_1(a_1, a_2, a_3, x_2), \quad \Delta_1(a_2, a_1, a_3, x_2), \quad \Delta_1(a_3, a_2, a_1, x_2)$$

has real roots, then X is rational.

(2) *When none of the polynomials (in x_2)*

$$\Delta_1(a_1, a_2, a_3, x_2), \quad \Delta_1(a_2, a_1, a_3, x_2), \quad \Delta_1(a_3, a_2, a_1, x_2)$$

has a real root, then $X(\mathbb{R})$ is connected and contains no line disjoint from any of the planes $\Pi_i, i = 1, 2, 3$ in X .

Proof. In the affine chart $\{x_1 = 1\}$, the general fiber $\pi^{-1}(x_2)$ is isomorphic to the quadric surface

$$(5.4) \quad \{u_1^2 + (1 - \frac{x_2^2}{4})u_2^2 + u_3^2 + \frac{\Delta_1(a_1, a_2, a_3, x_2)}{x_2^2 - 4}u_4^2 = 0\} \subset \mathbb{P}_{u_1, \dots, u_4}^3.$$

When $\Delta_1(a_1, a_2, a_3, x_2)$, as a polynomial in x_2 , has a real root t , the fiber (5.4) above $x_2 = t$ is a quadric cone (including the cases $t = \pm 2$), which contains a line intersecting Π_1 at one point. Thus, the line is disjoint from Π_2 . It follows that X is rational. By symmetry, when $\Delta_1(a_2, a_1, a_3, x_2)$ (respectively, $\Delta_1(a_3, a_2, a_1, x_2)$) has a real root, there exists a line in X disjoint from Π_2 (respectively, Π_3), and X is rational.

When none of these three polynomials has a real root, the image of π is connected but none of the fibers of π is a hyperboloid. Therefore, $X(\mathbb{R})$ is connected but contains no line disjoint from Π_1 , Π_2 or Π_3 . \square

Proposition 5.6. *Let X be a cubic given by (5.2), and*

$$\Delta_2(x) := \frac{a_2^2}{4}x^4 - (a + a_2^2 + a_3^2)x^3 + (4a - a_1)x^2 + (4a_1 - 1)x + 4 \in \mathbb{R}[x].$$

Then $X(\mathbb{R})$ is disconnected if and only if one of the following holds:

- (1) Δ_2 has 2 distinct real roots α_1 and α_2 , and $0 < \alpha_1 < \alpha_2 < 4$,
- (2) Δ_2 has 4 distinct real roots $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ and $0 < \alpha_1 < \alpha_2 < 4$,
- (3) Δ_2 has 4 distinct real roots $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ and $\alpha_1 < 0 < \alpha_3 < \alpha_4 < 4$.

Proof. In the affine chart $\{x_2 = 1\}$, when $x_1 \neq 0, 4$, the fiber $\pi^{-1}(x_1)$ is isomorphic to the quadric surface given by

$$S = \{u_1^2 + u_2^2 + ru_3^2 + \frac{x_1^2}{4r} \cdot \Delta_2(x_1)u_4^2 = 0\} \subset \mathbb{P}_{u_1, \dots, u_4}^3,$$

where

$$r = -\frac{x_1(x_1 - 4)}{4}.$$

One can check that the fiber $\pi^{-1}(x_1)$ is empty if and only if $r > 0$ and $\Delta_2(x_1) > 0$. Since X has no real singular points and $X_{\mathbb{C}}$ only contains three planes, $\Delta_2(x_1)$ can only have simple roots. It follows that $X(\mathbb{R})$ is disconnected in the cases described in the statement. \square

2A₂ + 4A₁ and 2A₃ + 4A₁-singularities.

Proposition 5.7. *Let $X \subset \mathbb{P}^4$ be a cubic such that $X_{\mathbb{C}}$ has singularities of type*

$$2A_2 + 4A_1, \quad \text{or} \quad 2A_3 + 4A_1.$$

Then X is rational over \mathbb{R} .

Proof. Geometrically, the four A₁-points lie on one plane and the line passing through the other two singular points is disjoint from this plane, see [CMTZ24, Section 9]. The plane and the line are defined over \mathbb{R} ; thus, X is rational over \mathbb{R} . \square

6. EIGHT SINGULAR POINTS

Proposition 6.1. *Let X be a cubic without real singular points and such that $X_{\mathbb{C}}$ has 8A₁-singularities. Then X is (stably) rational over \mathbb{R} if and only if it satisfies (H1) if and only if X contains three planes over \mathbb{R} .*

Proof. We recall the geometry of X , over \mathbb{C} : it has five planes Π_1, \dots, Π_5 and 8 singular points p_1, \dots, p_8 with the following inclusion relations:

$$\begin{aligned} \Pi_1 &\supset \{p_1, p_2, p_6, p_8\}, \\ \Pi_2 &\supset \{p_1, p_2, p_5, p_7\}, \\ \Pi_3 &\supset \{p_5, p_6, p_7, p_8\}, \\ \Pi_4 &\supset \{p_3, p_4, p_5, p_6\}, \\ \Pi_5 &\supset \{p_3, p_4, p_7, p_8\}. \end{aligned}$$

Enumerating involutions $\iota \in \mathfrak{S}_8$ preserving the configuration of singular points and planes as above, we find that Γ acts on the eight singular points via one of the following possibilities for ι :

- (1) $\iota = (1, 2)(3, 4)(5, 6)(7, 8)$: the planes Π_3, Π_4, Π_5 are defined over \mathbb{R} and the line through p_1 and p_2 is disjoint from Π_4 (cf. [CTZ24, Section 8]). It follows that X is rational over \mathbb{R} .
- (2) $\iota = (1, 2)(3, 4)(5, 7)(6, 8)$: the planes Π_1, Π_2, Π_3 are defined over \mathbb{R} and the line through p_3 and p_4 is disjoint from Π_1 , thus X is rational over \mathbb{R} .

(3) $\iota = (1, 2)(3, 4)(5, 8)(6, 7)$: only Π_3 is defined over \mathbb{R} . One can compute (cf. [CTZ24, Section 8]) that

$$H^1(\langle \iota \rangle, \text{Pic}(\tilde{X}_{\mathbb{C}})) = \mathbb{Z}/2.$$

It follows that X is not stably rational over \mathbb{R} .

□

Remark 6.2. All three cases in the proof of Proposition 6.1 can be indeed realized, e.g.,

- (1) $a_1(x_1^2 + x_3^2)x_4 + x_4x_5^2 + a_2(x_2^2x_5 - x_3^2x_5 + x_4^2x_5) + a_3x_3x_4x_5 = 0$,
- (2) $a_1(x_1^2 - x_3^2)x_4 + x_4x_5^2 + a_2(x_2^2x_5 + x_3^2x_5 + x_4^2x_5) + a_3x_3x_4x_5 = 0$,
- (3) $a_1(x_1^2 + x_3^2)x_4 + x_4x_5^2 + a_2(x_2^2x_5 + x_3^2x_5 + x_4^2x_5) + a_3x_3x_4x_5 = 0$,

with general parameters $a_1, a_2, a_3 \in \mathbb{R}$.

7. NON-ISOLATED SINGULARITIES

Let X be a real cubic threefold such that $X_{\mathbb{C}}$ has non-isolated singularities. As before, we assume that $X^{\text{sing}}(\mathbb{R}) = \emptyset$ and $X(\mathbb{C})$ is not a cone. It easily follows from [Yok02, Proposition 4.2] that X^{sing} is either a pointless conic or a pointless form of the rational normal quartic curve. We proceed to analyze these cases.

X^{sing} is a smooth conic. Over \mathbb{R} , the normal form of such X is

$$(7.1) \quad c + x_1q_1 + x_2q_2 + x_3q_3 + x_4(x_1^2 + x_2^2 + x_3^2) = 0,$$

where

$$\begin{aligned} c &= a_1x_4^3 + a_2x_4x_5^2 + a_3x_4^2x_5 + a_4x_5^3, \\ q_1 &= a_5x_4^2 + a_6x_4x_5 + a_7x_5^2, \\ q_2 &= a_8x_4^2 + a_9x_4x_5 + a_{10}x_5^2, \\ q_3 &= a_{11}x_4^2 + a_{12}x_4x_5 + a_{13}x_5^2, \end{aligned}$$

with general parameters $a_1, \dots, a_{13} \in \mathbb{R}$. We have

$$X^{\text{sing}} = \{x_1^2 + x_2^2 + x_3^2 = x_4 = x_5 = 0\} \subset \mathbb{P}^4.$$

Note that X contains the plane $\Pi = \{x_4 = x_5 = 0\}$. As before, projection from Π yields a quadric surface bundle

$$\pi : X \rightarrow \mathbb{P}_{x_4, x_5}^1.$$

Proposition 7.1. *Let X be a cubic given by (7.1). Then $X(\mathbb{R})$ is disconnected if and only if $\Delta(x)$ has four distinct real roots, where $\Delta(x)$ is the quartic polynomial given by*

$$\begin{aligned} & -((a_7 + a_{10})^2 + a_{13}^2)x^4 + (4a_4 - 2(a_6 + a_9)(a_7 + a_{10}) - 2a_{12}a_{13})x^3 + \\ & + (4a_2 - 2(a_5 + a_8)(a_7 + a_{10}) - (a_6 + a_9)^2 - 2a_{11}a_{13} - a_{12}^2)x^2 + \\ & + (4a_3 - 2(a_5 + a_8)(a_6 + a_9) - 2a_{11}a_{12})x + \\ & + 4a_1 - (a_5 + a_8)^2 - a_{11}^2 \end{aligned}$$

Proof. Set $x_4 = 1$. The fiber $\pi^{-1}(x_5)$ is isomorphic to the quadric surface given by

$$\{u_1^2 + u_2^2 + u_3^2 + \Delta(x_5)u_4^2 = 0\} \subset \mathbb{P}_{u_1, u_2, u_3, u_4}^3.$$

It is clear that its real locus is empty if and only if $\Delta(x_5) > 0$. It follows that $X(\mathbb{R})$ is disconnected if and only if $\Delta \in \mathbb{R}[x]$ as a quartic polynomial has four distinct real roots. \square

Remark 7.2. When $X(\mathbb{R})$ is connected, the rationality of X is open, see [CTP24]. We provide examples with $X(\mathbb{R})$

- disconnected: $a_1 = a_2 = a_3 = 100, a_4 = 5, a_5 = 20, a_6 = a_7 = a_8 = \dots = a_{13} = 1$.
- connected: $a_4 = -1, a_2 = 1, a_1 = a_3 = a_5 = a_6 = a_7 = a_8 = \dots = a_{13} = 0$.

The connected example also appeared in [CTP24, Section 11.3].

Chordal cubic. A cubic $X_{\mathbb{C}}$ singular along a rational normal quartic curve C is unique: it is the secant variety of C , also known as the *chordal cubic*, see, e.g., [All03] or [Yok02]. Over \mathbb{R} , and under the assumption $C(\mathbb{R}) = \emptyset$, there is a unique such cubic X , given by

$$(7.2) \quad \{x_1^2x_2 + x_1x_2^2 + x_2x_3^2 + x_1x_4^2 - 2x_3x_4x_5 - (x_1 + x_2)x_5^2 = 0\}.$$

Proposition 7.3. *The chordal cubic X given by (7.2) is rational.*

Proof. Let $C = X^{\text{sing}}$. The linear system $|\mathcal{O}_X(2) - C|$ has dimension 5 and gives a contraction

$$\varphi : X \dashrightarrow S,$$

where $S = \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ is the Veronese surface. Projection from a general plane Π in \mathbb{P}^4 gives a rational map

$$\pi : X \dashrightarrow \mathbb{P}^1.$$

The product

$$\pi \times \varphi : X \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^2$$

is a birational map. It follows that X is rational over \mathbb{R} . \square

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