# COHOMOLOGICAL OBSTRUCTIONS TO EQUIVARIANT UNIRATIONALITY

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Abstract. We study cohomological obstructions to equivariant unirationality, with special regard to actions of finite groups on del Pezzo surfaces and Fano threefolds.

## 1. Introduction

Let $X$ be a smooth projective rational variety with a generically free regular action of a finite group $G$. We say that $X$ is *$G$-unirational*, respectively, *projectively $G$-unirational*, if there exists a dominant $G$-equivariant rational map

$$\mathbb{P}(V) \dashrightarrow X,$$

where $V$ is a representation of $G$, respectively, of a projective representation of $G$. In the analogy between geometry over nonclosed fields and equivariant geometry, this should be viewed as being dominated by projective space, i.e., unirationality, versus being dominated by a Brauer-Severi variety.

An obvious obstruction to rationality of $X$ over a field $k$ is the absence of $k$-rational points – a birational invariant. In the equivariant context, existence of $G$-fixed points is not an equivariant birational invariant – only the existence of fixed points upon restriction to abelian subgroups is; we refer to this as Condition **(A)**.

In general, unirationality and $G$-unirationality are difficult to establish or exclude, when the obvious obstructions, such as the absence of $k$-rational points, respectively, failure of Condition **(A)**, vanish. In this note, we explore new cohomological obstructions to (projective) unirationality, and apply them to del Pezzo surfaces and Fano threefolds. The obstructions arise from considerations of the $G$-action on the Picard group $\mathrm{Pic}(X)$. Concretely, the Leray spectral sequence yields homomorphisms in group cohomology

$$\mathrm{H}^{j-2}(G, \mathrm{Pic}(X)) \xrightarrow{\delta_j} \mathrm{H}^j(G, k^\times), \quad j = 2, 3,$$

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where $G$ acts trivially on $k^\times$. The images of $\delta_j$ are stable birational invariants. Moreover, they vanish in the presence of $G$-fixed points and for $G$-unirational $X$, see Section 2.

Similar homomorphisms exist in the framework of birational geometry over nonclosed fields $k$, for Galois cohomology. However, their role in that context is limited: the images are trivial in presence of $k$-points, which is a (stable) birational invariant and an obvious necessary condition for $k$-unirationality. In the equivariant context, the invariants are quite subtle and informative – this highlights a stark difference between birational geometry over $k$ and over the stack $BG$.

In this paper, we classify generically free regular actions of finite groups $G$ on smooth projective $X$ such that

- $X^A \neq \emptyset$, for all abelian $A \subseteq G$, and
- $\mathrm{Am}^3(X, H) \neq 0$, for some $H \subseteq G$,

when $X$ is a del Pezzo surface (see Theorem 4) or a Kummer quartic double solid, i.e., a double cover of $\mathbb{P}^3$, ramified in a Kummer surface arising from the Jacobian of a genus 2 curve with maximal automorphisms (see Theorem 5). In all cases, we find that the obstruction arises from

$$\mathrm{Q}_8,$$

the quaternion group of order 8. As a corollary, such varieties are not $G$-unirational, nor projectively $G$-unirational.

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## 2. Cohomological obstructions

Let $k$ be an algebraically closed field of characteristic zero and $X$ a smooth projective rational variety over $k$, with a regular action of a finite group $G$. The Leray spectral sequence for $G$-actions yields an exact sequence (see, e.g., [10, Section 3]):

$$\begin{aligned} 0 \to \mathrm{Hom}(G, k^\times) \to \mathrm{Pic}(X, G) \to \mathrm{Pic}(X)^G \xrightarrow{\delta_2} \mathrm{H}^2(G, k^\times) \\ \xrightarrow{\gamma} \mathrm{Br}([X/G]) \xrightarrow{\beta} \mathrm{H}^1(G, \mathrm{Pic}(X)) \xrightarrow{\delta_3} \mathrm{H}^3(G, k^\times), \end{aligned} \tag{2.1}$$

where $\mathrm{Pic}(X, G)$ is the group of isomorphism classes of $G$-linearized line bundles on $X$, and $\mathrm{Br}([X/G])$ is the Brauer group of the quotient stack. This gives rise to the following invariants:

- $\mathrm{Am}^2(X, G) := \mathrm{Im}(\delta_2)$, the *Amitsur group*, and
- $\mathrm{Am}^3(X, G) := \mathrm{Im}(\delta_3)$.

The *Amitsur* group

$$\mathrm{Am}(X,G)=\mathrm{Am}^2(X,G),$$

defined in [1, Section 6] as the image of $G$-invariant divisor classes

$$\mathrm{Pic}(X)^G \xrightarrow{\delta_2} \mathrm{H}^2(G,k^\times),$$

is a stable $G$-birational invariant. The same holds for the higher Amitsur group $\mathrm{Am}^3(X,G)$, see [10, Section 3].

We note in passing that basic results in group cohomology allow to reduce the study of these invariants to $p$-Sylow subgroups: vanishing of $\mathrm{Am}^j(X,H)$ for $p$-Sylow subgroups $H$ of $G$, for all $p$, implies the vanishing for all subgroups of $G$. Furthermore, both groups vanish when $G$ has a fixed point on $X$, by functoriality. They vanish for $G$-varieties which are stably linearizable, or when $G$ is cyclic. We record the following refinement:

**Proposition 1.** *Let $Y\to X$ be a $G$-equivariant morphism of smooth projective varieties with regular $G$-actions. Then*

$$\mathrm{Am}^j(X,G)\subseteq \mathrm{Am}^j(Y,G),\quad j=2,3.$$

*Proof.* Containment follows from functoriality of the Leray spectral sequence:

$$\begin{array}{ccc} \mathrm{Pic}(X)^G & \xrightarrow{\delta_2} & \mathrm{H}^2(G,k^\times) \\ \downarrow & & \| \\ \mathrm{Pic}(Y)^G & \xrightarrow{\delta_2} & \mathrm{H}^2(G,k^\times) \end{array} \qquad \begin{array}{ccc} \mathrm{H}^1(G,\mathrm{Pic}(X)) & \xrightarrow{\delta_3} & \mathrm{H}^3(G,k^\times) \\ \downarrow & & \| \\ \mathrm{H}^1(G,\mathrm{Pic}(Y)) & \xrightarrow{\delta_3} & \mathrm{H}^3(G,k^\times) \end{array}$$

□

We are now in the position to formulate necessary conditions for

- $G$-unirationality:
  - Condition **(A)**: for all abelian $H\subseteq G$, one has $X^H\neq\emptyset$.
  - Amitsur: for all $H\subseteq G$, one has $\mathrm{Am}^j(X,H)=0$, $j=2,3$.
- projective $G$-unirationality:
  - Amitsur: for all $H\subseteq G$, one has that $\mathrm{Am}^2(X,H)$ is cyclic.
  - Amitsur: for all $H\subseteq G$, one has $\mathrm{Am}^3(X,H)=0$.

Indeed, for *linear* actions, we have

$$\mathrm{Am}^2(X,G)=0,$$

since, by definition, the generator of $\mathrm{Pic}(\mathbb{P}(V))$ is $G$-linearized. For *projectively linear* actions, the Amitsur group is cyclic. Furthermore, for linear and projectively linear actions, we have

$$\mathrm{H}^1(G,\mathrm{Pic}(\mathbb{P}(V)))=0,$$

which implies the vanishing

$$\mathrm{Am}^3(X,G)=0.$$

We recall the definition of the *Bogomolov multiplier* of a finite group:

$$\mathrm{B}^2(G):=\mathrm{Ker}\left(\mathrm{H}^2(G,k^\times)\to\bigoplus_A \mathrm{H}^2(A,k^\times)\right),$$

where $A$ runs over all abelian subgroups of $G$. This invariant emerged in the study of Noether's problem; the main result is that it equals the unramified Brauer group of the quotient $V/G$, where $V$ is a faithful representation of $G$ [3]. More generally, one may consider higher-degree versions

$$\mathrm{B}^n(G):=\mathrm{Ker}\left(\mathrm{H}^n(G,k^\times)\to\bigoplus_A \mathrm{H}^n(A,k^\times)\right),\quad n\ge 2.$$

For $n=3$, it is easy to obtain nonvanishing, see the tables below.

| GapID | $G$ | $\mathrm{B}^3(G)$ |
|---|---|---|
| (8,4) | $\mathrm{Q}_8$ | $\mathbb{Z}/2$ |
| (16,9) | $\mathrm{Q}_{16}$ | $\mathbb{Z}/2$ |
| (16,12) | $C_2\times\mathrm{Q}_8$ | $(\mathbb{Z}/2)^3$ |
| (16,13) | $\mathfrak{D}_4:C_2$ | $\mathbb{Z}/2$ |
| (81,3) | $C_3^2:C_9$ | $\mathbb{Z}/3$ |
| (81,8) | $\mathrm{He}_3.C_3$ | $\mathbb{Z}/3$ |
| (81,10) | $C_3.\mathrm{He}_3$ | $(\mathbb{Z}/3)^2$ |
| (81,13) | $C_3\times C_9:C_3$ | $\mathbb{Z}/3$ |
| (81,14) | $C_3.C_3^3$ | $\mathbb{Z}/3$ |

Even more generally, one may consider

$$\mathrm{B}^n(G,M):=\mathrm{Ker}\left(\mathrm{H}^n(G,M)\to\bigoplus_A \mathrm{H}^n(A,M)\right),\quad n\ge 2,$$

for an arbitrary $G$-module $M$; as explained in [11], the group

$$\mathrm{B}^2(G,\mathrm{Pic}(X)^\vee\otimes k^\times)$$

also receives an obstruction to $G$-unirationality, when the action satisfies Condition **(A)**.

The following proposition clarifies the connection between Bogomolov multipliers and Amitsur invariants:

**Proposition 2.** *Let $X$ be a smooth projective variety with a regular action of a finite group $G$, satisfying Condition* **(A)**. *Then*

$$\mathrm{Am}^j(X,G)\subseteq\mathrm{B}^j(G),\quad j=2,3.$$

In the following sections we present examples with obstructions to $G$-unirationality, based on the nonvanishing $\mathrm{Am}^3(X, G)$, for $G = \mathrm{Q}_8$, the smallest group with nonvanishing $\mathrm{B}^3(G)$.

We now recall the general formalism for the computation of $\mathrm{Am}^3(X, G)$, from [12, §4]: Choose an appropriate Zariski open subset $U \subset X$, with boundary divisors $D_\alpha$, $\alpha \in \mathcal{A}$, generating $\mathrm{Pic}(X)$. We have an exact sequence

$$0 \to R \to \bigoplus_{\alpha\in\mathcal{A}} D_\alpha \to \mathrm{Pic}(X) \to 0, \tag{2.2}$$

where $R$ is the module of relations between the $D_\alpha$. The diagram of exact sequences

$$\begin{array}{ccccc} & & \mathrm{H}^2(G, \mathbb{G}_m(U)) & & \\ & & \downarrow & & \\ 0 \to \mathrm{H}^1(G, \mathrm{Pic}(X)) & \longrightarrow & \mathrm{H}^2(G, R) & \longrightarrow & \mathrm{H}^2(G, \bigoplus_{\alpha\in\mathcal{A}} D_\alpha) \\ & \searrow^{\delta_3} & \downarrow & & \\ & & \mathrm{H}^3(G, k^\times) & & \end{array}$$

allows to compute $\delta_3$, where the vertical sequence arises from the exact sequence

$$0 \to k^\times \to \mathbb{G}_m(U) \to R \to 0.$$

## 3. Del Pezzo surfaces

In this section, we study cohomological obstructions to (projective) unirationality of regular, generically free, $G$-actions on smooth del Pezzo surfaces $X$, over algebraically closed fields $k$ of characteristic zero. Let

$$\deg(X) := (-K_X)^2 \in [9, \dots, 1]$$

be the degree of $X$. We begin with a summary of known results:

- $G$-actions are known, in principle [7].
- The groups $\mathrm{Am}^2(X, G)$ have been determined [1, Proposition 6.7]; these are trivial when $\mathrm{rk}\,\mathrm{Pic}(X)^G = 1$ and $\deg(X) \le 6$.
- The groups $\mathrm{H}^1(G, \mathrm{Pic}(X))$ have been determined, starting with [13], [17]; these are trivial when $\deg(X) \ge 5$. There are algorithms to compute them when $G$ is cyclic, in [2], [15], and also for general $G$, in [10]. E.g., for a del Pezzo surface with an involution fixing a (necessarily unique) smooth curve of genus $g$, one has

$$\mathrm{H}^1(G, \mathrm{Pic}(X)) = (\mathbb{Z}/2)^{2g}.$$

- There is an algorithm to compute $\mathrm{Br}([X/G])$, in [10] and [12]. It is based on the stabilizer stratification of the $G$-action. When $G$ is abelian and $\mathrm{rk}\,\mathrm{Pic}(X)^G = 1$, all possibilities of $\mathrm{Br}([X/G])$ have been determined in [14].
- When $X$ is a del Pezzo surfaces of degree 1, one always has $X^G \neq \emptyset$, in particular,

$$\mathrm{Am}^j(X,H) = 0, \quad \forall H \subseteq G, \quad j = 2,3.$$

- $G$-unirationality of del Pezzo surfaces of degree $\geq 3$ has been settled in [8], it is equivalent to Condition **(A)**.

We complement these results by analyzing

$$\mathrm{Am}^3(X,G),$$

for all del Pezzo surfaces $X$. In particular, we classify all actions with Amitsur obstructions to (projective) $G$-unirationality.

In the following, we rely on (refinements of) tables of possible actions, going back to [7]. We focus on $p$-groups, and del Pezzo surfaces of degree $4, 3, 2$.

The tables below list pairs $(X, G)$ such that

- $X$ is $G$-minimal, i.e., there are no equivariant contractions,
- $X^G = \emptyset$,
- $\mathrm{H}^1(G, \mathrm{Pic}(X)) \neq 0$,
- the $G$-action on $X$ fails Condition **(A)**, when $\deg(X) \geq 3$.

**Del Pezzo surfaces of degree 4.** Only 2-groups can give rise to nontrivial cohomology in this case. There are 5 types of actions, distinguished in [7, Section 6]; all 2-groups that appear are contained in the following two types:

(II) $X = \{\sum_{j=0}^4 x_j^2 = x_0^2 + ax_1^2 - x_2^2 + ax_3^2 = 0\}$, $a \neq 0, \pm 1$.

(III) $X = \{\sum_{j=0}^4 x_j^2 = x_0^2 + \zeta_4 x_1^2 - x_2^2 + \zeta_4 x_3^2 = 0\}$.

Clearly, Type (II) specializes to Type (III), and we are only listing new Type (III) actions in the table. In Type (II), there are 5 conjugacy classes of actions of $C_2^2$, indicated in the table, all have the same cohomological invariants.

| | $G$ | GapID | $\mathrm{H}^2(G,k^\times)$ | $\mathrm{Pic}(X)^G$ | $\mathrm{H}^1(G,\mathrm{Pic}(X))$ | $\mathrm{Br}([X/G])$ | $\mathrm{Am}^3(X,G)$ |
|---|---|---|---|---|---|---|---|
| II | $C_2^2(5)$ | (4,2) | $\mathbb{Z}/2$ | $\mathbb{Z}^2$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^2$ | 0 |
| II | $C_2^3$ | (8,5) | $(\mathbb{Z}/2)^3$ | $\mathbb{Z}^2$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^4$ | 0 |
| II | $C_2^3$ | (8,5) | $(\mathbb{Z}/2)^3$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^4$ | 0 |
| II | $C_2^3$ | (8,5) | $(\mathbb{Z}/2)^3$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^4$ | 0 |
| II | $C_2^3$ | (8,5) | $(\mathbb{Z}/2)^3$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^4$ | 0 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| III | $C_2 \times C_4$ | (8,2) | $\mathbb{Z}/2$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^2$ | 0 |
| III | $C_2 \times \mathfrak{D}_4$ | (16,11) | $(\mathbb{Z}/2)^3$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^4$ | 0 |

**Example 3.** When $G$ is abelian, $\mathrm{Br}([X/G])$ in the table above has been computed in [14]. As a demonstration of the general algorithm, here we explain the computation of $\mathrm{Br}([X/G])$ in the last row. Consider the $G$-action on the $X$ in Type (III) generated by

$$\mathrm{diag}(-1,1,1,1,1), \quad \mathrm{diag}(1,1,1,1,-1), \quad (\mathbf{x}) \mapsto (x_3, x_4, x_1, x_2, x_5).$$

The fixed loci stratification of this action is takes the form

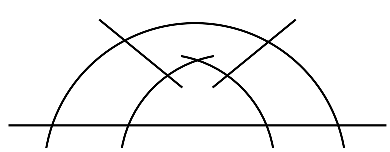
| | Strata | Stabilizer | Residue | dim | deg |
|---|---|---|---|---|---|
| 1 | $\mathfrak{p}_1$ | $\mathfrak{D}_4$ | triv | 0 | 1 |
| 2 | $\mathfrak{p}_2$ | $\mathfrak{D}_4$ | triv | 0 | 1 |
| 3-7 | $\mathfrak{p}_3 - \mathfrak{p}_7$ | $C_2^2$ | triv | 0 | 1 |
| 8 | $Q_1$ | $C_2$ | $C_2^2$ | 1 | 2 |
| 9 | $Q_2$ | $C_2$ | $C_2^2$ | 1 | 2 |
| 10 | $E_1$ | $C_2$ | $C_2^2$ | 1 | 4 |
| 11 | $E_2$ | $C_2$ | $\mathfrak{D}_4$ | 1 | 4 |

There are 7 curves with a nontrivial generic stabilizers. They are in the orbits of the curves

$$Q_1 = \{x_0 + x_2 = x_1 - x_3 = 0\} \cap X, \quad Q_2 = \{x_0 - x_2 = x_1 + x_3 = 0\} \cap X,$$

$$E_1 = \{x_0 = 0\} \cap X, \quad E_2 = \{x_4 = 0\} \cap X.$$

For each of these curves, the generic stabilizer is isomorphic to $C_2$ and the quotient by the residual action has genus 0. After blowing up the orbits of $\mathfrak{p}_1$ and $\mathfrak{p}_2$, the action is in the standard form. The configuration of curves with nontrivial generic stabilizers is illustrated as follows:



where the horizontal line comes from $E_2$; the big arc comes from $E_1$; the two small arcs come from $Q_1$ and $Q_2$, the two tilted lines come from the exceptional divisors. By [12, Corollary 4.6], this implies that

$$\mathrm{Br}([X/G]) = (\mathbb{Z}/2\mathbb{Z})^4.$$

**Cubic surfaces.** In this case, only 3-groups contribute to nontrivial cohomology. We follow [7, Table 4] for the classification of the actions, and focus on types with maximal automorphisms. They are

(I) Fermat cubic surface $X$, with $\mathrm{Aut}(X) = C_3^3 \rtimes \mathfrak{S}_4$,

(III) The cubic surface
$$X = \{6ax_2x_3x_4 + \sum_{i=1}^{4} x_i^3 = 0\}, \quad 20a^3 + 8a^6 = 1,$$
with $\mathrm{Aut}(X) = C_3^2 \rtimes \mathfrak{S}_3.C_2$.

The following table has been essentially computed in [10] and [14].

| | $G$ | GapID | $\mathrm{H}^2(G, k^\times)$ | $\mathrm{Pic}(X)^G$ | $\mathrm{H}^1(G, \mathrm{Pic}(X))$ | $\mathrm{Br}([X/G])$ | $\mathrm{Am}^3(X, G)$ |
|---|---|---|---|---|---|---|---|
| I | $C_3^2$ | (9,2) | $\mathbb{Z}/3$ | $\mathbb{Z}$ | $\mathbb{Z}/3$ | $(\mathbb{Z}/3)^2$ | 0 |
| I | $C_3^2$ | (9,2) | $\mathbb{Z}/3$ | $\mathbb{Z}$ | $\mathbb{Z}/3$ | $(\mathbb{Z}/3)^2$ | 0 |
| I | $C_3^3$ | (27,5) | $(\mathbb{Z}/3)^3$ | $\mathbb{Z}$ | $\mathbb{Z}/3$ | $(\mathbb{Z}/3)^3$ | $\mathbb{Z}/3$ |
| III | $C_3^2$ | (9,2) | $\mathbb{Z}/3$ | $\mathbb{Z}$ | $\mathbb{Z}/3$ | $(\mathbb{Z}/3)^2$ | 0 |
| III | $C_3^2$ | (9,2) | $\mathbb{Z}/3$ | $\mathbb{Z}$ | $\mathbb{Z}/3$ | $(\mathbb{Z}/3)^2$ | 0 |

The only action with nontrivial $\mathrm{Am}^3(X, G)$ is that of $G = C_3^3$ in Type (I); it fails Condition **(A)**.

**Del Pezzo surfaces of degree 2.** Again, only 2-groups matter. We follow [7, Table 6], and focus on types with maximal automorphisms. These are realized as $X \subset \mathbb{P}(2, 1, 1, 1)$, with equations

(I) $X = \{w^2 = x_1^3x_2 + x_2^3x_3 + x_3^3x_4\}$, and $\mathrm{Aut}(X) = C_2 \times \mathsf{PSL}_2(\mathbb{F}_{11})$.

(II) $X = \{w^2 = x_1^4 + x_2^4 + x_3^4\}$, and $\mathrm{Aut}(X) = C_2 \times (C_4^2 \rtimes \mathfrak{S}_3)$.

(III) $X = \{w^2 = x_1^4 + x_2^4 + x_3^4 + ax_1^2x_2^2\}$, for $a^2 = -12$, and $\mathrm{Aut}(X) = C_2 \times (\mathsf{SL}_2(\mathbb{F}_3) \rtimes C_2)$.

| | $G$ | GapID | $\mathrm{H}^2(G, k^\times)$ | $\mathrm{Pic}(X)^G$ | $\mathrm{H}^1(G, \mathrm{Pic}(X))$ | $\mathrm{Br}([X/G])$ | $\mathrm{Am}^3(X, G)$ |
|---|---|---|---|---|---|---|---|
| I | $C_2^3$ | (8,5) | $(\mathbb{Z}/2)^3$ | $\mathbb{Z}$ | $(\mathbb{Z}/2)^3$ | $(\mathbb{Z}/2)^6$ | 0 |
| I | $C_2^3$ | (8,5) | $(\mathbb{Z}/2)^3$ | $\mathbb{Z}$ | $(\mathbb{Z}/2)^3$ | $(\mathbb{Z}/2)^6$ | 0 |
| I | $C_2 \times C_4$ | (8,2) | $\mathbb{Z}/2$ | $\mathbb{Z}$ | $(\mathbb{Z}/2)^2$ | $(\mathbb{Z}/2)^3$ | 0 |
| I | $C_2 \times \mathfrak{D}_4$ | (16,11) | $(\mathbb{Z}/2)^3$ | $\mathbb{Z}$ | $(\mathbb{Z}/2)^2$ | $(\mathbb{Z}/2)^5$ | 0 |
| I | $\mathfrak{D}_4(2)$ | (8,3) | $\mathbb{Z}/2$ | $\mathbb{Z}^2$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^2$ | 0 |
| I | $\mathfrak{D}_4$ | (8,3) | $\mathbb{Z}/2$ | $\mathbb{Z}^2$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^2$ | 0 |
| II | $C_2^3(2)$ | (8,5) | $(\mathbb{Z}/2)^3$ | $\mathbb{Z}$ | $(\mathbb{Z}/2)^3$ | $(\mathbb{Z}/2)^6$ | 0 |
| II | $C_2 \times C_4(2)$ | (8,2) | $\mathbb{Z}/2$ | $\mathbb{Z}$ | $(\mathbb{Z}/2)^2$ | $(\mathbb{Z}/2)^3$ | 0 |
| II | $C_2 \times C_4(2)$ | (8,2) | $\mathbb{Z}/2$ | $\mathbb{Z}$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/4$ | $(\mathbb{Z}/2)^3$ | 0 |
| II | $C_2^2 \times C_4(2)$ | (16,10) | $(\mathbb{Z}/2)^3$ | $\mathbb{Z}$ | $(\mathbb{Z}/2)^2$ | $(\mathbb{Z}/2)^4$ | $\mathbb{Z}/2$ |
| II | $\mathrm{OD}_{16}$ | (16,6) | 0 | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | 0 |
| II | $C_2 \times \mathfrak{D}_4(2)$ | (16,11) | $(\mathbb{Z}/2)^3$ | $\mathbb{Z}$ | $(\mathbb{Z}/2)^2$ | $(\mathbb{Z}/2)^5$ | 0 |
| II | $C_2 \times Q_8$ | (16,12) | $(\mathbb{Z}/2)^2$ | $\mathbb{Z}$ | $(\mathbb{Z}/2)^2$ | contains $\mathrm{Q}_8$ | |
| II | $\mathfrak{D}_4 \rtimes C_2(2)$ | (16,13) | $(\mathbb{Z}/2)^2$ | $\mathbb{Z}$ | $(\mathbb{Z}/2)^2$ | contains $\mathrm{Q}_8$ | |
| II | $\mathfrak{D}_4 \rtimes C_2$ | (16,13) | $(\mathbb{Z}/2)^2$ | $\mathbb{Z}$ | $(\mathbb{Z}/2)^2$ | $(\mathbb{Z}/2)^2$ | $(\mathbb{Z}/2)^2$ |
| II | $C_2 \times C_4^2$ | (32,21) | $(\mathbb{Z}/2)^2 \oplus \mathbb{Z}/4$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | contains $C_2^2 \times C_4$ | |
| II | $C_2 \times \mathrm{OD}_{16}$ | (32,37) | $(\mathbb{Z}/2)^2$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | contains $C_2^2 \times C_4$ | |
| II | $C_4 \wr C_2$ | (32,11) | $\mathbb{Z}/2$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^2$ | 0 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| II | $C_4 \wr C_2$ | (32,11) | $\mathbb{Z}/2$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | contains $\mathrm{Q}_8$ | |
| II | $C_2 \times (\mathfrak{D}_4 \rtimes C_2)$ | (32,48) | $(\mathbb{Z}/2)^5$ | $\mathbb{Z}$ | $(\mathbb{Z}/2)^2$ | contains $C_2^2 \times C_4$ | |
| II | $C_2 \times (C_4 \wr C_2)$ | (64,101) | $(\mathbb{Z}/2)^3$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | contains $C_2^2 \times C_4$ | |
| II | $Q_8$ | (8,4) | 0 | $\mathbb{Z}^2$ | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}/2$ |
| II | $Q_8$ | (8,4) | 0 | $\mathbb{Z}^2$ | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}/2$ |
| II | $\mathfrak{D}_4(2)$ | (8,3) | $\mathbb{Z}/2$ | $\mathbb{Z}^2$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^2$ | 0 |
| II | $\mathfrak{D}_4(3)$ | (8,3) | $\mathbb{Z}/2$ | $\mathbb{Z}^2$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^2$ | 0 |
| II | $\mathfrak{D}_4 \rtimes C_2$ | (16,13) | $(\mathbb{Z}/2)^2$ | $\mathbb{Z}^2$ | $\mathbb{Z}/2$ | contains $\mathrm{Q}_8$ | |
| III | $C_2 \times C_4$ | (8,2) | $\mathbb{Z}/2$ | $\mathbb{Z}$ | $(\mathbb{Z}/2)^2$ | $(\mathbb{Z}/2)^3$ | 0 |
| III | $C_2 \times C_4$ | (8,2) | $\mathbb{Z}/2$ | $\mathbb{Z}$ | $\mathbb{Z}/2 \oplus \mathbb{Z}/4$ | $(\mathbb{Z}/4)^2$ | 0 |
| III | $\mathfrak{D}_4(2)$ | (8,3) | $\mathbb{Z}/2$ | $\mathbb{Z}^2$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^2$ | 0 |
| III | $\mathfrak{D}_4$ | (8,3) | $\mathbb{Z}/2$ | $\mathbb{Z}^2$ | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^2$ | 0 |

The Brauer group $\mathrm{Br}([X/G])$ was computed as in Example 3. Condition **(A)** is satisfied only for the action of $\mathrm{OD}_{16}$ in Type (II) and the two (nonconjugate) actions of $\mathrm{Q}_8$ in Type (II), see [16] for computational details. We summarize the analysis:

**Theorem 4.** *Let $X$ be a del Pezzo surface with a regular generically free action of a finite group $G$. Assume that $X$ is $G$-minimal and that $\mathrm{Am}^3(X, H) \neq 0$, for some $H \subset G$. Then, up to isomorphism, one of the following holds:*

1. *$X$ is the cubic surface given by*
$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$$
*and $G$ contains $H = C_3^3$ generated by*
$$\mathrm{diag}(1, \zeta_3, 1, 1), \quad \mathrm{diag}(1, 1, \zeta_3, 1), \quad \mathrm{diag}(1, 1, 1, \zeta_3).$$
2. *$X$ is a del Pezzo surface of degree 2 given by*
$$w^2 = x_1^4 + x_2^4 + x_3^4 + a x_1^2 x_2^2, \quad a^2 \neq 4,$$
*and $G$ contains one of the following $H$:*
   (a) $\mathrm{Q}_8$ *generated by*
$$(w, x_1, x_2, x_3) \mapsto (w, x_2, x_1, \zeta_4 x_3), \quad \mathrm{diag}(-1, -1, 1, \zeta_4),$$
   (b) $\mathrm{Q}_8$ *generated by*
$$(w, x_1, x_2, x_3) \mapsto (-w, x_2, x_1, \zeta_4 x_3),$$
$$(w, x_1, x_2, x_3) \mapsto (-w, x_2, -x_1, -x_3),$$
   (c) $C_2^2 \times C_4$ *generated by*
$$\mathrm{diag}(-1, 1, 1, 1), \quad \mathrm{diag}(-1, \zeta_4^3, \zeta_4, 1), \quad \mathrm{diag}(1, -1, 1, -1),$$
   (d) $C_2^2 \times C_4$ *generated by*
$$(w, x_1, x_2, x_3) \mapsto (w, \zeta_4^3 x_2, \zeta_4 x_1, x_3),$$
$$\mathrm{diag}(-1, 1, 1, 1), \quad \mathrm{diag}(1, \zeta_4, \zeta_4, -1),$$

(e) $\mathfrak{D}_4 \rtimes C_2$ *generated by*

$$(w, x_1, x_2, x_3) \mapsto (w, \zeta_4 x_2, \zeta_4 x_1, -x_3),$$
$$\mathrm{diag}(-1, -1, 1, -1), \quad \mathrm{diag}(1, \zeta_4^3, \zeta_4, 1).$$

*The above actions satisfy* $\mathrm{Am}^3(X, H) \neq 0$*; in particular, the* $G$*-actions are not projectively unirational. Moreover, the actions in 2(a) and 2(b) are the only ones where Condition* **(A)** *is satisfied for* $H$*.*

We explain these cases in more details.

Case 1: This has been addressed in [12, Section 5.3], yielding

$$\mathrm{Am}^2(X, H) = 0, \quad \mathrm{H}^2(H, k^\times) = (\mathbb{Z}/3)^3, \quad \mathrm{H}^1(H, \mathrm{Pic}(X)) = \mathbb{Z}/3.$$
$$\mathrm{Br}([X/H]) = (\mathbb{Z}/3)^3, \quad \mathrm{Am}^3(X, H) = \mathbb{Z}/3.$$

Case 2(a) and 2(b): In both cases, we have

$$\mathrm{H}^2(H, k^\times) = \mathrm{Am}^2(X, H) = 0, \quad \mathrm{H}^1(H, \mathrm{Pic}(X)) = \mathbb{Z}/2,$$

and the $H$-action on $X$ is in standard form. We compute the stabilizer stratification:

| | Strata | Stabilizer | Residue | dim | deg |
|---|---|---|---|---|---|
| 1–4 | $\mathfrak{p}_i$ | $C_4$ | triv | 0 | 1 |
| 5 | $\mathfrak{p}_5$ | $C_4$ | triv | 0 | 1 |
| 6 | $E$ | $C_2$ | $C_2^2$ | 1 | 4 |

The only divisor with nontrivial generic stabilizer is an $H$-invariant curve $E$ of genus 1, with a generic stabilizer $C_2$. Using Riemann-Hurwitz, we compute the genus $\mathrm{g}(E/H) = 0$. It follows that

$$\mathrm{Br}([X/H]) = 0, \quad \mathrm{Am}^3(X, H) = \mathbb{Z}/2.$$

Case 2(c) and 2(d): In both cases, we have

$$\mathrm{Am}^2(X, H) = 0, \quad \mathrm{H}^2(H, k^\times) = (\mathbb{Z}/2)^5, \quad \mathrm{H}^1(H, \mathrm{Pic}(X)) = (\mathbb{Z}/2)^2,$$

and the $H$-action on $X$ is in standard form. We see from [14, Section 4.2, Case 2.G24] that $\mathrm{Br}([X/H]) = (\mathbb{Z}/2)^4$, which implies that

$$\mathrm{Am}^3(X, H) = \mathbb{Z}/2.$$

Case 2(e): We have

$$\mathrm{Am}^2(X, H) = 0, \quad \mathrm{H}^2(H, k^\times) = (\mathbb{Z}/2)^2, \quad \mathrm{H}^1(H, \mathrm{Pic}(X)) = (\mathbb{Z}/2)^2.$$

We compute the stabilizer stratification:

| | Strata | Stabilizer | Residue | dim | deg |
|---|---|---|---|---|---|
| 1 | $\mathfrak{p}_1$ | $Q_8$ | triv | 0 | 1 |
| 2 | $\mathfrak{p}_2$ | $C_2$ | triv | 0 | 1 |
| 3 | $\mathfrak{p}_3$ | $C_2$ | triv | 0 | 1 |
| 4 | $\mathfrak{p}_4$ | $C_2$ | triv | 0 | 1 |
| 5 | $E$ | $C_2$ | $C_2^2$ | 1 | 4 |

The only curve with a nontrivial generic stabilizer is an $H$-invariant curve $E$ of genus 1. The points $\mathfrak{p}_1, \dots, \mathfrak{p}_4 \notin E$. We compute $\mathrm{g}(E/H) = 1$. Note that the $H$-action on $X$ is not in standard form. To achieve standard form, we need to blow up the orbit of $\mathfrak{p}_1$. But since $\mathfrak{p}_1 \notin E$, the exceptional divisors are rational curves disjoint from $E$, and thus they do not contribute to $\mathrm{Br}([X/H])$. It follows that

$$\mathrm{Br}([X/H]) = (\mathbb{Z}/2)^2, \quad \mathrm{Am}^3(X, H) = (\mathbb{Z}/2)^2.$$

Using the same method, we compute $\mathrm{Br}([X/G])$ for all $p$-groups $G$ not containing one of the cases above and conclude that these are the only cases with $\mathrm{Am}^3(X, H) \neq 0$. We list the results in the tables below. One can also check that among these six cases, Condition **(A)** is satisfied only in Case 2(a) and 2(b).

## 4. Quartic double solids

Let $X \to \mathbb{P}^3$ be a double cover ramified in a quartic surface $S$. A *very general* nodal $X$ with at most 7 nodes fails stable rationality [18, Theorem 1.1], see also [9]. *Any* nodal $X$ with at most most 6 nodes is irrational; it is rational when the number of nodes is at least 11 [6].

We consider $S$ of Kummer type, i.e., $S = \mathrm{J}(C)/2$, the quotient of the Jacobian of a genus two curve, modulo the standard involution. In this case, $X$ has 16 nodes, the maximal number of nodes on such $X$, in characteristic zero. We choose $X$ with maximal $\mathrm{Aut}(X)$, considered in [5]. They correspond to curves

$$C_1 : \{y^2 = x(x^4 - 1)\}, \qquad (4.1)$$
$$C_2 : \{y^2 = x^5 + 1\}$$

and we denote by $X_1$ and $X_2$ the corresponding threefolds. Their automorphisms are known

$$\mathrm{Aut}(X_1) = C_2 \times (C_2^4 \rtimes \mathfrak{S}_4), \quad \mathrm{Aut}(X_2) = C_2 \times (C_2^4 \rtimes C_5).$$

The Sylow subgroups of $\mathrm{Aut}(X_2)$ are $C_2^5$ and $C_5$, which have trivial $\mathrm{B}^3$. Thus, we focus on the first case. Let $X = X_1$. Explicitly, $X$ is given by

$$X_1 = \{w^2 = x_1^4 + x_2^4 + x_3^4 + x_4^4 - 4\zeta_4 x_1 x_2 x_3 x_4\} \subset \mathbb{P}(2, 1, 1, 1, 1), \quad \zeta_4 = e^{\frac{2\pi i}{4}}.$$

Next, we compute (via `magma`)

$$\mathrm{H}^1(G, \mathrm{Pic}(\tilde{X})), \tag{4.2}$$

for all subgroups of $\mathrm{Aut}(X)$; here $\tilde{X}$ is the blowup of the 16 nodes of $X$, a smooth model of $X$. We obtain:

**Theorem 5.** *Let $X \to \mathbb{P}^3$ be a double cover ramified in $S = \mathrm{J}(C)/2$, with $C$ given by* (4.1)*, and $G \subseteq \mathrm{Aut}(X)$. Assume that the $G$-action on $\tilde{X}$ satisfies Condition* **(A)**. *Then*

$$\mathrm{Am}^3(\tilde{X}, H) \neq 0 \text{ for some } H \subset G$$

*if and only if $G$ contains a $\mathrm{Q}_8$ generated by one of the following:*

$$(w, x_1, x_2, x_3, x_4) \mapsto (w, x_2, x_1, \zeta_4^3 x_3, \zeta_4 x_4),\ \mathrm{diag}(-1, -1, 1, \zeta_4, \zeta_4) \tag{4.3}$$

*or*

$$\begin{aligned} &(w, x_1, x_2, x_3, x_4) \mapsto (-w, x_2, x_1, \zeta_4 x_3, \zeta_4^3 x_4), \\ &(w, x_1, x_2, x_3, x_4) \mapsto (-w, x_2, -x_1, -x_3, x_4). \end{aligned} \tag{4.4}$$

*Proof.* The proof relies on the algorithm to compute $\mathrm{Am}^3(\tilde{X}, G)$ outlined in Section 2. We explain the main steps here and refer the reader to [16] for computational details.

By [5], the class group $\mathrm{Cl}(X)$ is generated by 32 irreducible surfaces $\Pi_i$ which map to 16 planes in $\mathbb{P}^3$. Let $\tilde{X}$ be the blowup of the 16 nodes of $X$. There is an exact sequence

$$0 \to R \to \bigoplus_{i=1}^{48} \mathbb{Z} \cdot D_i \to \mathrm{Pic}(\tilde{X}) \to 0, \tag{4.5}$$

where

- $D_i = \tilde{\Pi}_i$, $i = 1, \dots, 32$, and $\tilde{\Pi}_i$ is the strict transform of $\Pi$ in $\tilde{X}$,
- $D_{32+i} = E_i$, $i = 1, \dots, 16$, and $E_i$ are exceptional divisors above singular points $\mathfrak{p}_i$ of $X$,
- $R$ is the relation space spanned by

$$\sum_{i \in I} \left( \tilde{\Pi}_i + \sum_{t \in P_i} E_t \right) - \sum_{j \in J} \left( \tilde{\Pi}_j + \sum_{s \in P_j} E_s \right) = 0, \tag{4.6}$$

  for any $I, J \subset \{1, \dots, 32\}$ such that $|I| = |J| = 4$ and

$$\sum_{i \in I} \Pi_i, \quad \sum_{j \in J} \Pi_j \in |-2K_X|.$$

  Each set $P_i \subset \{1, \dots, 16\}$ consists of indices such that $t \in P_i$ if and only if $\mathfrak{p}_t \in \Pi_i$.

Using this presentation, we find that *all* groups $G$ such that

- (4.2) is nontrivial;
- Condition **(A)** is satisfied for the $G$-action on $X$;
- there is no $G$-fixed smooth point on $X$;
- $\mathrm{B}^3(G)$ is nontrivial;

contain a subgroup conjugate to one of the two groups $G = \mathrm{Q}_8$ given in (4.3) and (4.4).

For these two groups, we implement the general formalism explained in Section 2 to compute $\mathrm{Am}^3(\tilde{X}, G)$. For convenience, the computation uses a periodic resolution of $G = \mathrm{Q}_8$ given in [4, Chapter XII.7], to compute cohomology groups with coefficients in a general $G$-module $M$. Choose generators $x, y$ of $G$ such that

$$G = \langle x, y \mid x^2 = y^2, \quad xyxy^{-1} = 1 \rangle.$$

The group $\mathrm{H}^i(G, M)$ is the $i$-th cohomology of the periodic complex

$$M \xrightarrow{\begin{pmatrix}\Delta_x & \Delta_y\end{pmatrix}} M^2 \xrightarrow{\begin{pmatrix}L_x & L_{yx}\\ -L_y & -\Delta_x\end{pmatrix}} M^2 \xrightarrow{\begin{pmatrix}\Delta_x\\ -\Delta_{yx}\end{pmatrix}} M \xrightarrow{\Sigma} M \xrightarrow{\begin{pmatrix}\Delta_x\\ \Delta_y\end{pmatrix}} \cdots \tag{4.7}$$

where $\Delta_x := 1 - x$, $L_x := 1 + x$, and $\Sigma$ denotes the sum of all group elements in $G$.

Set $D = \cup_i D_i$ and $U = \tilde{X} \setminus D$, we have an exact sequence

$$1 \to k^\times \to \mathbb{G}_m(U) \to R \to 0. \tag{4.8}$$

Up to scalar, each element in $R$ of the form (4.6) lifts to a function $f_I/f_J \in \mathbb{G}_m(U)$, where $f_I$ and $f_J$ are degree-2 polynomials (on $X$) in the variables $w, x, y, z$, with weights $(2, 1, 1, 1)$, such that

$$\sum_{i \in I} \Pi_i = \{f_I = 0\} \cap X, \quad \sum_{j \in J} \Pi_j = \{f_J = 0\} \cap X.$$

To determine the image of $\delta_3$, we use the following commutative diagram mentioned in Section 2

$$\begin{array}{ccccc}
 & & \mathrm{H}^2(G, \mathbb{G}_m(U)) & & \\
 & & \downarrow & & \\
0 \to \mathrm{H}^1(G, \mathrm{Pic}(\tilde{X})) & \longrightarrow & \mathrm{H}^2(G, R) & \longrightarrow & \mathrm{H}^2(G, \bigoplus_{i=1}^{48} \mathbb{Z} \cdot D_i) \\
 & \searrow^{\delta_3} & \downarrow & & \\
 & & \mathrm{H}^3(G, k^\times) & &
\end{array}$$

First, we consider $G = \mathrm{Q}_8$ given by (4.3). Using (4.5), we find that

$$\mathrm{H}^1(G, \mathrm{Pic}(\tilde{X})) = (\mathbb{Z}/2)^2. \tag{4.9}$$

Combining (4.7) and (4.5), we identify (4.9) as elements in $R^2$, which can then be lifted to elements in $\mathbb{G}_m(U)^2$ via (4.8). Applying the differential $(\Delta_x \; -\Delta_{xy})$ in (4.7), we find that two elements of (4.9) map to $-1 \in \mathbb{G}_m(U)$ and the other two elements map to $1 \in \mathbb{G}_m(U)$. Since the $G$-action on $X$ satisfies Condition **(A)** and $\mathrm{B}^3(G) = \mathbb{Z}/2$, we know that

$$\mathrm{Am}^3(\tilde{X}, G) = \mathbb{Z}/2.$$

When $G = \mathrm{Q}_8$ is given by (4.4), the same steps as above yield

$$\mathrm{H}^1(G, \mathrm{Pic}(\tilde{X})) = \mathrm{Am}^3(\tilde{X}, G) = \mathbb{Z}/2.$$

□

**Remark 6.** The computation in the proof of Theorem 5 suggests that

$$\mathrm{Br}([\tilde{X}/G]) = \begin{cases} \mathbb{Z}/2 & \text{when } G \text{ is given by (4.3),} \\ 0 & \text{when } G \text{ is given by (4.4).} \end{cases}$$

We display the fixed locus stratification of $X$ for the action of (4.3):

| | Strata | Stabilizer | Residue | dim | deg |
|---|---|---|---|---|---|
| 1–2 | $\mathfrak{q}_i$ | $C_4$ | triv | 0 | 1 |
| 3–4 | $\mathfrak{q}_i$ | $C_4$ | triv | 0 | 1 |
| 5 | $E_1$ | $C_4$ | $C_2$ | 1 | 4 |
| 6 | $E_2$ | $C_2$ | $C_2^2$ | 1 | 4 |

and that of (4.4):

| | Strata | Stabilizer | Residue | dim | deg |
|---|---|---|---|---|---|
| 1–2 | $\mathfrak{q}_i$ | $C_4$ | triv | 0 | 1 |
| 3–4 | $\mathfrak{q}_i$ | $C_4$ | triv | 0 | 1 |
| 5–8 | $\mathfrak{q}_i$ | $C_4$ | triv | 0 | 1 |
| 9 | $E_1$ | $C_2$ | $C_2^2$ | 1 | 4 |
| 10 | $E_2$ | $C_2$ | $C_2^2$ | 1 | 4 |

Note that, unlike in the case of surfaces, we obtain a nontrivial Brauer group of the quotient stack despite the absence of divisors with nontrivial generic stabilizers. This suggests that one needs to blow up additional strata to see divisors that contribute to $\mathrm{Br}([\tilde{X}/G])$, see [12, Section 4 and 8] for another such instance and an explanation.

**Remark 7.** Considering the $G$-invariant hyperplane section of $X$ given by $x_4 = 0$ in (4.3) and (4.4), we recover the $G = \mathrm{Q}_8$-actions on the degree 2 del Pezzo surface from Case 2(a) and 2(b) in Theorem 4. Applying

Proposition 1, we obtain another proof that these surfaces are not $G$-unirational.

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