Final Take Home Examination.

Due before the end of term.

Q1. Let $f(t) \geq 0$ be a weight on $[0, 2\pi]$. Let

$$\sigma^2 = \inf \left[\int_0^{2\pi} |1 - \sum_{j \neq 0} a_j e^{ijt}|^2 f(t) dt \right]$$

be the error in approximating 1 by functions involving only non zero frequencies in $L_2(f dt)$. When is $\sigma^2 > 0$ and what is it?

Q2. The Riesz transforms map $L_\infty \to BMO$. Why do they map $BMO \to BMO$?

Q3. A function $f(x)$ on R is said to be in VMO if

$$\lim_{h \to 0} \sup_x \left[\frac{1}{2h} \int_{x-h}^{x+h} |f(y) - \frac{1}{2h} \int_{x-h}^{x+h} f(z) dz| dy \right] = 0$$

Can you find a function in BMO that is not in VMO? Show that uniformly continuous functions are in VMO and in fact are dense in VMO.

Q4. For the permutation group of n objects, besides $\pi(g) = 1$ and $\pi(g) = \sigma(g)$ the sign of the permutation g, show directly that there are no other 1 dimensional representations.

Q5. Can you find a function f such that both f and its Fourier transform $\widehat{f}$ have compact support? Given an arbitrary positive function $\phi(t)$ on $[0, \infty]$ decreasing to zero as $t \to \infty$ can you find f such that both f and $\widehat{f}$ satisfy

$$|f(x)| \leq C\phi(|x|)$$

$$|\widehat{f}(x)| \leq C\phi(|x|)$$

for some finite constant C?