Take Home Final.

Due on Dec 18

Q1. Consider a population of fixed size N that consists k types of size $n_1(t), n_2(t), \ldots, n_k(t)$ repectively in generation t. At generation $t+1$ the population reproduces according to the folowing rule. Each member of the new generation can be of type $1, 2 \ldots, k$ with probabilities $\frac{n_1(t)}{N}, \frac{n_2(t)}{N}, \ldots, \frac{n_k(t)}{N}$ repectively. The types of the N members of generation $t+1$ are chosen independently with these probabilities. This defines a Markov Chain with transition probabilities $\pi(\mathbf{m}, \mathbf{n})$ on the space $\mathcal{E}_N$ consisting of $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ with $n_1 + n_2 + \cdots + n_k = N$.

a). Write down explicitly the transition probability $\pi(\mathbf{m}, \mathbf{n})$.

b). For smooth functions f on R^k, show that the limit

$$(\mathcal{A}f)(x) = \lim_{\substack{N\to\infty \\ \frac{\mathbf{m}}{N}\to x}} N \sum_{\mathbf{n}} [f(\frac{\mathbf{n}}{N}) - f(\frac{\mathbf{m}}{N})] \; \pi(\mathbf{m}, \mathbf{n})$$

exists and evaluate $\mathcal{A}$ explicitly.

c). Consider the rescaled process $x(t) = \frac{\mathbf{n}(Nt)}{N}$ starting at time 0 from $a_N = \frac{\mathbf{n}}{N}$, with a distribution of P_{N,a_N}. Show that P_{N,a_N} is totally bounded and any limit P with $a_N \to x$ is a solution to the martingale problem for $\mathcal{A}$ that lives on the simplex $\{x = (x_1, x_2, \ldots, x_k) : x_i \ge 0, \sum x_i = 1\}$ and starts at time 0 from $a \in \mathcal{S}$.

d). Show that the solution to the martingale problem for $\mathcal{A}$ is unique, by verifying that the equation

$$\frac{\partial f(t,x)}{\partial t} = \mathcal{A}f \;; f(0,x) = f_0(x)$$

has solution which is a polynomial in $x = (x_1, x_2 \ldots, x_k)$ of degree n with coefficients that are smooth functions of t, provided the initial data $f_0(x)$ is a polynomial of n.

e). What happens to the process $x(t)$ as $t \to \infty$ under P_x? If one of many things can happen try to determine the respective probabilities as functions of x.

f). How will things change if between generations there is a possibility of mutation where each individual of type i can change its type to j with a small probability $\frac{p_{i,j}}{N}$ and remain the same type with prbability $1 - \frac{1}{N}\sum_{j \ne i} p_{i,j}$? (Different individuals act independently.)

Q2. Let $l(t)$ be the local time at the origin of the one dimensional Brownian motion $\beta(t)$.

$$l(t) = \int_0^t \delta(\beta(s)) ds$$

Define

$$\tau(t) = \{\sup s : l(s) \le t\}$$

Show that $\tau(t)$ is a right continuous process with independent increments and find its Levy-Khintchine representation.

Q3. The Brownina bridge $x(t)$ is the Gaussian process on $[0,1]$ with $E[x(t)] = 0$ and $E[x(s)x(t)] = \min(s,t) - st$. Show that its distribution is the same as that of $\beta(t) - t\beta(1)$. Show that it is a Markov process, and in fact a diffusion process with generator $\frac{1}{2}\frac{d^2}{dx^2} + b(t,x)\frac{d}{dx}$. Determine $b(t,x)$ explicitly. Show that the Brownian bridge is the conditional distribution of the Brownina motion on $[0,1]$ given that $\beta(1) = 0$. In other words the transition probability density $q(s,x;t,y)$ of the Brownian bridge is given by

$$q(s,x;t,y) = \frac{p(s,x;t,y)p(t,y;1,0)}{p(s,x;1,0)}$$

where

$$p(s,x;t,y) = \frac{1}{\sqrt{2\pi t}} \exp[-\frac{(y-x)^2}{2(t-s)}]$$

Q4. Under what conditions on the function $h(r)$ will the diffusion process with generator

$$\frac{1}{2}\Delta + h(r) < \frac{x}{r} \cdot \nabla >$$

where $(r = \sqrt{(x_1^2 + \cdots + x_d^2)}$, have an invariant probabiliy measure on R^d? Assume that $h(r)$ is a smooth bounded function of r with $h(0) = 0$.